Proiezioni ortogonali su Sottovarietà lineari

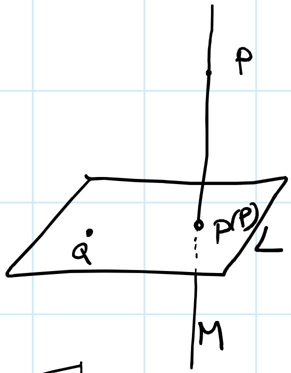
Definizione: in $\mathbb{R}^n$ $\qquad L: Q+V_L \qquad V_L \leq \mathbb{R}^n$

Dato un punto $P \in \mathbb{R}^n$ la proiezione ortogonale di P su L è

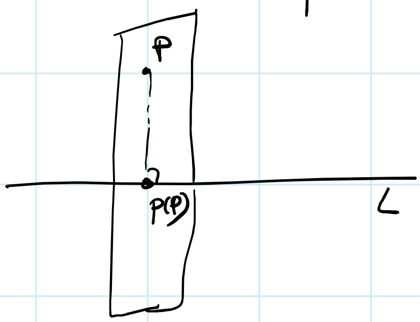
$p(P) = L \cap M \qquad L: Q+V_L$

$M: P+V_L^{\perp}$

Esempio: $\mathbb{R}^3$ $\quad L$ piano $\dim(L)=2$



$p(P) = L \cap M$

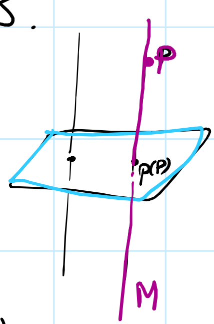


Se L è una retta $\dim(L)=1$

$p(P) = L \cap M$

Esercizio: determinare la proiezione ortogonale del punto $P = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ sul piano $\pi: 2x - y + 3z = 8$.

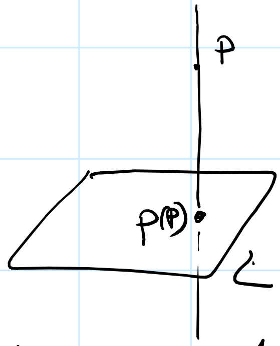
Svolg:



$P \notin \pi \qquad 2 \cdot 7 - 1 + 3 \cdot 3 \neq 8$

$p(P) = \pi \cap M \qquad M = P + V_\pi^{\perp} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\rangle$

$$\begin{cases} 2x-y+3z=8 \\ \begin{pmatrix}7\\1\\3\end{pmatrix}+\left\langle\begin{pmatrix}2\\-1\\3\end{pmatrix}\right\rangle \end{cases} \qquad p(P)=\begin{pmatrix}7+2a\\1-a\\3+3a\end{pmatrix} \qquad a\in\mathbb{R}$$

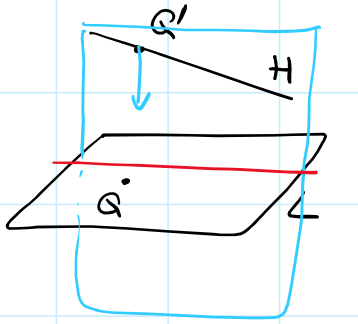


$$2(7+2a)-(1-a)+3(3+3a)=8$$

$$14+4a-\cancel{1}+a+\cancel{9}+9a=\cancel{8} \qquad 14a=-14 \qquad a=-1$$

$$\boxed{p(P)=\begin{pmatrix}5\\2\\0\end{pmatrix}}$$

Proiezione ortogonale di una sottovarietà H sulla sottovarietà lineare L.



$$H=Q'+V_H \qquad L=Q+V_L$$

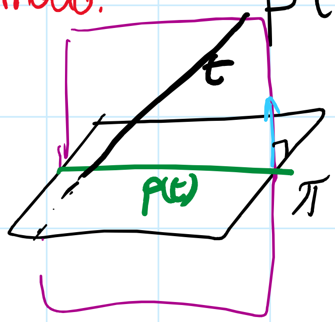
$$p(H)=L\cap M \qquad M=Q'+\langle V_H, V_L^{\perp}\rangle$$

Esempio: determinare la proiezione ortogonale della retta

$t:\begin{pmatrix}7\\1\\3\end{pmatrix}+\left\langle\begin{pmatrix}1\\1\\0\end{pmatrix}\right\rangle$ sul piano $\pi: 2x-y+3z=8$.

