STRUCTURE of $(T_f)'$, $f \in BV(a,b)$

$$(T_f)' = \mu^+ - \mu^- = \mu_{ac} + \mu_J + \mu_c$$

$$\mu_{ac} = \underbrace{(f')^+ dx - (f')^- dx}$$

f' is the a.e derivative
$(f')^+$ is the positive part, $(f')^-$ negative part.

μ_{ac} is the difference of 2 measures ABSOLUTELY CONTINUOUS with respect to Lebesgue, with density the positive and negative part of f' (derivative almost everywhere)

$$f_{ac}(x) = \int_a^x f'(t)\,dt + f(a^+) \in W^{1,1}(a,b).$$

If $\mu_J, \mu_c = 0 \Rightarrow (T_f)' = T_{f'}$ $\quad$ f' weak derivative

$f \in W^{1,1}(a,b)$.

$\mu_J \perp \mathcal{L}$ and concentrated on jumps

$J_f = \{ y \in (a,b) \quad f(y^+) \neq f(y^-) \}$ $\quad$ <u>J_f is COUNTABLE</u>

$$\mu_J = \sum_{y \in J_f} [f(y^+) - f(y^-)] \, \delta_y$$

$$f_J(x) = \sum_{y \in J_f, \, y \leq x} [f(y^+) - f(y^-)]$$

$f - f_J$ is CONTINUOUS $\quad f - f_J - f_{ac} = f_c$ (cantorian function)

f_c is continuous, $f_c' = 0$ almost everywhere

$(T_{f_c})' = \mu_c \perp \mathcal{L}$

Therefore

if $f \in BV(a,b) \to$ (1) f admits a trace in a, b

$f(a^+), f(b^-)$

(2) f can be extended to a $BV(\mathbb{R})$ function.

(take $$\bar{f}(x) = \begin{cases} f(a^+) & x \leq a \\ f & a < x < b \\ f(b^-) & x \geq b \end{cases}$$

take $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi \equiv 1$ on $(a-1, b+1)$

$\phi \equiv 0$ on $(-\infty, a-2) \cup (b+2, +\infty)$

$0 \leq \phi \leq 1$

$\Rightarrow$ $f \cdot \phi \in BV(\mathbb{R})$, $f\phi = f$ on (a,b)

$\|f\phi\|_{BV(\mathbb{R})} \leq C \|f\|_{BV((a,b))}$

$supp(\phi f) \subseteq [a-2, b+2]$.

(3) COMPACTNESS THEOREM IN BV(a,b)

Let $(a,b)\subseteq \mathbb{R}$ bounded and $f_k \in BV(a,b)$ $\|f_k\|_{BV}\leq C$

then up to a subsequence $f_k \to f$ in $L^1(a,b)$

where $f\in BV(a,b)$ and $V(f,(a,b)) \leq \liminf_k V(f_k,(a,b))$

sometimes called Helly's theorem

it is based on <u>HELLY'S SELECTION THEOREM</u>

Let g_k a sequence of monotone (non decreasing) functions on a interval, which are uniformly bdd. Then there is a subsequence which converge POINTWISE to a monotone function g.

actually it can be obtained as in dim > 1

(by a corollary of Rellich-Kondrachov)

DENSITY THEOREM in $BV(U)$ $U \subseteq \mathbb{R}^n$.

Let $U \subseteq \mathbb{R}^n$ $\quad f \in BV(U) \iff \exists f_k \in C^\infty(U) \cap W^{1,1}(U)$

such that

$f_k \to f$ in $L^1(U)$ $\quad V(f_k, U) \to V(f, U)$

(NOT $f_k \to f$ in $BV(U)$!! otherwise $f \in W^{1,1}(U)$ ").

$W^{1,1}(U)$ is dense in $BV(U)$ with respect to the STRICT CONVERGENCE

proof by mollification: $f * \rho^N \to f$ in L^1 $\Rightarrow \lim_N V(f * \rho^N, U) \geq V(f, U)$

$V(f * \rho^N, U) \leq V(f, U)$ since

$$\int_U f * \rho^N \operatorname{div} \Phi \, dx = \int_U f \, \rho^N * \operatorname{div} \Phi \, dx = \int_U f \operatorname{div}(\rho^N * \Phi) \, dx$$

Using this, we deduce all the theorems (recalling the results for $W^{1,1}(U)$).

(GNS) $n>1$ $\forall f \in BV(\mathbb{R}^n)$ $\exists C \leq 1$

$$\|f\|_{L^{n/n-1}} \leq C V(f, \mathbb{R}^n)$$

(approximate f by $g_k \in W^{1,1}(\mathbb{R}^n)$ $\|g_k - f\|_{L^1} \to 0$

$V(f, \mathbb{R}^n) = \lim_k V(g_k, \mathbb{R}^n) = \lim_k \| |Dg_k| \|_{L^1}$ C

$g_k \to f$ a.e. by FATOU $\liminf_k \int_{\mathbb{R}^n} |g_k|^{\frac{n}{n-1}} dx \geq \int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx$

U open bdd of class C^1

$\to$ $\exists$ TRACE : $T: BV(U) \longrightarrow L^1(\partial U)$
$f \longmapsto Tf$ (CONTINUOUS)

$\longrightarrow$ $\exists$ EXTENSION $E_V: BV(U) \longrightarrow BV(\mathbb{R}^n)$ $f = E_V f$ in U
$\forall V \supset\supset U$ $f \longrightarrow E_V(f)$ supp $E_V f \subseteq V$

HELLY'S THEOREM (compactness in BV) (R–K)

U open bdd of class C^1 ($n=1$ open interval)

$f_k \in BV(U)$ with $\|f_k\|_{L^1} + V(f_k, U) \le C$

Up to a subsequence $f_k \to f$ in $L^p(U)$ $\forall\, p \in [1, \frac{n}{n-1})$

$f \in BV(U) \cap L^{n/n-1}(U)$, $V(f,U) \le \liminf_k V(f_k, U)$.

proof by density $\forall k$ $\exists\, g_k \in W^{1,1}(U)$ $\|f_k - g_k\|_{L^1} \le \frac{1}{k}$

$$|V(f_k,U) - V(g_k,U)| \le \frac{1}{k} \implies \|g_k\|_{W^{1,1}} \le \bar{C}$$

$\implies$ (RK.) $g_k \to f$ in $L^q(U)$ $\forall\, 1 \le q < 1^* = \frac{n}{n-1}$, $f \in L^{\frac{n}{n-1}}(U)$

and since $V(g_k, U) \le C \to f \in BV(U)$ $V(f,U) \le \liminf_k V(g_k, U)$

Then $V(f,U) \le \liminf_k V(f_k, U)$ $\|f_k - f\|_{L^1(U)} \to 0$

Finally $\|f_k - f\|_{L^{n/n-1}} \underset{\uparrow}{\leq} C\|f_k - f\|_{BV(U)} \leq C\|f_k\|_{BV} + C\|f\|_{BV}$

by extension and (GNS) $\qquad \leq \bar{C}$

$\Rightarrow \|f_k - f\|_{L^q} \leq \|f_k - f\|_{L^1}^{\theta} \|f_k - f\|_{L^{n/n-1}}^{1-\theta}$ by interpolation.

$\Rightarrow \|f_k - f\|_{L^q} \to 0 \quad \forall q < 1^* = \frac{n}{n-1}$.