Esercizio

Esercizio 4. Nello spazio euclideo di dimensione 3 si consideri la retta r:

$$\begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}1\\-1\\2\end{pmatrix}\rangle$$

1. Determinare equazioni cartesiane della retta r. Determinare equazioni parametriche della retta s passante per $P=\begin{pmatrix}1\\3\\-2\end{pmatrix}$ ed ortogonale al piano $\pi: 2x+y+z=8$. (2 pti)
2. Determinare le bisettrici delle rette r ed s (nella forma $P+V$). (2 pti)
3. Calcolare la distanza tra la retta r e il piano $\sigma: 2x-z=1$. (1 pto)
4. Dati $v=\begin{pmatrix}1\\-1\\2\end{pmatrix}$ e $w=\begin{pmatrix}2\\1\\1\end{pmatrix}$ calcolare $v\times w$. (1 pto)

Svolgimento:

$$r: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}1\\-1\\2\end{pmatrix}\rangle$$

$$s \quad \begin{cases} P=\begin{pmatrix}1\\3\\-2\end{pmatrix}\in s \\ s \perp \pi: 2x+y+z=8 \end{cases} \qquad s: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}2\\1\\1\end{pmatrix}\rangle \qquad V_s = V_\pi^\perp$$

$$\boxed{s: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}2\\1\\1\end{pmatrix}\rangle} \qquad \boxed{\begin{cases} x=1+2a\\ y=3+a\\ z=-2+a\end{cases}}$$

Equazione parametriche di s.

$a\in\mathbb{R}$

2)

$$r: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}1\\-1\\2\end{pmatrix}\rangle \qquad s: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}2\\1\\1\end{pmatrix}\rangle$$

$$\boxed{\begin{aligned} b_1&: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle w_r+w_s\rangle \\ b_2&: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle w_r-w_s\rangle\end{aligned}}$$

$w_r\in V_r=\langle\begin{pmatrix}1\\-1\\2\end{pmatrix}\rangle$

$w_s\in V_s=\langle\begin{pmatrix}2\\1\\1\end{pmatrix}\rangle$

$\|w_r\|=\|w_s\|\neq 0$

$$w_r=\begin{pmatrix}1\\-1\\2\end{pmatrix} \quad w_s=\begin{pmatrix}2\\1\\1\end{pmatrix} \quad \|w_r\|=\sqrt{1+1+4}=\|w_s\|=\sqrt6$$

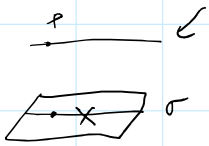
$$\boxed{\begin{aligned} b_1&: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}3\\0\\3\end{pmatrix}\rangle \\ b_2&: \begin{pmatrix}1\\3\\-2\end{pmatrix}+\langle\begin{pmatrix}-1\\-2\\1\end{pmatrix}\rangle\end{aligned}}$$

3) $\sigma: 2x-z=1$ $\quad d(r,\sigma)$

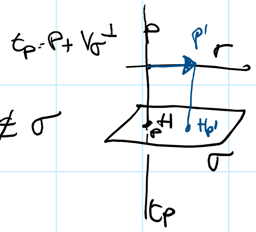
$$r: \underset{P}{\begin{pmatrix}1\\3\\-2\end{pmatrix}}+\langle\begin{pmatrix}1\\-1\\2\end{pmatrix}\rangle, \quad r/\!/\sigma \iff V_r=\langle\begin{pmatrix}1\\-1\\2\end{pmatrix}\rangle\subseteq V_\sigma: 2x-z=0$$

$$\iff \begin{pmatrix}1\\-1\\2\end{pmatrix}\in V_\sigma \quad 2\cdot 1-2=0 \text{ sì}$$

$\Rightarrow r/\!/\sigma$



$P \in \sigma$? $\begin{pmatrix}1\\3\\-2\end{pmatrix}$ $\boxed{\sigma: 2x-z=1}$ $2\cdot 1+2=4\neq 1$ $P\notin\sigma$