Svolg.:

1° modo: $p(t)=\pi\cap M \qquad M=\begin{pmatrix}7\\1\\3\end{pmatrix}+\left\langle\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\-1\\3\end{pmatrix}\right\rangle$



Equazione cartesiana di M

$$\begin{pmatrix}7\\1\\3\end{pmatrix} + \begin{pmatrix}a\\a\\0\end{pmatrix} + \begin{pmatrix}2b\\-b\\3b\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x-7\\y-1\\z-3\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}2\\-1\\3\end{pmatrix}$$

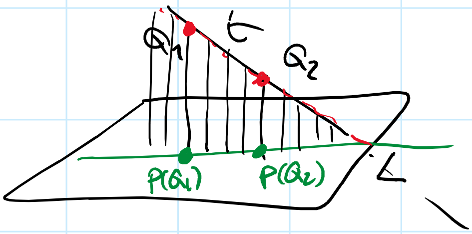
$$\det\begin{pmatrix}x-7 & 1 & 2\\ y-1 & 1 & -1\\ z-3 & 0 & 3\end{pmatrix} = 0 \qquad \begin{matrix}+\\-\\+\end{matrix}$$

$$3(x-7) - 3(y-1) + (z-3)(-3) = 0$$

$$3x - 21 - 3y + 3 - 3z + 9 = 0 \qquad M: 3x - 3y - 3z = 9$$

$$\boxed{p(t) = \begin{cases} 2x - y + 3z = 8 \\ x - y - z = 3 \end{cases}} \qquad \begin{matrix}\pi\\ \\ M\end{matrix}$$

2° modo:



$$t = Q_1 \vee Q_2$$

con Q_1, Q_2 punti di t distinti

$$p(t) = p(Q_1) \vee p(Q_2) \qquad Q_1 = \begin{pmatrix}7\\1\\3\end{pmatrix} \qquad p(Q_1) = \begin{pmatrix}5\\2\\0\end{pmatrix}$$

$$t: \overset{Q_1}{\begin{pmatrix}7\\1\\3\end{pmatrix}} + \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle \qquad Q_2 = Q_1 + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}7\\1\\3\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}8\\2\\3\end{pmatrix}$$

Per casa: calcolare $p(Q_2)$ proiezione ortogonale di Q_2 sul piano π. Verificate che $p(t) = p(Q_1) \vee p(Q_2)$

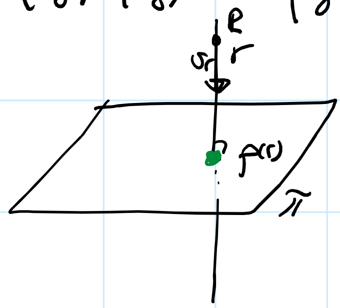
Esercizio: determinare la proiezione ortogonale della retta

$$r: \underset{R}{\begin{pmatrix}2\\1\\8\end{pmatrix}} + \left\langle \underset{v_r}{\begin{pmatrix}1\\1\\8\end{pmatrix}} \right\rangle$$ sul piano $\pi: x+y+8z=1$

Svolgimento:

$p(r) = \pi \cap M$

$$M = \underset{R}{\begin{pmatrix}2\\1\\8\end{pmatrix}} + \left\langle \underset{v_r}{\begin{pmatrix}1\\1\\8\end{pmatrix}}, \underset{V_\pi^\perp}{\begin{pmatrix}1\\1\\8\end{pmatrix}} \right\rangle = \begin{pmatrix}2\\1\\8\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\8\end{pmatrix} \right\rangle$$



$$\begin{cases} x+y+8z=1 \\ \begin{pmatrix}2\\1\\8\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\8\end{pmatrix} \right\rangle \end{cases} \qquad \begin{pmatrix}2+a\\1+a\\8+8a\end{pmatrix}$$

$$(2+a) + (\cancel{1}+a) + 64 + 64a = \cancel{1}$$

$$66 + 66a = 0 \qquad a = -1$$

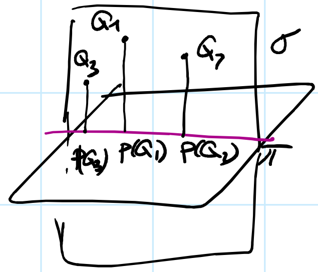
$$\boxed{p(r) = \begin{pmatrix}1\\0\\0\end{pmatrix}}$$

Esercizio: determinare la proiezione ortogonale del piano $\sigma: x - z = 5$ sul piano $\pi: x+y+z=7$.

