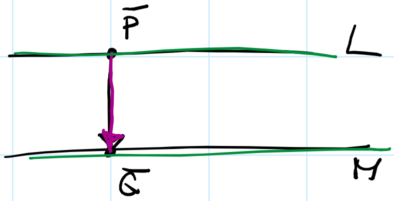
Distanze e punti di minima distanza $\mathbb{R}^3$

Def: $L: P + V_L$ $\qquad d(L,M) = \min_{\substack{P' \in L \\ Q' \in M}} d(P', Q')$

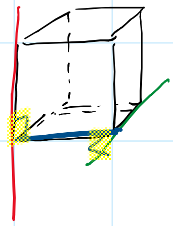
$M: Q + V_M$

Esistono $\bar{P} \in L$ e $\bar{Q} \in M$ che realizzano la minima distanza

$d(L,M) = d(\bar{P}, \bar{Q})$ $\qquad (\bar{P}, \bar{Q})$ viene detta **cmd coppia di minima distanza.**

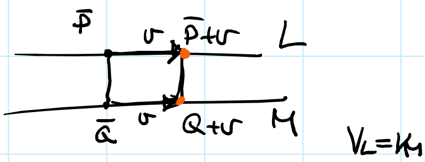


$$\begin{cases} (\bar{Q} - \bar{P}) \in V_L^{\perp} \\ (\bar{Q} - \bar{P}) \in V_M^{\perp} \end{cases}$$



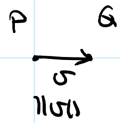
La cmd è unica se $V_L \cap V_M = \{\vec{0}\}$, mentre se $V_L \cap V_M \neq \{\vec{0}\}$ (esempio rette parallele, una retta // a un piano...) data $(\bar{P}, \bar{Q})$ una cmd tutte le altre sono

$(\bar{P} + v, \bar{Q} + v)$ con $v \in V_L \cap V_M$



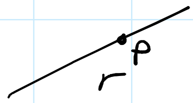
Distanze in $\mathbb{R}^3$ e cmd.

1) Punto-Punto $\quad d(P,Q) = \|Q - P\|$ unica cmd (P,Q)



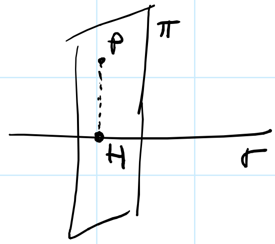
2) Punto-Retta $\quad d(P, r)$

Se $P \in r$ $\quad d(P,r) = 0$ $\quad$ cmd (P,P)



Se $P \notin r$ $\quad H = \pi \cap r$ con

$\pi: P + V_r^{\perp}$ $\quad$ unica cmd (P,H)



$d(P,r) = d(P,H)$

Esempio: $r: \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle$ $\qquad P = \begin{pmatrix}1\\3\\1\end{pmatrix}$

determinare la $d(P,r)$ e una cmd.

Svolg.: unica cmd (P,H) con $H = \pi \cap r$ $\qquad \pi: \begin{pmatrix}1\\3\\1\end{pmatrix} + \left\langle \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle^{\perp}$

$\pi: \ 2x + z \underset{\substack{\uparrow \\ P = \begin{pmatrix}1\\3\\1\end{pmatrix} \in \pi}}{=} 2 + 1 = 3$ $\qquad \pi: 2x + z = 3$ $\qquad V_\pi^{\perp} = \left\langle \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle$

$H = \pi \cap r \begin{cases} 2x + z = 3 \\ \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle \end{cases}$ $\qquad 2(1+2a) + (1+a) = 3$ $\qquad \cancel{2} + 4a + \cancel{1} + a = \cancel{3}$

$H = \begin{pmatrix}1+2a\\1\\1+a\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + a\begin{pmatrix}2\\0\\1\end{pmatrix}$ $\qquad 5a = 0 \quad a = 0$