$d(r,\sigma)=d(P,\sigma)$ $P=\begin{pmatrix}1\\3\\-2\end{pmatrix}\in r$ $d(P,\sigma)=\dfrac{|2\cdot 1+2-1|}{\sqrt{4+0^2+1}}=\dfrac{3}{\sqrt5}$

cmd (P, H_P)

Esercizio: dati i punti $A=\begin{pmatrix}1\\2\\1\end{pmatrix}$ e $B=\begin{pmatrix}1\\4\\1\end{pmatrix}$ determinare un punto C tale che

$$\begin{cases} ① \text{ il triangolo } A,B,C \text{ sia contenuto nel piano } \pi: x-z=0 \\ ② \ C\hat{A}B=\pi/2 \\ ③ \ \|C-A\|=\sqrt2. \end{cases}$$

Svolgimento:

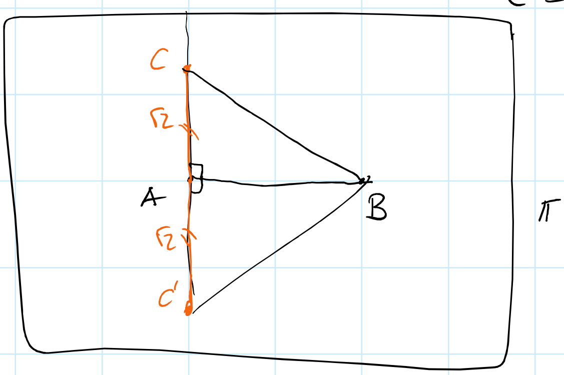
$C=\begin{pmatrix}x\\y\\z\end{pmatrix}$ ① $\pi: x-z=0$

$A=\begin{pmatrix}1\\2\\1\end{pmatrix}\in\pi$ $1-1=0$

$B=\begin{pmatrix}1\\4\\1\end{pmatrix}\in\pi$ $1-1=0$

$C=\begin{pmatrix}x\\y\\z\end{pmatrix}\in\pi \Leftrightarrow x-z=0$ $z=x$

$C=\begin{pmatrix}x\\y\\x\end{pmatrix}$



② $C\hat{A}B=\dfrac{\pi}{2}$ $(B-A)\cdot(C-A)=0$

$\begin{pmatrix}0\\2\\0\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-2\\x-1\end{pmatrix}=0$ $0\cdot(\cancel{x-1})+2(y-2)+0(\cancel{x-1})=0$ $\boxed{y=2}$

$C=\begin{pmatrix}x\\2\\x\end{pmatrix}$

③ $\|C-A\|=\sqrt2$ $C-A=\begin{pmatrix}x\\2\\x\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}x-1\\0\\x-1\end{pmatrix}$

$$\|C-A\| = \sqrt{(x-1)^2+(x-1)^2} = \sqrt{2(x-1)^2} = \sqrt{2}\,|x-1| = \sqrt{2} \iff |x-1|=1$$

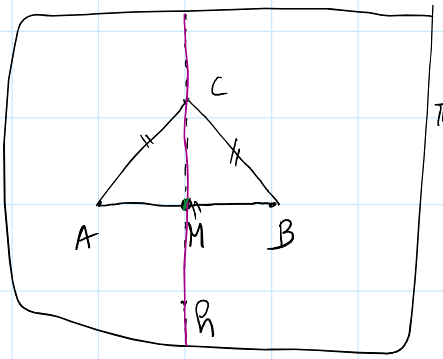
$$\Rightarrow \quad \begin{array}{ll} x-1=1 & x=2 \\ x-1=-1 & x=0 \end{array} \qquad \boxed{\begin{array}{l} C=\begin{pmatrix}2\\2\\2\end{pmatrix} \\ C'=\begin{pmatrix}0\\2\\0\end{pmatrix}\end{array}}$$

Esempio:

Dati A,B punti $A \neq B$ contenuti nel piano π

Determinare $C \in \pi$ tale che il triangolo A,B,C sia equilatero.

