# Fascio di piani paralleli o improprio

$\pi: \ 3x - y + 2z = 8$

$\pi \cap \pi_K \quad \left(\begin{array}{ccc|c} 3 & -1 & 2 & 8 \\ 3 & -1 & 2 & K \end{array}\right)$

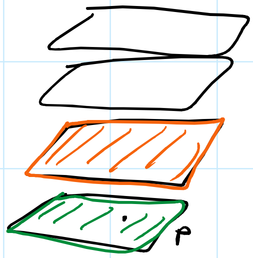
Determinare tutti i piani // al piano $\pi$.

Un piano $\sigma // \pi \iff V_\sigma = V_\pi : 3x - y + 2z = 0$

$$\boxed{\pi_K: 3x - y + 2z = K \quad \text{con } K \in \mathbb{R}}$$

## Esercizio:

Determinare i piano // a $2x - 5z = 1$ e passante per il punto $P = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.



## Svolgimento:

$\pi_K : \overbrace{2x - 5z}^{6 = K} = K$

$P \in \pi_K \qquad 2 \cdot 3 - 5 \cancel{0} = \boxed{6 = K}$

$$\boxed{\pi: 2x - 5z = 6}$$

## Esercizio:

Date le rette $r: \begin{cases} x = 1 \\ y + z = 0 \end{cases}$ e $s: \begin{cases} x = 3 \\ y - z = 0 \end{cases}$

e i punti $P = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ e $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

1) Determinare la posizione reciproca fra r ed s.

2) Dimostrare che $P \notin r$, $P \notin s$, $O \notin r$, $O \notin s$.

3) Determinare, se esiste, una retta t incidente sia r che s e passante per P.

4) Determinare, se esiste, una retta ℓ incidente sia r che s e passante per O.

Svolgimento:

1) $r: \begin{cases} x=1 \\ y+z=0 \end{cases}$ e $s: \begin{cases} x=3 \\ y-z=0 \end{cases}$ $\quad r \cap s \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \end{array}\right) = A|b$

Riduciamo $\begin{array}{c} 1^\circ \\ 2^\circ \\ 4^\circ-2^\circ \\ 3^\circ-1^\circ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$ $\quad rg(A)=3 \neq rg(A|b)=4$

$\Rightarrow$ r e s sono sghembe

$r \cap s = \emptyset \qquad V_r \cap V_s \begin{cases} x=0 \\ y+z=0 \\ x=0 \\ y-z=0 \end{cases} \Rightarrow V_r \cap V_s = \{\vec{0}\} \Rightarrow$ sono sghembe

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2) $P = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ $\qquad P \notin r \quad \begin{cases} 3=1 \\ 1+0=0 \end{cases}$ NO imp.

$P \notin s \quad \begin{cases} 3=3 \\ 1-0=0 \end{cases}$ imp.

$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad O \notin r \quad \begin{cases} 0=1 \\ 0+0=0 \end{cases}$ imp.

$O \notin s \quad \begin{cases} 0=3 \\ 0-0=0 \end{cases}$ imp.

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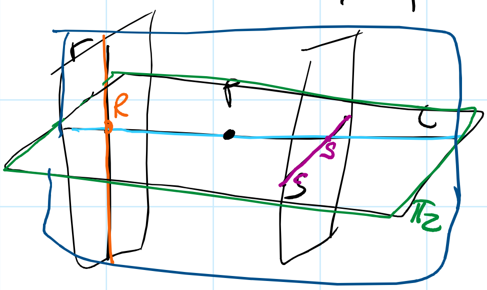
3) Determinare $t$ retta tale che
$$\boxed{\begin{cases} P\in t \\ t\cap r\neq\emptyset \\ t\cap s\neq\emptyset\end{cases}}$$

$\Rightarrow$ Se esiste la retta $t$ essa

$$\begin{cases} t\subseteq r\vee P = \pi_1 \\ t\subseteq s\vee P = \pi_2\end{cases}$$

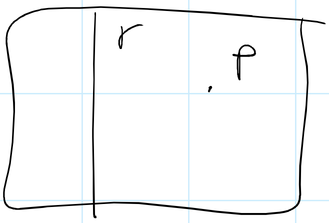
sono piani incidenti in una retta, quindi se $t$ esiste

$$\boxed{t=\pi_1\cap\pi_2}$$




$$\boxed{\pi_1 = r\vee P}$$

$$\begin{cases} x=1 \\ y+z=0\end{cases} \qquad \begin{cases} x-1=0 \\ y+z=0\end{cases}$$