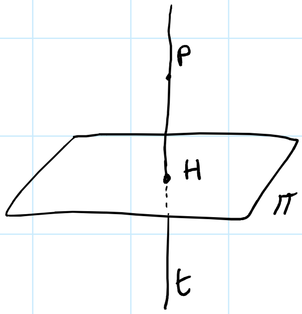
$H = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\qquad$ cmd $\left(\overset{P}{\begin{pmatrix}1\\3\\1\end{pmatrix}}, \overset{H}{\begin{pmatrix}1\\1\\1\end{pmatrix}} \right)$ $\qquad d(P,r) = d(P,H) = \|H - P\| =$

$= \left\| \begin{pmatrix}0\\-2\\0\end{pmatrix} \right\| = \sqrt{0+4+0} = 2$

3) Punto - Piano $\quad P, \pi$

Formule $\quad P = \begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}$ $\qquad \pi: ax + by + cz = d$

$$d(P,\pi) = \frac{|ap_1 + bp_2 + cp_3 - d|}{\sqrt{a^2+b^2+c^2}}$$



unica cmd (P,H) $\qquad H = t \cap \pi$ con

$t: P + V_\pi^{\perp}$

Esempio: $P=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad \pi: x+y+z=4.$

Formule $\quad d(P,\pi)=\dfrac{|1+0+0-4|}{\sqrt{1^2+1^2+1^2}}=\dfrac{|-3|}{\sqrt{3}}=\sqrt{3}$

cmd (P,H) $\quad H=t\cap\pi$ con $t:\begin{pmatrix}1\\0\\0\end{pmatrix}+\left\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle$

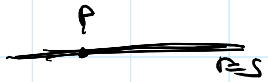
$$H=t\cap\pi\begin{cases}\begin{pmatrix}1\\0\\0\end{pmatrix}+\left\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle & \begin{pmatrix}1+a\\a\\a\end{pmatrix}\\ x+y+z=4 & (1+a)+a+a=4 \quad 3a=3 \quad a=1\end{cases}$$

$H=\begin{pmatrix}2\\1\\1\end{pmatrix}$ $\qquad$ cmd $\left(\overset{P}{\begin{pmatrix}1\\0\\0\end{pmatrix}},\overset{H}{\begin{pmatrix}2\\1\\1\end{pmatrix}}\right)$ $\qquad d(P,\pi)=d(P,H)=\|H-P\|=$

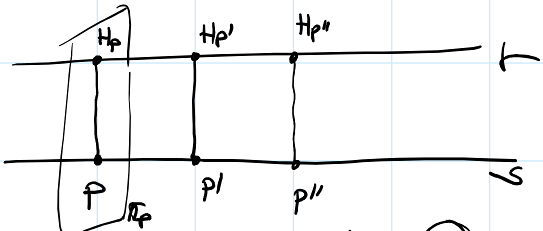
$=\left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

4) Retta - Retta $\quad r,s$

ⓐ se $r=s$ $\quad d(r,s)=0$ $\quad$ cmd (P,P) con $P\in r=s$



ⓑ se $\begin{cases} r\parallel s\\ r\cap s=\emptyset\end{cases}$ $\qquad$ rette parallele disgiunte



$d(r,s)=d(r,P)$

ritorniamo a calcolare una distanza punto retta ② e

cmd (P,H_P) sono infinite con $P\in s$ qualunque

$$H_P=r\cap\pi_P \qquad \pi_P: P+V_r^{\perp}$$

Se $(\bar{P},H_{\bar{P}})$ è cmd tutte le altre sono

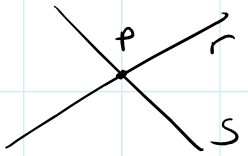
$(\bar{P}+v,H_{\bar{P}+v})$ con $v\in V_r\cap V_s=V_r=V_s$ perché $r\parallel s$

Esempio:

$r: \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle \qquad s = \overset{P}{\begin{pmatrix}1\\3\\1\end{pmatrix}} + \left\langle \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle \qquad r \parallel s$

$d(r,s) = d(r,P)$ vedi esempio caso 2.