Svolgimento:



$M = \frac{A+B}{2}$ $\qquad \sigma = \frac{A+B}{2} + \langle B-A \rangle^{\perp}$ asse di A,B

$\boxed{h = \sigma \cap \pi}$

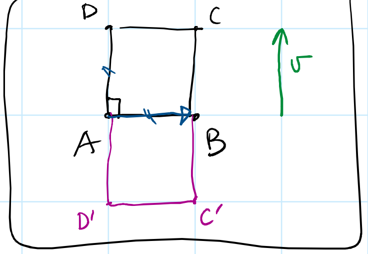
$C \in h \qquad \|C-M\| = \frac{\sqrt{3}}{2}\|B-A\|$

$\|C-A\| = \|B-A\| = \|C-B\|$

$$\boxed{C = M + \frac{\sqrt{3}}{2}u \qquad \begin{cases} \|u\|=1 \\ u \in \langle B-A\rangle^{\perp} \\ u \in V_{\pi} \end{cases}}$$

Esercizio:

Costruire C,D punti in π tali che A,B,C,D sia un quadrato.



$$\boxed{\begin{array}{l} D = A + v \\ C = B + v \end{array}} \qquad \begin{cases} v \in V_{\pi} \\ v \cdot (B-A) = 0 \\ \|v\| = \|B-A\| \end{cases}$$

Prodotto vettoriale $\mathbb{R}^3$

$v \times w \in \mathbb{R}^3$ con v e w vettori di $\mathbb{R}^3$

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad w = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \qquad \det(v, w, v \times w) \geq 0$

$$\boxed{v \times w = \det \begin{pmatrix} a & a' & e_1 \\ b & b' & e_2 \\ c & c' & e_3 \end{pmatrix}} = \begin{pmatrix} bc' - b'c \\ a'c - ac' \\ ab' - a'b \end{pmatrix}$$

$+$
$-$
$+$

$= e_1(bc' - b'c) - e_2(ac' - a'c) + e_3(ab' - a'b)$

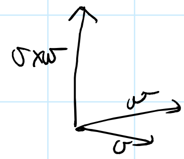
$\det(v \; w \; \mathcal{E})$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xe_1 + ye_2 + ze_3$

Esempio:

$v = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \qquad v \times w = \det \begin{pmatrix} 1 & 0 & e_1 \\ 3 & 2 & e_2 \\ 1 & -1 & e_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}$

Esercizio $\quad v = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad v \times w = \det \begin{pmatrix} 1 & 2 & e_1 \\ -1 & 1 & e_2 \\ 2 & 1 & e_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$



$v \cdot (v \times w) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} = -3 - 3 + 6 = 0$

$w \cdot (v \times w) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} = -6 + 3 + 3 = 0$

$\|v \times w\| =$ Area del parallelogramma formato da v e w.

Proprietà del prodotto vettoriale

1) $\det(v \; w \; \mathcal{E}) = 0$ se v e w sono linearmente dipendenti.

$\boxed{v \times w = \vec{0} \iff v \text{ e } w \text{ sono linearmente dipendenti}}$

$v=\begin{pmatrix}1\\3\\1\end{pmatrix}$ $w=\begin{pmatrix}2\\6\\2\end{pmatrix}=2\begin{pmatrix}1\\3\\1\end{pmatrix}$ $v\times w=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\det\begin{pmatrix}1&2&e_1\\3&6&e_2\\1&2&e_3\end{pmatrix}$

2) $\boxed{w\times v=-v\times w}$ $\det\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}=-1$

3) $\boxed{v\times(aw_1+bw_2)=a(v\times w_1)+b(v\times w_2)}$ $a,b\in\mathbb{R}$, $v,w_1,w_2\in\mathbb{R}^3$

$\boxed{(av_1+bv_2)\times w=a(v_1\times w)+b(v_2\times w)}$ $v_1,v_2,w\in\mathbb{R}^3$

Prodotto misto

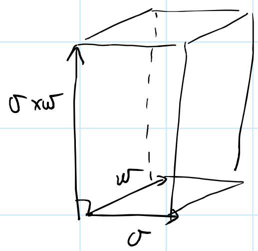
4) Siano $v,w,z\in\mathbb{R}^3$ $\boxed{(v\times w)\cdot z=\det(v\ w\ z)}$