Svolg.:

$$V_\sigma^\perp = \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\rangle \quad V_\pi^\perp = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \quad \begin{pmatrix}1\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 1+0-1=0 \quad \boxed{\sigma \perp \pi}$$

$\sigma: S+V_\sigma$



$p(\sigma)=\pi\cap M$

$M=S+\langle V_\sigma, \cancel{V_\pi^\perp}\rangle$ essendo $V_\pi^\perp=\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle$ $\begin{pmatrix}1\\1\\1\end{pmatrix}\in V_\sigma: x-z=0$
$1-1=0$

dato che $V_\pi^\perp\subseteq V_\sigma \Longrightarrow M=S+V_\sigma=\sigma$

$\boxed{p(\sigma)=\pi\cap\sigma}$ $\begin{cases}x+y+z=7\\x-z=5\end{cases}$

$\sigma=Q_1\vee Q_2\vee Q_3$ con Q_1,Q_2,Q_3 punti di σ non allineati

$p(\sigma)=p(Q_1)\vee p(Q_2)\vee p(Q_3)$

Esempio:

Determinare la proiezione del piano $\sigma: x=4$ sul piano

$\pi: x+y+z=7.$

Svolgimento: notiamo che $\boxed{\sigma \text{ non è ortogonale a } \pi}$ perché

$V_\sigma^\perp=\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\rangle$ $V_\pi^\perp=\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle$ $\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1+0+0\neq 0$

$V_\sigma=\boxed{x=0}$ $\begin{pmatrix}0\\y\\z\end{pmatrix}$

$\sigma: S+V_\sigma=\begin{pmatrix}4\\0\\0\end{pmatrix}+\langle\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\rangle$

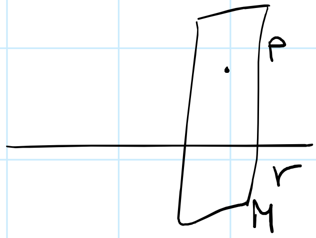
$p(\sigma)=\pi\cap M$ $M=S+\langle V_\sigma, V_\pi^\perp\rangle$

$M=\begin{pmatrix}4\\0\\0\end{pmatrix}+\underbrace{\langle\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle}_{\dim=3}=\mathbb{R}^3$ $\boxed{p(\sigma)=\pi\cap\mathbb{R}^3=\pi}$

Esercizio:

Calcolare la proiezione ortogonale del punto $P=\begin{pmatrix}1\\3\\1\end{pmatrix}$ sulla retta

$r: \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$. $\qquad V_r = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$



Svolgimento:

$\boxed{p(P) = r \cap M \qquad M = P + V_r^{\perp}}$

$\dim(V_r^{\perp}) = 2$ quindi M è il piano $\perp$ ad r passante per P

$M: x+z=K$ $\qquad$ imponiamo il passaggio per $P=\begin{pmatrix}1\\3\\1\end{pmatrix}$

$1+1=K \qquad \Longrightarrow M: x+z=2$

$p(P) = r \cap M \quad \begin{cases} \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle \\ x+z=2 \end{cases}$

$\boxed{\begin{pmatrix}a\\0\\a\end{pmatrix} \quad a+a=2 \Rightarrow a=1}$

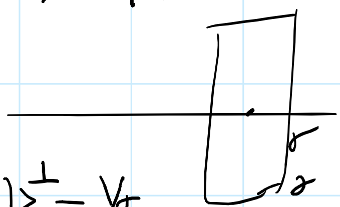
$\boxed{p(P) = \begin{pmatrix}1\\0\\1\end{pmatrix}}$

Esercizio:

determinare la proiezione ortogonale del piano

$\sigma: x+3y=1$ sulla retta $r: \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\3\\0\end{pmatrix} \right\rangle$.