Rette incidenti in un punto:



$r \cap s = \{P\}$

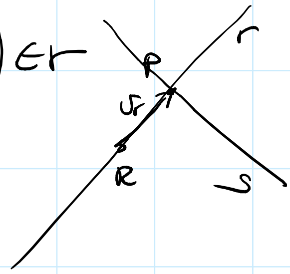
$d(r,s) = 0$ e

cmd (P,P) con $P = r \cap s$

Esempio:

$r: \overset{R}{\begin{pmatrix}1\\0\\0\end{pmatrix}} + \left\langle \overset{v_r}{\begin{pmatrix}2\\1\\1\end{pmatrix}} \right\rangle \qquad \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} \in r$

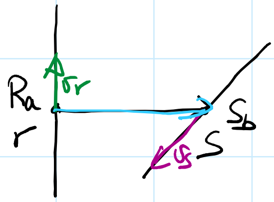
$s: \begin{pmatrix}3\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle \qquad \begin{pmatrix}3\\1\\1\end{pmatrix} \in s$



cmd $(P,P) = \left(\begin{pmatrix}3\\1\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix} \right)$

Distanza fra rette sghembe: r, s

cmd $(\bar{R}, \bar{S})$



$r: R + \langle v_r \rangle \qquad R_a = R + a v_r$

$s: S + \langle v_s \rangle \qquad S_b = S + b v_s$

$$\begin{cases} (S_b - R_a) \cdot v_r = 0 \\ (S_b - R_a) \cdot v_s = 0 \end{cases}$$

Risolvendo il sistema si trova $\begin{pmatrix}\bar{a}\\\bar{b}\end{pmatrix} \in \mathbb{R}^2$

cmd $(R_{\bar{a}}, S_{\bar{b}}) \qquad d(r,s) = d(R_{\bar{a}}, S_{\bar{b}}) = \| S_{\bar{b}} - R_{\bar{a}} \|.$

Esempio:

$r: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle \qquad s: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\-1\end{pmatrix} \right\rangle$

determinare cmd e distanza fra r e s.

Svolgimento:

$r: R + \langle v_r \rangle \qquad R_a = R + a v_r = \begin{pmatrix}1\\0\\0\end{pmatrix} + a\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\a\\-a\end{pmatrix}$

$s: S + \langle v_s \rangle \qquad S_b = S + b v_s = \begin{pmatrix}0\\1\\5\end{pmatrix} + b\begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\5-b\end{pmatrix}$

$$S_b - R_a = \begin{pmatrix}0\\1\\5-b\end{pmatrix} - \begin{pmatrix}1\\a\\-a\end{pmatrix} = \begin{pmatrix}-1\\1-a\\5-b+a\end{pmatrix}$$

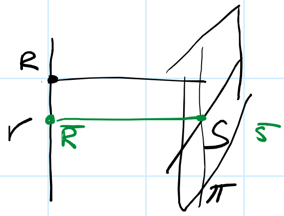
$$\begin{cases}(S_b - R_a)\cdot v_r = 0\\ (S_b - R_a)\cdot v_s = 0\end{cases} \qquad \begin{pmatrix}-1\\1-a\\5-b+a\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = 0 \quad \begin{pmatrix}-1\\1-a\\5-b+a\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix} = 0 \qquad \begin{cases}(1-a)-(5-b+a)=0\\ -(5-b+a)=0\end{cases}$$

$$\begin{cases}a=1\\ 5-b+a=0\end{cases} \qquad \begin{cases}a=1\\ b=6\end{cases} \qquad \text{cmd}(R_1, S_6) = \left(\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$$

$$d(r,s) = d(R_1, S_6) = \|S_6 - R_1\| = \left\|\begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\| = \left\|\begin{pmatrix}-1\\0\\0\end{pmatrix}\right\| = 1$$

Distanza tra rette sghembe:

Se r ed s sono sghembe $d(r,s) = d(R,\pi)$ con π piano contenente s e parallelo a r.



