Q 1

$U \subseteq \mathbb{R}^n$ open bdd of class C^1 $\quad p \geq 1$.

$$B = \{ f \in W^{1,p}(U) \,,\ \|f\|_{W^{1,p}(U)} \leq 1 \}$$

In which space this set is COMPACT?

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Answer: for $p = 1$ never compact

for $1 < p < n$ $\rightarrow$ B is compact in $L^q(U)$ $\forall\, q \in [1, p^*)$

for $p = n$ $\rightarrow$ B is compact in $L^q(U)$ $\forall\, q \in [1, +\infty)$

for $p > n$ $\rightarrow$ B is compact in $L^q(U)$ $\forall\, q \in [1, +\infty]$

B is compact in $C(\overline{U})$

B is compact in $C^{0,\alpha}(U)$ $\forall\, \alpha < 1 - \frac{n}{p}$.

proof by GNS $\quad p < n \quad B \subseteq L^q(U) \quad \forall q \in [1, p^*]$

$p = n \quad B \subseteq L^q(U) \quad \forall q \in [1, +\infty)$

by Morrey $\quad p > n \quad B \subseteq C^{0,1-\frac{n}{p}}(U)$

(so $B \subseteq C(\bar{U})$, $B \subseteq L^q(U)$ $q \in [1, +\infty]$)

by corollary of Rellich-Kondrachov

$f_k \in B \Rightarrow$ up to subsequence $f_k \to f$ strongly in $L^q(U)$ $\forall q < p^*$ if $p < n$, $\forall q < +\infty$ if $p = n$, $\forall q \leq +\infty$ $p > n$.

Moreover if $p > 1$ $\quad f \in W^{1,p}(U)$

(whereas if $p = 1$ $\quad f \in BV(U)$ but in general $f \notin W^{1,1}(U)$

so $f \notin B \rightarrow B$ is NOT CLOSED in $L^1(U)$)

$p > 1$ $f \in W^{1,p}(U)$ and $Df_k \longrightarrow Df$ weakly in L^p.

$f_k \longrightarrow f$ strongly in L^p

$\Longrightarrow$ $f_k \to f$ weakly in $W^{1,p}(U)$

$\|f\|_{W^{1,p}} \leq \liminf_k \|f_k\|_{W^{1,p}} = 1$ (the norm is LSC. w.r.t. weak convergence)

$\hookrightarrow f \in B$. So B is closed in $L^q(U)$ $\forall\ q < p^*$ $(p < n)$

$\forall\ q < +\infty$ $(p \geq n)$

$p > n$ by Morrey $f_k \in B \to f_k \in C^{0,1-\frac{n}{p}}(U)$ and

$\|f_k\|_{C^{0,1-\frac{n}{p}}} \leq \overline{C}$

$\Longrightarrow$ A.A. $f_k \to f$ uniformly (so in $C(\bar{U})$)

$f_k \to f$ in $C^{0,\alpha}(U)$ $\forall\ \alpha < 1 - \frac{n}{p}$

Q2 $B = \{ f \in W^{1,p}(\mathbb{R}^n) \quad \|f\|_{W^{1,p}} \leq 1 \}$.

In which spaces it is closed?

ANSWER $p = 1$ no spaces

$1 < p < n$ it is closed in $L^q(U)$ $p \leq q \leq p^*$

$p = n$ " " " in $L^q(U)$ $p \leq q < +\infty$

$p > n$ it is closed in $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$

it is closed in $L^q(U)$ $\forall\, p \leq q \leq +\infty$

proof $f_k \in B$ $\qquad \mathbb{R}^n = \bigcup_{N=1}^{+\infty} B(0,N)$

we apply the previous in every $B(0,N)$

and then extract a subsequence by diagonalization up to subseq

$$\textcircled{*}\quad \begin{aligned} f_k &\longrightarrow f \quad \text{locally in } L^q(U) \quad q < p^* \\ f_k &\overset{*}{\rightharpoonup} f \quad \text{locally weakly in } L^{p^*}(U) \\ Df_k &\longrightarrow Df \quad \text{locally weakly in } L^p(U) \end{aligned}$$

$f \in W^{1,p}(\mathbb{R}^n) \qquad f \in B$

How to control that a set C is closed in $(X, \| \|_X)$?

$C \subseteq X \qquad$ if c_k is a sequence of elements in C

$c_k \to c \in X \qquad \|c_k - c\|_X \to 0$

Then $c \in C$.

(if $f_k \to \bar{f}$ strongly in $L^q(\mathbb{R}^n)$ $\quad f_k \in B \quad$ then $\bar{f}$ is also the local limit f obtained in $\textcircled{*}$ $\Rightarrow \bar{f} \in B$.

