COMPUTABILITY (22/12/2025)

The shape of a typical EXAM

Computability
January 28, 2025

basic notions
theorem statements
proofs
small variations

Exercise 1

a. Provide the definition of a semi-decidable predicate.

b. Show that if $P(\vec{x})$ is semi-decidable then there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$.

c. Let $P(\vec{x}, y)$ be a predicate. Is it the case that if the predicate $Q(\vec{x}) \equiv \forall y. P(\vec{x}, y)$ is decidable then $P(\vec{x}, y)$ is semi-decidable? Prove it or provide a counterexample.

constructions of PR / R
diagonalisation
smn

Exercise 2

Is there a non-computable function $f : \mathbb{N} \to \mathbb{N}$ with the property that $f(x) = \varphi_x(x)$ for infinitely many inputs, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = \varphi_x(x)\}$ is infinite? Justify your answer by providing an example of such function, if it exists, or by proving that it does not exist, otherwise.

classify sets (recursive / r.e.), saturatedness

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is monotone when $\forall x, y \in dom(f)$ if $x \leqslant y$ then $f(x) \leqslant f(y)$. Classify the following set from the point of view of recursiveness

$$A = \{x \in \mathbb{N} \mid \varphi_x \text{ monotone}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 4

Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} \mid \exists y \in W_x.\, x + y \in E_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

Note: Each exercise contributes with the same number of points (8) to the final grade.

$\boxed{\text{ORAL EXAM}}$: optional, needed for distinction (lode)
focused on theory / proofs
range: +/- 4

Exercise 1

a. Provide the definition of a semi-decidable predicate.

b. Show that if $P(\vec{x})$ is semi-decidable then there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$.

c. Let $P(\vec{x}, y)$ be a predicate. Is it the case that if the predicate $Q(\vec{x}) \equiv \forall y. P(\vec{x}, y)$ is decidable then $P(\vec{x}, y)$ is semi-decidable? Prove it or provide a counterexample.

a. Provide the definition of a semi-decidable predicate.

A predicate $P(\vec{x}) \subseteq \mathbb{N}^k$ is semi-decidable if the corresponding semi-characteristic function

$$sc_p : \mathbb{N}^k \to \mathbb{N}$$

signature

$$sc_p(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$

argument

is computable.

b. Show that if $P(\vec{x})$ is semi-decidable then there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$.

Let $P(\vec{x}) \subseteq \mathbb{N}^k$ be semi-decidable, i.e. the semi-characteristic function $sc_p : \mathbb{N}^k \to \mathbb{N}$ is computable

Let $e \in \mathbb{N}$ such that $\varphi_e^{(k)} = sc_p$. Then

$$\begin{aligned}
P(\vec{x}) &\equiv \text{“} sc_p(\vec{x}) = 1 \text{”} \\
&\equiv \text{“} sc_p(\vec{x})\downarrow \text{”} \\
&\equiv \text{“} \varphi_e^{(k)}(\vec{x})\downarrow \text{”} \\
&\equiv \text{“} P_e(\vec{x}) \text{ halts” } \\
&\equiv \exists t.\ H^{(k)}(e, \vec{x}, t) \\
&\equiv \exists t.\ Q(\vec{x}, t)
\end{aligned}$$

where $Q(\vec{x}, t) \equiv H^{(k)}(e, \vec{x}, t)$ decidable since $H^{(k)}$ is decidable in fact

$$\chi_Q(\vec{x},t) = \chi_{H(x)}(e,\vec{x},t) \qquad \text{computable by composition.}$$

$\uparrow$ computable $\quad$ $\uparrow$ constant

c. Let $P(\vec{x}, y)$ be a predicate. Is it the case that if the predicate $Q(\vec{x}) \equiv \forall y. P(\vec{x}, y)$ is decidable then $P(\vec{x}, y)$ is semi-decidable? Prove it or provide a counterexample.

The answer is negative :

let $\quad P(x,y) \equiv$ " $x \in \bar{K}$ and $y = 0$ "

$\rightarrow \quad Q(x) \equiv \forall y.\ P(x,y) \equiv$ false $\qquad$ decidable

$\rightarrow \quad P(x,y)$ not semidecidable

otherwise $\quad \underset{\displaystyle x \in \bar{K}}{\underset{\parallel\!\!\wr}{\exists y.\ P(x,y)}}$ semi-decidable , which is not.

NOTE : The question is not

Given $P(\vec{x},y)$ if $Q(\vec{x}) \equiv \exists y.\ P(\vec{x},y)$ is semi-decidable

then $P(\vec{x},y)$ semi-decidable??