$\Rightarrow$ $v\times w\perp v$ $(v\times w)\cdot v=\det(v\ w\ v)=0$

$v\times w\perp w$ $(v\times w)\cdot w=\det(v\ w\ w)=0$

$$\boxed{\boxed{(v\times w)\cdot\underbrace{(v\times w)}_{=z}}=\det(v\ w\ v\times w)}$$

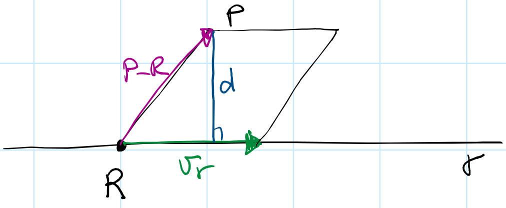
$\|v\times w\|^2=\det(v\ w\ v\times w)\geq 0$



$\text{Vol }P(v\ w\ v\times w)=\|v\times w\|^2=\det(v\ w\ v\times w)$

5) $\text{Area }P(v,w)=\|v\times w\|$

Distanza Punto-Retta in $\mathbb{R}^3$



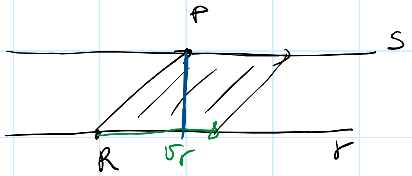
$r:\ R+\langle v_r\rangle$

$r: \begin{pmatrix}1\\5\\\alpha\end{pmatrix} + \left\langle \boxed{\begin{pmatrix}1\\1\\1\end{pmatrix}} \right\rangle \qquad P = \begin{pmatrix}0\\0\\\alpha\end{pmatrix} \qquad R \in r \quad R = \begin{pmatrix}1\\5\\\alpha\end{pmatrix} \qquad v_r = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$d(P,r) = \frac{\text{Area } P(v_r, P-R)}{\|v_r\|} = \frac{\| v_r \times (P-R) \|}{\|v_r\|} =$$

$$= \frac{\left\| \det \begin{pmatrix} 1 & -1 & e_1 \\ 1 & -5 & e_2 \\ 1 & 0 & e_3 \end{pmatrix} \right\|}{\sqrt{1+1+1}} = \frac{\left\| \begin{pmatrix}5\\-1\\-4\end{pmatrix} \right\|}{\sqrt{3}} = \frac{\sqrt{25+1+16}}{\sqrt{3}} = \sqrt{\frac{42}{3}}$$

2) $d(r,s)$ $\qquad r: \begin{pmatrix}1\\5\\\alpha\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \qquad s: \overset{P}{\begin{pmatrix}0\\0\\\alpha\end{pmatrix}} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \qquad r /\!/ s$



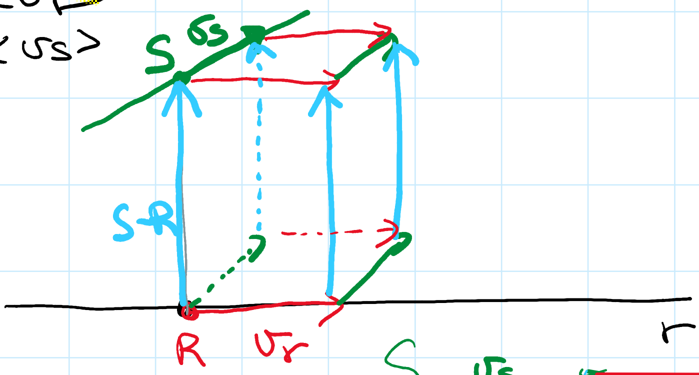
Se $\begin{cases} r /\!/ s \\ r \cap s = \emptyset \end{cases}$ $\qquad d(r,s) = \dfrac{\| v_r \times (P-R) \|}{\|v_r\|}$