Svolg:



$p(\sigma) = r \cap M = r \cap \sigma$

$V_r^{\perp} = \left\langle \begin{pmatrix}1\\3\\0\end{pmatrix} \right\rangle^{\perp} = V_\sigma$

$M = S + \langle V_\sigma, V_r^{\perp} \rangle = S + \langle V_\sigma \rangle = \sigma$

E

Esercizio:

Si consideri la retta $r: \begin{pmatrix}2\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

1) Determinare, se esiste, l'equazione cartesiana di un piano π contenente r (1) ed ortogonale al vettore (2) $v = \begin{pmatrix}1\\0\\1\end{pmatrix}$

2) Verificare che $r \subseteq \sigma$ con $\sigma: x - z = 2$ e determinare le equazioni cartesiane delle rette s_1 e s_2 contenute in σ e distanti $\sqrt{2}$ da r.

3) Determinare la distanza fra r e $t: \begin{cases} x+y+z=1 \\ 2x+y+z=1 \end{cases}$.

Svolgimento:

1) (2) $\pi \perp v = \begin{pmatrix}1\\0\\1\end{pmatrix}$ $\qquad x+z=k$

(1) $r \subseteq \pi$ $\qquad r: \underset{R}{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $\qquad R \in \pi \quad 2+0=k$

$\begin{pmatrix}2\\1+a\\0\end{pmatrix}$ è un generico punto di r.

$\pi: x+z=2$
$2+0=2$ sì $\Rightarrow r \subseteq \pi$ $\qquad \boxed{\pi: x+z=2}$

2) Verificare $r: \overset{R}{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ è contenuta in $\sigma: x-z=2$.

1° modo $\quad r \subseteq \sigma \iff \begin{cases} R \in \sigma \\ V_r \subseteq V_\sigma \end{cases}$ $\quad R = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in \sigma \quad 2-0=2$

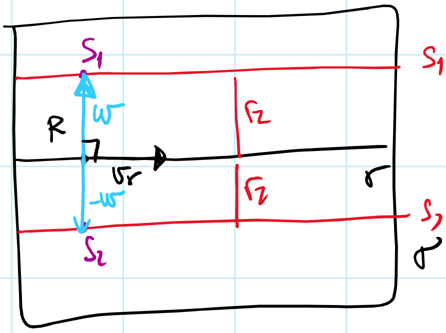
$V_r = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle \subseteq V_\sigma : x - z = 0$

2° modo $\quad r \subseteq \sigma \quad \begin{pmatrix} 2 \\ 1+a \\ 0 \end{pmatrix}$ è un generico punto di r

$\sigma : x - z = 2 \qquad 2 - 0 = 2$ sì $\Rightarrow r \subseteq \sigma.$

3° modo

$r \cap \sigma \quad rg(A) = rg(A|b) = 2 \qquad r \subseteq \sigma.$



$r : \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle \qquad R = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$V_r = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$

$s_1 : S_1 + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$

$s_2 : S_2 + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$

$\sigma : x - z = 2$

$S = R + w \qquad \begin{cases} ① & w \in V_\sigma \\ ② & w \cdot v_r = 0 \\ ③ & \|w\| = \sqrt{2} \end{cases} \qquad w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

① $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V_\sigma : x - z = 0 \qquad z = x \qquad \boxed{w = \begin{pmatrix} x \\ y \\ x \end{pmatrix}}$ ①

② $w \cdot v_r = 0 \qquad \begin{pmatrix} x \\ y \\ x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = y = 0 \qquad \boxed{w = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix}}$ ① ②

③ $\|w\| = \sqrt{2} \qquad \|w\| = \left\| \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} \right\| = \sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2}\,|x| = \sqrt{2} \Rightarrow |x| = 1$

se $x = 1 \quad \boxed{w = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \longrightarrow S_1 = R + w = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

oppure

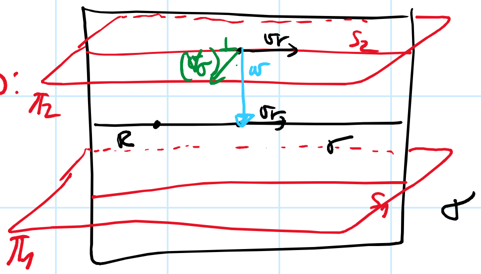
se $x = -1 \quad \boxed{w' = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}} \longrightarrow S_2 = R + w' = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$w' = -w$

S_1: $S_1 + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$ $\boxed{S_1: \begin{cases} x = 3 \\ z = 1 \end{cases}}$

S_2: $S_2 + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$ $\boxed{S_2: \begin{cases} x = 1 \\ z = -1 \end{cases}}$

2° metodo:



Cerchiamo π_1 e π_2 piani ortogonali a σ $\overset{/\!/ r}{e}$ distanti $\sqrt{2}$ da r.