Esempio:

$r: \overset{R}{\begin{pmatrix}1\\0\\0\end{pmatrix}} + \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle \qquad s: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\-1\end{pmatrix} \right\rangle$

piano π contenente s e parallelo a r è

$\pi: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle$ eq. cartesiana $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\1\\5\end{pmatrix} + a\begin{pmatrix}0\\0\\-1\end{pmatrix} + b\begin{pmatrix}0\\1\\-1\end{pmatrix}$

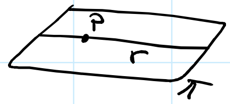
$$\begin{cases}x=0\\ y=1+b\\ z=5-a-b\end{cases} \quad b=y-1 \quad \begin{cases}x=0\\ z=5-a-y+1\end{cases} \quad a=6-y-z \quad \boxed{x=0}$$

$\pi: x=0 \qquad d(r,s) = d(R,\pi) = d\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, x=0\right) = \frac{|1|}{\sqrt{1^2}} = 1$

5) Retta-Piano $\quad r, \pi$

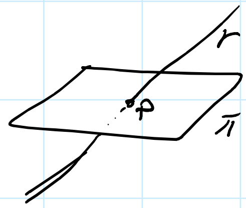
Se $r \subseteq \pi \quad d(r,\pi)=0 \quad$ Cmd (P,P) con $P \in r$

non è unica

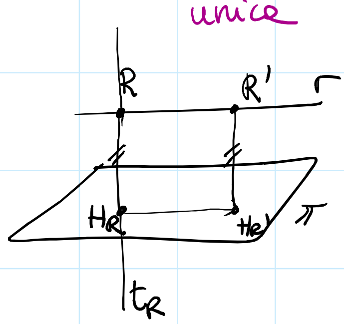


Se $r \cap \pi = \{P\} \quad d(r,\pi)=0 \quad$ Cmd (P,P)

unica



Se $\begin{cases} r \parallel \pi \\ r \cap \pi = \emptyset \end{cases}$



$d(r,\pi) = d(R,\pi)$ con $R \in r$

Cmd (R, H_R) con $R \in r \qquad H_R = \pi \cap t_R \qquad t_R: R + V_\pi^\perp$

sono infinite

6) Piano-Piano $\quad \pi_1, \pi_2$

•) Se $\pi_1 = \pi_2 \quad d(\pi_1,\pi_2)=0 \quad$ Cmd (P,P) con $P \in \pi_1 = \pi_2$

sono infinite

•) Se $\pi_1 \cap \pi_2 = r \quad d(\pi_1,\pi_2)=0$

$d(\pi_1,\pi_2) = d(R,R) = 0$

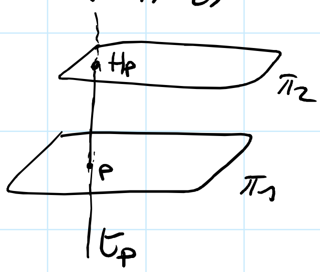
Cmd (R,R) con $R \in r$

sono infinite.

•) Se $\begin{cases} \pi_1 \parallel \pi_2 \\ \pi_1 \cap \pi_2 = \emptyset \end{cases}$ $\qquad d(\pi_1,\pi_2) = d(P,\pi_2)$ con $P \in \pi_1$

Cmd (P, H_P)

sono infinite



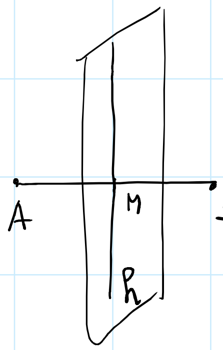
$P \in \pi_1 \quad H_P = t_P \cap \pi_2$

con $t_P: P + V_{\pi_1}^\perp$

Essendo $\pi_1 \parallel \pi_2$

$V_{\pi_1} = V_{\pi_2}$

Asse di un segmento



Luogo geometrico dei punti equidistanti da A e B

Asse: $\frac{A+B}{2} + \langle B-A \rangle^{\perp}$

Esempio: $A = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ $B = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$