$\pi_1$ appartiene al fascio $\boxed{(*)\ \pi_{\alpha,\beta}:\ \alpha(x-1)+\beta(y+z)=0}$ e passante per $P=\begin{pmatrix}3\\1\\0\end{pmatrix}$

$$\alpha(3-1)+\beta(1+0)=0$$

$$2\alpha+\beta=0 \qquad \beta=-2\alpha \qquad \boxed{\begin{matrix}\alpha=1\\ \beta=-2\end{matrix}}\ (*)$$

$$\pi_1=\pi_{1,-2}=1\cdot(x-1)-2(y+z)=0 \qquad \boxed{\pi_1:\ x-2y-2z=1}$$

$$s:\begin{cases} x=3 \\ y-z=0\end{cases} \qquad \begin{cases} x-3=0 \\ y-z=0\end{cases}$$

$\pi_2=P\vee s$ appartiene al fascio $\boxed{(*)\ \alpha(x-3)+\beta(y-z)=0}$ e passante per $P=\begin{pmatrix}3\\1\\0\end{pmatrix}$

$$\alpha(\cancel{3-3})+\beta(1-0)=0$$

$$\pi_2:\ 1(x-3)=0 \qquad \boxed{\pi_2:\ x=3} \qquad \beta=0 \qquad \boxed{\begin{cases}\alpha=1\\ \beta=0\end{cases}}\ (*)$$

$t: \begin{cases} x-2y-2z=1 \\ x=3 \end{cases} \quad \begin{cases} 3-2y-2z=1 \\ x=3 \end{cases} \quad \begin{cases} 2=2y+2z \\ x=3 \end{cases} \quad \begin{cases} y+z=1 \\ x=3 \end{cases}$

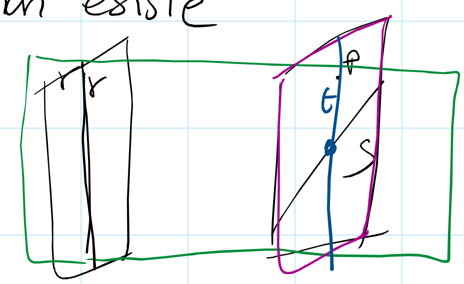
$P=\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \quad t: \begin{cases} y+z=1 \\ x=3 \end{cases} \quad r: \begin{cases} x=1 \\ y+z=0 \end{cases} \quad s: \begin{cases} x=3 \\ y-z=0 \end{cases}$

$\begin{cases} P\in t \\ t\cap r\neq\emptyset \\ t\cap s\neq\emptyset \end{cases} \quad P\in t \quad \begin{cases} 1+0=1 \\ 3=3 \end{cases}$ sì

$t\cap r \quad \begin{cases} y+z=1 \\ x=3 \\ x=1 \\ y+z=0 \end{cases} \quad t\cap r=\emptyset \quad \begin{cases} r/\!/t \\ r\cap t=\emptyset \end{cases} \leftarrow rg(A)=2\neq rg(A|b)=3$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

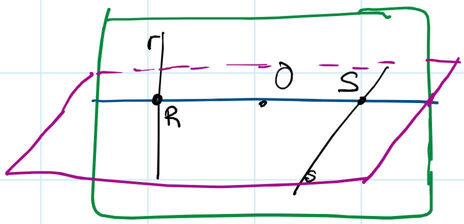
$\Rightarrow$ t non esiste




$t\subseteq \overbrace{r\vee P}^{\pi_1} \quad t,r\subseteq\pi_1$

$t\subseteq \underbrace{s\vee P}_{\pi_2} \quad t,s\subseteq\pi_2$

(a) $\ell \quad \begin{cases} O\in\ell \\ \ell\cap r\neq\emptyset \\ \ell\cap s\neq\emptyset \end{cases} \quad \ell=\sigma_1\cap\sigma_2 \quad \boxed{\sigma_1=O\vee r} \quad \boxed{\sigma_2=O\vee s}$




$\sigma_1=O\vee r \quad r\begin{cases} x=1 \\ y+z=0 \end{cases} \quad \boxed{\alpha(x-1)+\beta(y+z)=0}$ passante per O $\quad O=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\alpha(0-1)+\beta(0+0)=0$