$\pi: \ ax + by + cz = d$ $\sigma: \ \underline{x - z} = 2$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$ $\pi \perp \sigma$ $a - c = 0$ $c = a$

$\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ $\boxed{\pi: \ ax + by + az = d}$

$V_r \subseteq V_\pi$ $V_r = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$ $V_\pi: \ ax + by + az = 0$

$0 + b + 0 = 0 \implies b = 0$

$\pi: \ ax + az = d$ $\boxed{\pi: \ x + z = k}$

$a \neq 0$

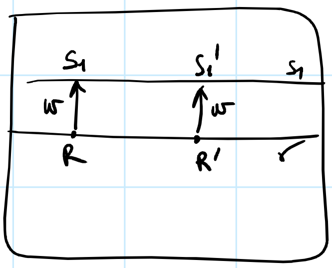
$d(r, \pi) = \sqrt{2}$ $r: \underset{R}{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}} + \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$ $d(r, \pi) = d(R, \pi)$ perché $r /\!/ \pi$

$d(R, \pi) = \dfrac{|2 + 0 - k|}{\sqrt{1 + 1}} = \dfrac{|2 - k|}{\sqrt{2}} = \sqrt{2}$ $|2 - k| = 2$

$2 - k = 2 \implies k = 0$

$2 - k = -2 \implies k = 4$

$$\boxed{\begin{array}{lll} S_2: \begin{cases} x + z = 0 \\ x - z = 2 \end{cases} & & \begin{cases} x = 1 \\ z = -1 \end{cases} \\ S_1: \begin{cases} x + z = 4 \\ x - z = 2 \end{cases} & & \begin{cases} x = 3 \\ z = 1 \end{cases} \end{array}}$$



$\|w\| = \sqrt{2}$ $w \in V_\sigma$ $w \perp \sigma_r$

3) Determinare la distanza fra le retta

$r: \begin{pmatrix}2\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ e $t: \begin{cases} x+y+z=1 \\ 2x+y+z=1 \end{cases}$

$\begin{cases} x=2 \\ z=0 \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{array}\right)$ $\begin{array}{c} 1° \\ 3°\text{-}1° \\ 4°\text{-}3° \\ 2° \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $\begin{array}{c} \\ \\ \\ 3°\text{-}1 \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$

r e t sono sghembe.

$r \cap t = \emptyset$ $\begin{pmatrix}2\\1+a\\0\end{pmatrix}$ $t \begin{cases} 2+\cancel{1}+a+0=\cancel{1} \\ 4+\cancel{1}+a+0=\cancel{1} \end{cases}$ $\begin{cases} a=-2 \\ a=-4 \end{cases}$ impossibile DISGIUNTE

$V_r \cap V_t = \{\vec{0}\}$ $V_r = \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $V_t: \begin{cases} x+y+z=0 \\ 2x+y+z=0 \end{cases}$ $\begin{cases} 0+a+0=0 \\ 0+a+0=0 \end{cases}$ $a=0$

$\begin{pmatrix}0\\a\\0\end{pmatrix} \xrightarrow[\text{se } a=0]{} \begin{pmatrix}0\\0\\0\end{pmatrix}$

Cerchiamo π contenente t e parallelo ad r.

$t: \begin{cases} x+y+z=1 \\ 2x+y+z=1 \end{cases}$ $\begin{cases} x+y+z=1 \\ 2°\text{-}1°\; x=0 \end{cases}$ $\begin{cases} y+z=1 \\ x=0 \end{cases}$

$\pi_{\alpha,\beta}: \alpha(\cancel{y+z}-1)+\beta x=0$ $\pi_{\alpha,\beta}$ contiene t

$\pi_{\alpha,\beta} /\!/ r$ $V_r \subseteq V_{\pi_{\alpha,\beta}}$

$\left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \subseteq \alpha(y+z)+\beta x=0$

$\alpha(1+0)+\beta \cdot 0=0$ $\alpha=0$

$\beta=1$

$\boxed{\pi: x=0}$ $\boxed{d(r,t)=d(R,\pi)=\frac{|2|}{\sqrt{1}}=2}$

$r: \boxed{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$