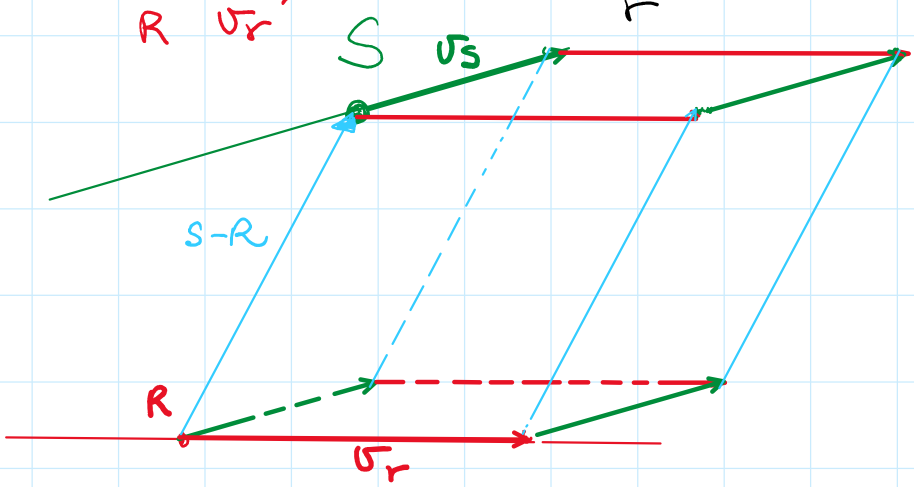
con $P \in s$, $R \in r$ $\qquad \langle v_r \rangle = V_r = V_s$

Distanza fra rette sghembe

$r: R + \langle v_r \rangle$

$s: S + \langle v_s \rangle$





$$d(r,s) = \frac{\text{Volume P.} \langle v_r \; v_s \; S-R \rangle}{\text{Area par.} \langle v_r, v_s \rangle} =$$

$$\boxed{= \frac{|\det(v_r \; v_s \; S-R)|}{\| v_r \times v_s \|} = d(r,s)}$$

Esempio: $r: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$ $\quad s: \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

determinare $d(r,s)$ distanza fra r e s.

Svolgimento:

r e s sono sghembe $\iff 3 = \dim(r \vee s) = \dim \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \right\rangle$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad v_r \qquad v_s \qquad S-R$

sono sghembe

$$d(r,s) = \frac{\left| \det \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\|} = \frac{|\det(v_r \; v_s \; S-R)|}{\| v_r \times v_s \|} =$$

$$= \frac{|-1|}{\left\| \det \begin{pmatrix} 0 & 0 & e_1 \\ 1 & 0 & e_2 \\ -1 & 1 & e_3 \end{pmatrix} \right\|} = \frac{1}{\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|} = 1$$

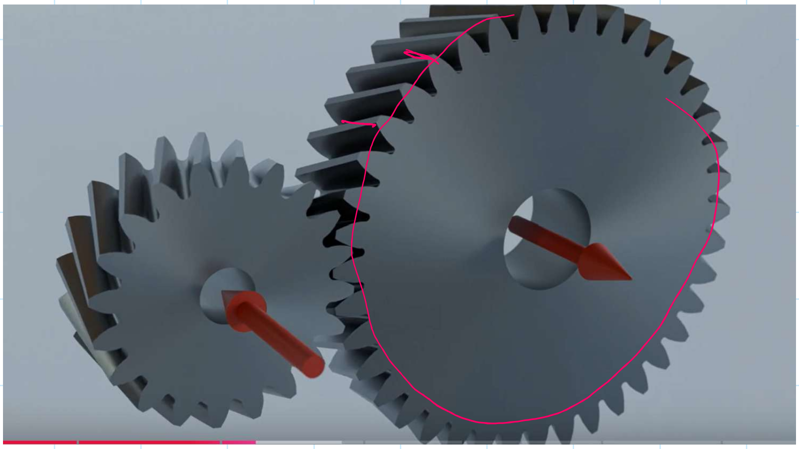
$r: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle \qquad R_a = \begin{pmatrix} 1 \\ a \\ -a \end{pmatrix} \qquad a \in \mathbb{R}$

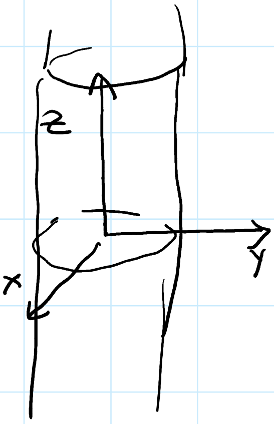
$s: \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad S_b = \begin{pmatrix} 0 \\ 1 \\ 5+b \end{pmatrix}$