NO : $\quad P(x,y) \equiv y \in \bar{K}$ and $x \in K$

- $Q(x) \equiv \exists y.\ P(x,y) \equiv x \in K \qquad$ semi-decidable

- $P(x,y)$ not semi-decidable ,

  otherwise $\underset{\displaystyle y \in \bar{K}}{\underset{\parallel\!\!\wr}{\exists x.\ P(x,y)}}$ would be semi-decidable (structure theorem)

Exercise 2

Is there a non-computable function $f : \mathbb{N} \to \mathbb{N}$ with the property that $f(x) = \varphi_x(x)$ for infinitely many inputs, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = \varphi_x(x)\}$ is infinite? Justify your answer by providing an example of such function, if it exists, or by proving that it does not exist, otherwise.

Yes, such a function exists and can be defined by diagonalisation

$$f(x) = \begin{cases} \varphi_x(x) & \text{if } x \text{ odd} \\ \varphi_{x/2}(x) + 1 & \text{if } x \text{ even and } \varphi_{x/2}(x)\downarrow \\ 0 & \text{if } x \text{ even and } \varphi_{x/2}(x)\uparrow \end{cases}$$

- f not computable, since it is total and different from all total computable functions: in fact for all $e \in \mathbb{N}$ if φ_e is total then $f(\underbrace{2e}_{\text{even}}) = \varphi_e(2e) + 1 \neq \varphi_e(2e)$ hence $f \neq \varphi_e$

- $\forall n \in \mathbb{N}$

$$f(\underbrace{2n+1}_{\text{odd}}) = \varphi_{2n+1}(2n+1)$$

which means $\{x \mid f(x) = \varphi_x(x)\} \supseteq$ odd numbers

so it is infinite.

SIMPLER SOLUTION

$$f(x) = \begin{cases} \varphi_x(x) & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

- f is not computable.

Take $g(x) = f(x) + 1 = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in K \\ 1 & \text{otherwise} \end{cases}$

g is not computable, total and different from all total computable function ($\forall e$ if $\underbrace{\varphi_e \text{ total}}_{\leadsto e \in K}$ $g(e) = \varphi_e(e) + 1 \neq \varphi_e(e)$)

Hence f is not computable (otherwise $g(x) = f(x) + 1$ would be so)

- $\{x \mid f(x) = \varphi_x(x)\} = K$ infinite (otherwise it would be recursive)

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is monotone when $\forall x, y \in dom(f)$ if $x \leqslant y$ then $f(x) \leqslant f(y)$. Classify the following set from the point of view of recursiveness

$$A = \{x \in \mathbb{N} \mid \varphi_x \text{ monotone}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

conjecture : $\bar{A}$ r.e. (I need to find y_1, y_2 with $y_1 \leq y_2$ and $\varphi_x(y_1)\downarrow > \varphi_x(y_2)\downarrow$

$\bar{A}$ not recursive

$\Downarrow$

A not r.e. (hence not recursive)

- A is saturated by definition

  $A = \{x \mid \varphi_x \in \mathcal{A}\}$ $\qquad$ $\mathcal{A} = \{f \mid f \text{ monotone}\}$

- $A, \bar{A}$ not recursive by Rice's theorem

  $A \neq \emptyset$ $\qquad$ e.g. id (identity) is computable and $id \in \mathcal{A}$

  (in fact $\forall x, y$ if $x \leq y$ $\quad id(x) = x \leq y = id(y)$)

  $A \neq \mathbb{N}$ $\qquad$ e.g. $\bar{sg}$ is computable and $\bar{sg} \notin \mathcal{A}$

  (in fact $0 \leq 1$ but $\bar{sg}(0) = 1 \not\leq 0 = \bar{sg}(1)$)

  $\Rightarrow$ by Rice $A, \bar{A}$ not recursive.

- $\bar{A}$ is recursively enumerable

  $SC_{\bar{A}}(x)$ searches for y_1, y_2 s.t. $y_1 \leq y_2$ and $\varphi_x(y_1) = z_1$ $>$ $\varphi_x(y_2) = z_2$

$$SC_{\bar{A}}(x) = \mathbb{1}\left(\mu(y_1, y_2, z_1, z_2, t).\ S(x, y_1, z_1, t) \wedge S(x, y_2, z_2, t) \wedge \wedge \underbrace{y_1 \leq y_2}_{y_2 = y_1 + h} \wedge \underbrace{z_1 > z_2}_{z_1 = z_2 + K + 1}\right)$$

$$= \mathbb{1}(\mu(\underset{1}{y_1}, \underset{2}{z_2}, \underset{3}{h}, \underset{4}{k}, \underset{5}{t}).\ S(x, y_1, z_2+k+1, t) \wedge S(x, y_1+h, z_2, t))$$