Asse del segmento AB è $\frac{A+B}{2} + \langle B-A \rangle^{\perp}$

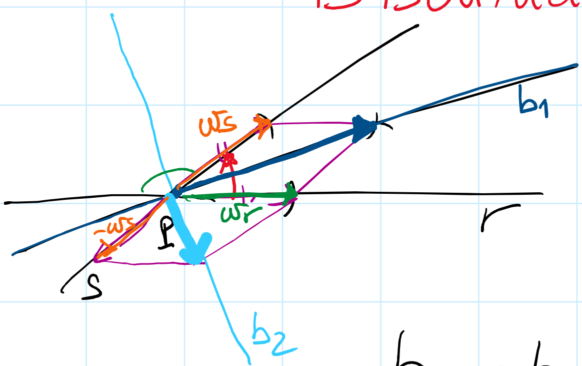
$M = \frac{A+B}{2} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ $B-A = \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}$ $\left\langle \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \right\rangle$

$\pi: \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix} \right\rangle^{\perp} = \underset{A}{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}} + \left\langle \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \right\rangle^{\perp}$ $2x - 3y + 0 \cdot z = K$

$2 \cdot 3 - \cancel{3 0} + \cancel{0 1} = K$

$\boxed{\pi: 2x - 3y = 6,}$

Bisettrici



$r: P + \langle v_r \rangle$

$s: P + \langle v_s \rangle$

Cerchiamo le bisettrici b_1 e b_2 delle rette r, s incidenti in P.

$b_1 \perp b_2$

$\boxed{\begin{array}{l} b_1: P + \langle w_r + w_s \rangle \\ b_2: P + \langle w_r - w_s \rangle \end{array}}$

Cerchiamo $\boxed{\begin{array}{l} w_r \in \langle v_r \rangle \\ w_s \in \langle v_s \rangle \\ \|w_r\| = \|w_s\| \end{array}}$

Esempio:

1) $r: \overset{P}{\begin{pmatrix}5\\1\\8\end{pmatrix}} + \left\langle \overset{v_r}{\begin{pmatrix}2\\1\\1\end{pmatrix}} \right\rangle$ $\quad v_r = \begin{pmatrix}2\\1\\1\end{pmatrix}$ $\quad \|v_r\| = \sqrt{4+1+1} = \sqrt{6}$

$s: \underset{P}{\begin{pmatrix}5\\1\\8\end{pmatrix}} + \left\langle \underset{v_s}{\begin{pmatrix}-2\\1\\-1\end{pmatrix}} \right\rangle$ $\quad v_s = \begin{pmatrix}-2\\1\\-1\end{pmatrix}$ $\quad \|v_s\| = \sqrt{4+1+1} = \sqrt{6}$

essendo $\|v_r\| = \|v_s\|$

$w_r = v_r$

$w_s = v_s$

Bisettrici $\quad b_1: \begin{pmatrix}5\\1\\8\end{pmatrix} + \langle w_r + w_s \rangle = \begin{pmatrix}5\\1\\8\end{pmatrix} + \left\langle \begin{pmatrix}0\\2\\0\end{pmatrix} \right\rangle$

$b_2: \begin{pmatrix}5\\1\\8\end{pmatrix} + \langle w_r - w_s \rangle = \begin{pmatrix}5\\1\\8\end{pmatrix} + \left\langle \begin{pmatrix}4\\0\\2\end{pmatrix} \right\rangle$

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2) $r: \begin{pmatrix}1\\1\\3\end{pmatrix} + \left\langle \overset{v_r}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \right\rangle$ $\quad v_r = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad \|v_r\| = 1$ $\quad w_r = v_r = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$s: \begin{pmatrix}1\\1\\3\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$ $\quad v_s = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad \|v_s\| = \sqrt{3}$ $\quad w_s = \dfrac{v_s}{\|v_s\|} = \dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\|w_r\| = \|w_s\| = 1$ Prendiamo i versori

$b_1: \begin{pmatrix}1\\1\\3\end{pmatrix} + \langle w_r + w_s \rangle = \begin{pmatrix}1\\1\\3\end{pmatrix} + \left\langle \begin{pmatrix}1/\sqrt{3}\\1+\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix} \right\rangle$

$b_2: \begin{pmatrix}1\\1\\3\end{pmatrix} + \langle w_r - w_s \rangle = \begin{pmatrix}1\\1\\3\end{pmatrix} + \left\langle \begin{pmatrix}-1/\sqrt{3}\\1-1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix} \right\rangle$