$\begin{cases} (S_b - R_a) \cdot v_r = 0 \\ (S_b - R_a) \cdot v_s = 0 \end{cases} \qquad d(r,s) = d(R_{\bar a}, S_{\bar b})$

$\qquad\qquad\qquad\qquad\qquad\qquad \text{Cmd } (R_{\bar a}, S_{\bar b})$

Ruote dentate con assi paralleli

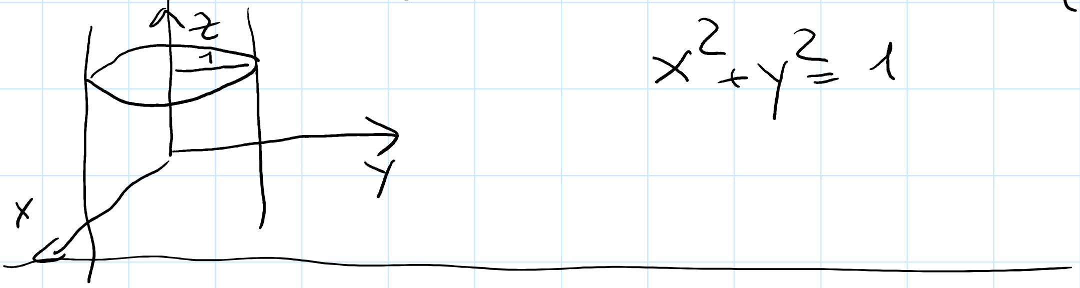




$$x^2+y^2=1$$

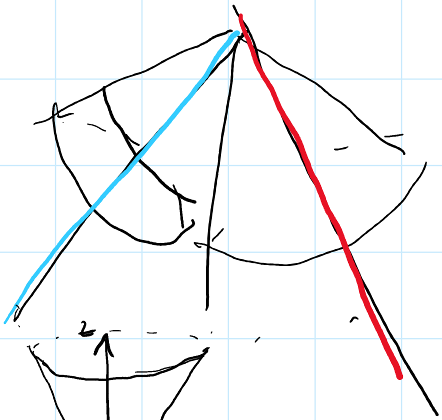
Se $r // s$ possiamo usare dei cilindri

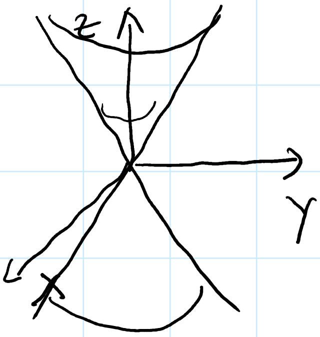
L'equazione del cilindro di asse di rotazione $\begin{cases} x=0 \\ y=0 \end{cases}$