$$= \mathbb{1}(\mu\omega.\ S(x, (\omega)_1, (\omega)_2+(\omega)_4+1, (\omega)_5) \wedge S(x, (\omega)_1+(\omega)_3, (\omega)_2, (\omega)_5))$$

$$= \mathbb{1}(\mu\omega.\ |\chi_S(x, (\omega)_1, (\omega)_2+(\omega)_4+1, (\omega)_5) \cdot \chi_S(x, (\omega)_1+(\omega)_3, (\omega)_2, (\omega)_5) - 1|)$$

computable

hence $\bar{A}$ is r.e.

Thus A not r.e. (since otherwise $A, \bar{A}$ r.e. hence recursive).

and therefore A not recursive.

Exercise 4

Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} \mid \exists y \in W_x .\, x + y \in E_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

conjecture : B r.e. , not recursive

$\bar{B}$ not r.e. (hence not recursive)

B not saturated

- $\boxed{B \text{ r.e.}}$

look for y, z such that $\varphi_x(y)\downarrow$ and $\varphi_x(z) = x+y$

$$sc_B(x) = \mathbb{1}\left(\mu(y,z,t).\ H(x,y,t) \wedge S(x,z,x+y,t)\right)$$

$$= \mathbb{1}\left(\mu\omega.\ \underbrace{H(x,(\omega)_1,(\omega)_3) \wedge S(x,(\omega)_2, x+(\omega)_1,(\omega)_3)}_{\text{decidable}}\right)$$

the sc_B is computable.

- $\boxed{B \text{ not recursive}}$

$K \leq_m B$ hence, since K not recursive, B is not recursive.

define

$$g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases} = y \cdot sc_K(x)$$

[idea : $g(x,y) = \varphi_{s(x)}(y) \rightsquigarrow 0 \in W_{s(x)}$ $\varphi_{s(x)}(s(x)) = s(x) + 0$]

g computable, hence, by smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t. $\varphi_{s(x)}(y) = g(x,y)$ $\forall x, y$

s is the reduction function $K \leq_m B$

- if $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = y$ $\forall y$

hence $y = 0 \in W_{S(x)} = \mathbb{N}$ and $\varphi_{S(x)}(S(x)) = S(x) = 0 + S(x)$

hence $0 + S(x) \in E_{S(x)}$

Thus $S(x) \in B$

- if $x \notin K$ then $\varphi_{S(x)}(y) = g(x,y) \uparrow \quad \forall y$

  hence obviously there is no $y \in \underset{\emptyset}{\underset{\parallel}{W_{S(x)}}}$ s.t. $y + S(x) \in \underset{\emptyset}{\underset{\parallel}{E_{S(x)}}}$

  Thus $S(x) \notin B$

Hence $K \leq_m B$ and B not recursive.

Thus $\bar{B}$ is not r.e. (otherwise, since B is r.e., B would be recursive).

Hence $\bar{B}$ not recursive.

* Is B saturated? NO

Idea: show there is $e \in \mathbb{N}$ s.t.

$$\varphi_e(y) = \begin{cases} e & \text{if } y = 0 \\ \uparrow & \text{otherwise} \end{cases} \qquad (*)$$

then, using $(*)$

- $e \in B$ $\quad \exists y = 0 \in W_e$ s.t. $e + 0 \in E_e$

  (since $\varphi_e(0) = e = e + 0$)

- there are infinitely many indices for the same function,

  hence if $e' \neq e$ s.t. $\varphi_e = \varphi_{e'}$

  $e' \notin B$ since if $y \in W_{e'} = W_{e'} = \{0\} \rightsquigarrow y = 0$

  and $e' + 0 = e' \notin E_{e'} = E_e = \{e\}$

  $\neq e'$

We show (*) and conclude

$$g(x,y) = \begin{cases} x & \text{if } y=0 \\ \uparrow & \text{otherwise} \end{cases} = x + \mu z. y$$

computable, hence there is $s: \mathbb{N} \to \mathbb{N}$ total computable (by smn)

s.t. $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} x & \text{if } y=0 \\ \uparrow & \text{otherwise} \end{cases}$$

Since s total computable, by the 2nd recursion theorem, there is $e \in \mathbb{N}$

s.t. $\varphi_e = \varphi_{s(e)}$ Hence

$$\varphi_e(y) = \varphi_{s(e)}(y) = g(e,y) = \begin{cases} e & \text{if } y=0 \\ \uparrow & \text{otherwise.} \end{cases}$$

as desired.

$\square$