$-\alpha=0 \quad \boxed{\begin{matrix} \alpha=0 \\ \beta=1 \end{matrix}}$

$\sigma_1: \; y+z=0$

$\sigma_2 = O \vee s$

$s: \begin{cases} x=3 \\ y-z=0 \end{cases}$

$\boxed{\alpha(x-3)+\beta(y-z)=0}$ passante per $O\begin{pmatrix}0\\0\\0\end{pmatrix}$

$\alpha(0-3)+\beta(0-0)=0$

$-3\alpha=0$ $\quad \boxed{\begin{matrix}\alpha=0\\ \beta=1\end{matrix}}$

$\sigma_2: \ y-z=0$

$\ell = \begin{cases} y+z=0 \\ y-z=0 \end{cases}$

$\begin{cases} O\in\ell & ① \\ \ell\cap r\neq\emptyset & ② \\ \ell\cap s\neq\emptyset & ③ \end{cases}$

① $O\in\ell$ $\quad \begin{cases} 0+0=0 \\ 0-0=0 \end{cases}$

② $\ell\cap r$ $\quad \begin{cases} y+z=0 \\ y-z=0 \\ x=1 \\ y+z=0 \end{cases}$ $\quad \ell\cap r = \begin{pmatrix}1\\0\\0\end{pmatrix} = R$

③ $\ell\cap s$ $\quad \begin{cases} y+z=0 \\ y-z=0 \\ x=3 \\ y-z=0 \end{cases}$ $\quad \ell\cap s = \begin{pmatrix}3\\0\\0\end{pmatrix} = S$

La retta $\ell$ esiste ed è $\begin{cases} y=0 \\ z=0 \end{cases}$

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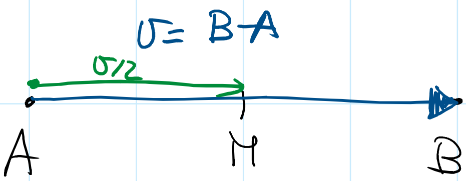
# Punto medio di un segmento




$A = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad B = \begin{pmatrix}3\\3\\5\end{pmatrix}$

M punto medio tra A e B

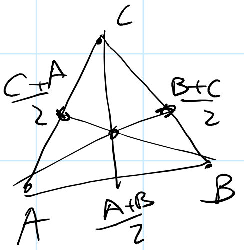
$\boxed{M = \dfrac{A+B}{2}}$

$M = \underline{A + \dfrac{B-A}{2}} = A + \dfrac{B}{2} - \dfrac{A}{2} = \dfrac{A}{2} + \dfrac{B}{2}$

$M = \dfrac{1}{2}\left[\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}3\\3\\5\end{pmatrix}\right] = \begin{pmatrix}2\\2\\3\end{pmatrix}$

# Baricentro di un triangolo

$T = \dfrac{A+B+C}{3}$




$\frac{1}{2}A + \frac{1}{2}B$

$\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$

Combinazioni baricentriche
somma dei coeff. 1

$\frac{1}{2} + \frac{1}{2} = 1$

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

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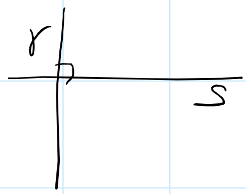
# Geometria Euclidea.

Uno spazio euclideo standard di dimensione $n$ è $\mathbb{R}^n$ come spazio affine con $V = \mathbb{R}^n$ dotato del prodotto scalare usuale.

## Ortogonalità:

① $r$ e $s$ rette si dicono ortogonali se

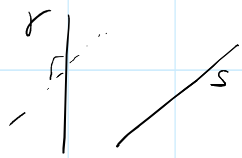



$r: R + \langle v_r \rangle$

$s: S + \langle v_s \rangle$

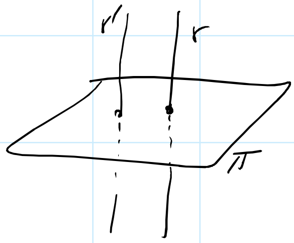
$v_r \cdot v_s = 0$

$v_r \neq \vec{0}$
$v_s \neq \vec{0}$



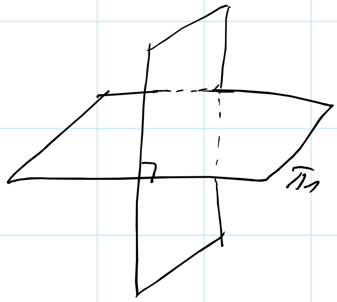

② In $\mathbb{R}^3$ una retta $r: R + \langle v_r \rangle$ e un piano $\pi: P + V_\pi$ si dicono ortogonali