$$x^2+y^2=1$$

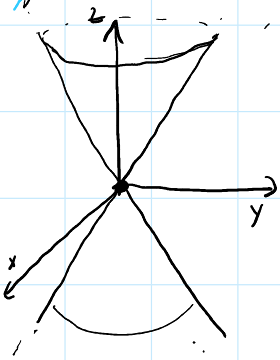
Ruote dentate con assi concorrenti





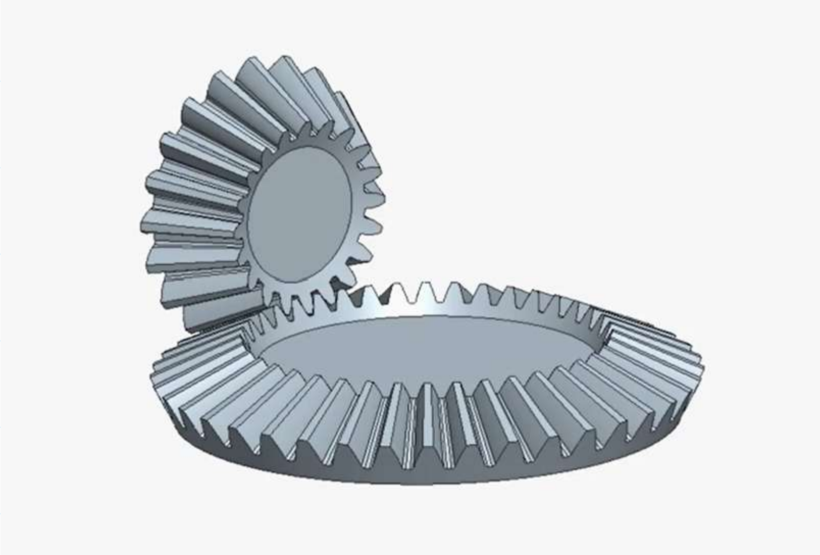
$$z^2=x^2+y^2$$

$$\begin{cases} y=0 \\ z^2=x^2 \end{cases}$$



Equazione di un cono:

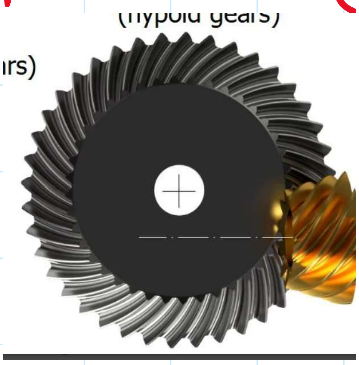
$$z^2=x^2+y^2$$

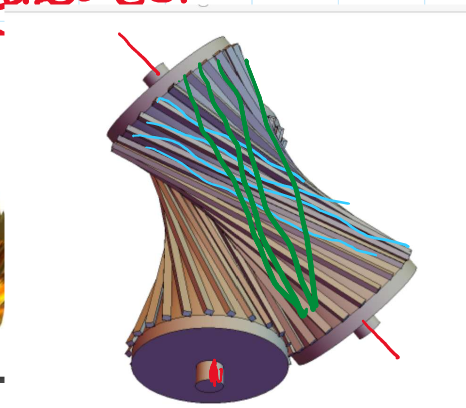


Ruote dentate

con sezioni coniche

Ruote dentati per assi sghembi:





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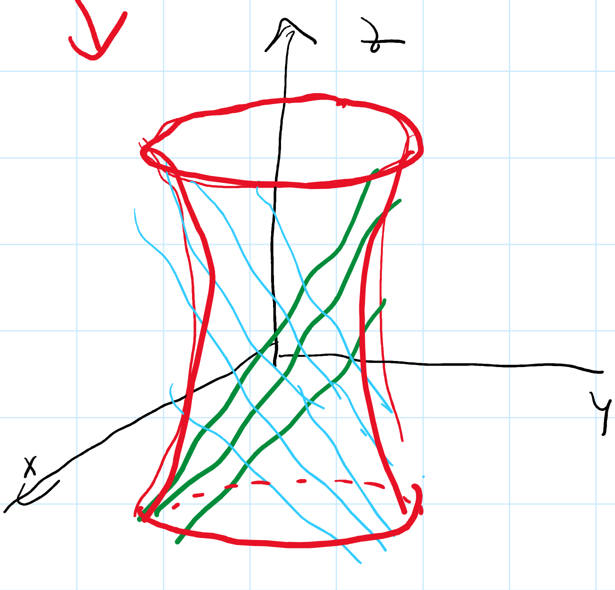

$\mathcal{C}: \ z^2 = x^2 + y^2 - 1$

quindi

$$z^2 - y^2 = x^2 - 1$$

$$t(z^2 - y^2) = t(x^2 - 1)$$

$\mathcal{C} \cap x = 0 : \ y^2 - z^2 = 1$ iper.

$\mathcal{C} \cap z = 0 : \ x^2 + y^2 = 1$

$$\boxed{t(z-y)}(z+y) = t\boxed{(x-1)}(x+1)$$

$$\begin{cases} t(z-y) = x-1 \\ z+y = t(x+1) \end{cases}$$

I° schiera di rette

$$\begin{cases} t(z-y) = x+1 \\ z+y = t(x-1) \end{cases}$$

II° schiera di rette

Due schiere di rette sghembe