$f = \bar{f}$

Q3 $\quad g \in L^2(U) \quad U$ open bdd class C^1

$$\int_U g(x)\,dx = 0$$

$$C = \left\{ f \in W^{1,2}(U) \quad \int_U f(x)\,dx = 0 \right\}$$

$$E(f) = \int_U |Df|^2 + f \cdot g \, dx$$

show that $\min E(f)$ exists in C, the minimizer is unique and write the equation satisfied by the minimizer.

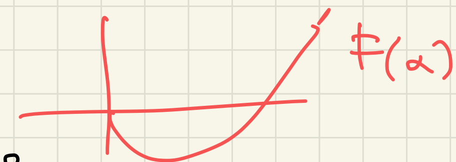
answer $\quad \forall f \in W^{1,2}(U) \quad E(f) \geq \| |Df| \|_{L^2}^2 - \|f\|_{L^2} \|g\|_{L^2}$

by Hölder inequality

for $f \in C$ by Poincaré

$$E(f) \geq \| |Df| \|_2^2 - \|g\|_2 \|f\|_2 \geq \| |Df| \|_{L^2}^2 - C\|g\|_{L^2} \| |Df| \|_{L^2}$$

let $F(\alpha) = \alpha^2 - C\|g\|_{L^2} \alpha$ $\quad F(\alpha) : [0, +\infty) \longrightarrow \mathbb{R}$



$$E(f) \geq \inf_{f \in C} \| |Df| \|_{L^2}^2 - C \|g\|_2 \| |Df| \|_{L^2} =$$

$$= \inf_{f \in C} F(\| |Df| \|_{L^2}) \geq \inf_{\alpha \geq 0} F(\alpha) > -\infty$$

so $E(f)$ is bdd by below

let $c := \inf_{f \in C} E(f)$. $\left(\geq \inf_{\alpha \geq 0} F(\alpha) \right)$

Let f_k be a minimizing sequence

$c \leq E(f_k) \leq c + \frac{1}{k} \Rightarrow$ by Poincaré & Hölder

$$\Rightarrow \underbrace{\| \, |Df_k| \, \|_{L^2}^2 - C\|g\|_2 \, \| \, |Df_k| \, \|_{L^2}^2}_{"F(\| |Df_k| \|_{L^2})} \leq c + 1$$

$\Rightarrow \exists \, \bar{C} \quad \| \, |Df_k| \, \|_{L^2} \leq \bar{C} \Rightarrow$ Poincaré $\Rightarrow \|f_k\|_{W^{1,p}} \leq \bar{C}$

$\Rightarrow$ Corollary of R-K $\Rightarrow$ up to subsequence

$$f_k \to \bar{f} \text{ in } L^2 \qquad \bar{f} \in W^{1,2}(U)$$
$$Df_k \rightharpoonup D\bar{f} \text{ in } L^2$$

since $f_k \to \bar{f}$ in $L^2 \Rightarrow f_k \to \bar{f}$ in $L^1 \Rightarrow \int_U f_k \, dx \to \int_U \bar{f} \, dx$

$\Rightarrow f \in C$

since $f_k \to \bar{f}$ in L^2 $\quad \int_U g f_k \, dx \to \int_U g \bar{f} \, dx$

moreover $\quad \int_U |Df_k|^2 dx \geq \int_U |D\bar{f}|^2 dx + \int_U 2 D\bar{f} \cdot (Df_k - D\bar{f}) dx$

$\Longrightarrow$ by weak convergence $\quad \liminf_k \int_U |Df_k|^2 dx \geq \int_U |D\bar{f}|^2 dx$

$\bar{C} \leq E(\bar{f}) \leq \liminf_k E(f_k) = \bar{C}$

$\Longrightarrow$ $\bar{f}$ is a minimizer $\quad E(\bar{f}) = \min_{f \in C} E(f)$.

$\bar{f}$ is UNIQUE

by contradiction let f_1, f_2 minimizers

$\lambda \in (0,1) \quad E(\lambda f_1 + (1-\lambda) f_2) = \int_U |\lambda Df_1 + (1-\lambda) Df_2|^2 + \lambda f_1 g + (1-\lambda) f_2 g$

$$< \int_U \lambda |Df_1|^2 + \lambda f_1 g + (1-\lambda)|Df_2|^2 + (1-\lambda) f_2 g =$$

by convexity

$$= \lambda E(f_1) + (1-\lambda) E(f_2) = C$$

$\Rightarrow$ $E(\lambda f_1 + (1-\lambda) f_2) < C = \min E(f)$ contradiction

the minimizer is UNIQUE.