Se $\langle v_r \rangle = V_\pi^\perp$

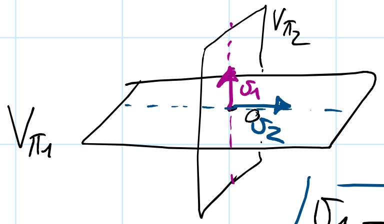


③ In $\mathbb{R}^3$ $\quad \pi_1: a_1x+b_1y+c_1z=d_1$

$\pi_2: a_2x+b_2y+c_2z=d_2$



$$\begin{pmatrix} a_1\\ b_1\\ c_1\end{pmatrix}\cdot\begin{pmatrix} a_2\\ b_2\\ c_2\end{pmatrix}=0 \qquad v_1\cdot v_2=0$$




$$\boxed{v_1 \perp v_2}$$

$$\boxed{v_1=\begin{pmatrix} a_1\\ b_1\\ c_1\end{pmatrix} \quad V_{\pi_1}: a_1x+b_1y+c_1z=0}$$

$v_2=\begin{pmatrix} a_2\\ b_2\\ c_2\end{pmatrix} \quad V_{\pi_2}: a_2x+b_2y+c_2z=0$

$V_{\pi_2}^{\perp}\subseteq V_{\pi_1}$

$V_{\pi_1}^{\perp}\subseteq V_{\pi_2}$

Esempio: $\mathbb{R}^3$ $\quad \pi_1: x=0 \quad v_1=\begin{pmatrix} 1\\ 0\\ 0\end{pmatrix} \quad v_1\cdot v_2=0 \quad \pi_1\perp\pi_2$

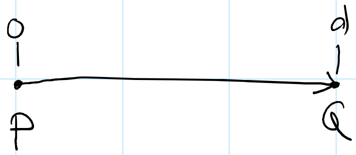
$\pi_2: y=0 \quad v_2=\begin{pmatrix} 0\\ 1\\ 0\end{pmatrix}$

Es: $\pi_1: x-y+3z=0 \quad v_1=\begin{pmatrix} 1\\ -1\\ 3\end{pmatrix} \qquad \pi_2: x+y=5 \quad v_2=\begin{pmatrix} 1\\ 1\\ 0\end{pmatrix}$

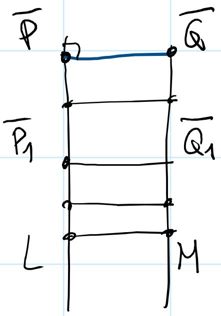
$$v_1\cdot v_2=\begin{pmatrix} 1\\ -1\\ 3\end{pmatrix}\cdot\begin{pmatrix} 1\\ 1\\ 0\end{pmatrix}=1\cdot 1+(-1)\cdot 1+3\cdot 0=1-1=0$$

# Distanze fra sottovarietà lineare in $\mathbb{R}^n$

Dati $P, Q$ punti di $\mathbb{R}^n$




$d(P,Q)=\|Q-P\|$

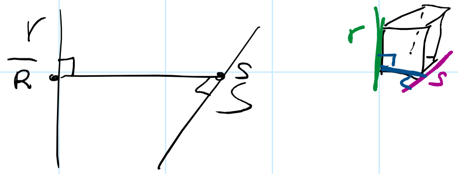



$$\boxed{d(L,M)=\min_{\substack{P\in L\\ Q\in M}} d(P,Q)}=d(\overline{P},\overline{Q})=d(\overline{P_1},\overline{Q_1})$$

Una coppia $(\bar{P},\bar{Q})$ con $\bar{P}\in L$ e $\bar{Q}\in M$ tali che

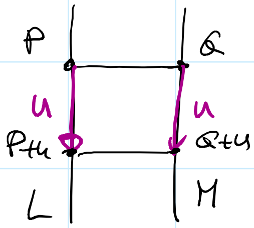
$d(L,M)=d(\bar{P},\bar{Q})$ viene detta coppia di punti di minima distanza.

Oss:

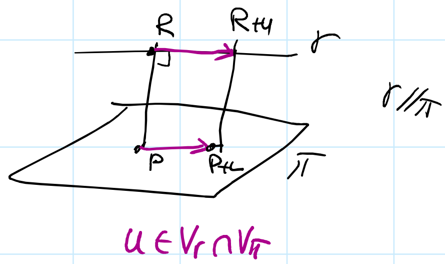


Se le sottovarietà lineari

$L$ e $M$ hanno $V_L\cap V_M=\{\vec{0}\}$ allora la coppia di punti di minima distanza è unica, altrimenti se $u\in V_L\cap V_M$ con $u\neq\vec{0}$ la coppia di punti di minima distanza non è unica se



$(P,Q)$ è coppia di minima distanza cmd allora anche $(P+u,Q+u)$ è c.m.d.

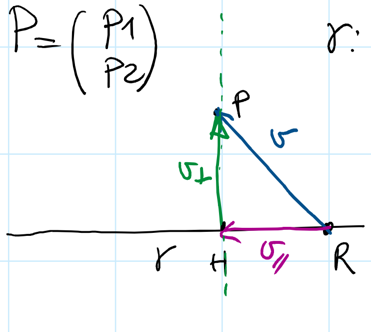


## Distanze in $\mathbb{R}^2$ $\quad L,M$

1) Distanza punto-punto: $d(P,Q)=\|Q-P\|$

2) Distanza punto-retta:

$P=\begin{pmatrix}p_1\\p_2\end{pmatrix}$ $\quad r: ax+by=c$ con $rg(a\ b)=1$



$d(P,r)=\|v_\perp\|$

$\boxed{P=\begin{pmatrix}p_1\\p_2\end{pmatrix}}$ $\quad\boxed{R=\begin{pmatrix}x\\y\end{pmatrix}\ ax+by=c}$

$r: ax+by=c$ cerchiamo $\underline{u \overset{\|u\|=1}{\text{versore}}}$ $\perp$ alla retta $r$

$$v_\perp := (v\cdot u)u \qquad \|v_\perp\| = \|(v\cdot u)u\| = |v\cdot u|\ \|u\| = |v\cdot u|$$

$d(P,r) = |v\cdot u|$ $\qquad$ $r: ax+by=c$ $\qquad u = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \end{pmatrix}$

$R = \begin{pmatrix} x \\ y \end{pmatrix} \in r$

$$|v\cdot u| = |(P-R)\cdot u| = \frac{1}{\sqrt{a^2+b^2}}\left|\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix}\right)\cdot\begin{pmatrix} a \\ b \end{pmatrix}\right| = \frac{\left|\begin{pmatrix} p_1-x \\ p_2-y \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix}\right|}{\sqrt{a^2+b^2}} =$$

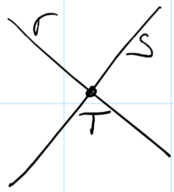
$$= \frac{|ap_1 - ax + bp_2 - by|}{\sqrt{a^2+b^2}} = \frac{|ap_1+bp_2-c|}{\sqrt{a^2+b^2}}$$

Formula:

$$\boxed{P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad r: ax+by=c \qquad d(P,r) = \frac{|ap_1+bp_2-c|}{\sqrt{a^2+b^2}}}$$

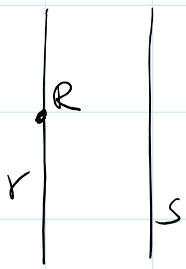
3) Distanza retta-retta $r, s$

Se $r\cap s \neq \emptyset$ allora $d(r,s)=0$




$d(r,s) = d(T,T)$

Se $r /\!/ s$ e $r\cap s = \emptyset$




$d(r,s) = d(R,s)$ con $R\in r$

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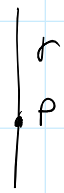
# Punti di minima distanza

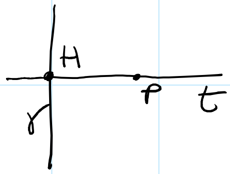
1) $d(P,Q)$ $\qquad$ la cmd $(P,Q)$

2) $d(P,r)$ è la cmd

Se $P \in r$ è la cmd $(P,P)$



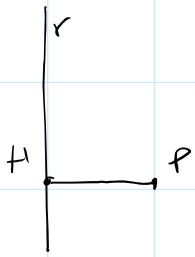

Se $P \notin r$




$d(P,r)$

è la cmd $(P,H)$ con

$H = t \cap r$ $t$ retta passante per $P$ e ortogonale a $r$




$d(P,r) = d(P,H)$ $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ $r: ax+by=c$

$t = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \left\langle \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle$

$P + \langle v_t \rangle$ $\langle v_t \rangle = \langle v_r \rangle^{\perp}$