Let $\phi \in C_c^\infty(U)$ $\qquad \tilde{\phi} = \phi(x) - \frac{1}{|U|}\int_U \phi(y)\,dy \in C^\infty(U)$

$$\int_U \tilde{\phi}\,dx = 0$$

$\bar{f} \in C \Rightarrow \bar{f} + \varepsilon\tilde{\phi} \in C \quad \forall \varepsilon \in \mathbb{R}$

$\lim_{\varepsilon \to 0} \frac{E(\bar{f} + \varepsilon\tilde{\phi}) - E(\bar{f})}{\varepsilon} = 0$ (by minimality of $\bar{f}$).

$$E(\bar f+\varepsilon\tilde\phi)-E(\tilde f)=\int_U |D\bar f+\varepsilon D\phi|^2-|D\bar f|^2+\cancel{\int_U \bar f g}+\varepsilon g\phi+$$

$D\phi = D\tilde\phi$

$$+\underbrace{\int_U \varepsilon g\left(\frac{1}{|U|}\int_U \phi(y)dy\right)}_{\overset{\shortparallel}{0}}-\cancel{\int_U \bar f g\,dx}=$$

since $\int_U g\,dx=0$

$$=\int_U 2\varepsilon D\bar f\cdot D\phi+\varepsilon^2|D\phi|^2+\varepsilon g\phi\,dx.$$

$$\lim_{\varepsilon\to 0}\frac{E(\bar f+\varepsilon\tilde\phi)-E(\bar f)}{\varepsilon}=0=\underbrace{\int_U 2D\bar f\cdot D\phi}_{=-2\Delta T_{\bar f}(\phi)}+g\cdot\phi\,dx \Rightarrow$$

$\forall \phi\in C_c^\infty(U)$ $\quad +2\Delta T_{\bar f}(\phi)=\int_U g\phi\,dx$

$\Rightarrow \bar f$ solves in the sense of distr. $\boxed{+2\Delta\bar f=g}$.

(P4) Poincaré inequality in dim 1.

1) $\forall f \in W^{1,p}(a,b)$ with $\int_a^b f(x)\,dx$

$$\|f\|_{L^p} \le C\|f'\|_{L^p} \implies \int_a^b |f(x)|^p\,dx \le C^p \int_a^b |f'(x)|^p\,dx$$

2) $\forall f \in W^{1,p}_0(a,b)$ $\quad (f(a)=0=f(b))$

$$\|f\|_{L^p} \le C\|f'\|_{L^p}$$

1 (2 is analogous)

To prove it we just use the fact that $f \in W^{1,p}(a,b) \to f$ is absolutely continuous $\implies f(x) = f(a) + \int_a^x f'(t)\,dt$

$$\int_a^b f(x)\,dx = 0 = f(a)(b-a) + \overbrace{\int_a^b \int_a^x f'(t)\,dt\,dx}^{\to \text{EXCHANGE THE ORDER OF INTEGRALS}} =$$

$$= f(a)(b-a) + \int_a^b (b-t)f'(t)\,dt$$

$$f(a) = -\int_a^b \frac{b-t}{b-a} f'(t)\,dt$$

$$f(x) = f(a) + \int_a^x f'(t)\,dt = \int_a^x f'(t)\left[1 - \frac{b-t}{b-a}\right]dt - \int_x^b f'(t)\,\frac{b-t}{b-a}\,dt$$

$$= \int_a^x f'(t)\,\underbrace{\left(\frac{t-a}{b-a}\right)}_{0 \le \frac{t-a}{b-a} \le 1} dt - \int_x^b f'(t)\,\underbrace{\frac{b-t}{b-a}}_{0 \le \frac{b-t}{b-a} \le 1}\,dt$$

$$|f(x)| \le \int_a^x |f'(t)|\,dt + \int_x^b |f'(t)|\,dt = \int_a^b |f'(t)|\,dt$$

$$|f(x)|^p \le \left[\int_a^b |f'(t)|\,dt\right]^p = (b-a)^p \left[\frac{1}{b-a}\int_a^b |f'(t)|\,dt\right]^p \le \text{JENSEN}$$

$$\le (b-a)^p \,\frac{1}{b-a} \int_a^b |f'(t)|^p\,dt = (b-a)^{p-1} \int_a^b |f'(t)|^p\,dt$$

$$\Longrightarrow \int_a^b |f(x)|^p dx \le \int_a^b (b-a)^{p-1} \int_a^b |f'(t)|^p dt \, dx =$$

$$= (b-a)^p \int_a^b |f'(t)|^p dt$$

$$\Longrightarrow \|f\|_{L^p(a,b)} \le (b-a) \|f'\|_{L^p(a,b)} .$$

Case 2 $\quad f \in W_0^{1,p}(a,b) \Longrightarrow f(x) = \int_a^x f'(t) dt$

$$|f(x)|^p \le (b-a)^{p-1} \int_a^b |f'(t)|^p dt$$

$\cdots \rightarrow \quad \cdots$