Let $\mathcal{P}(\mathbb{R}) = \{\mu \text{ Borel measure } \quad \mu(\mathbb{R}) = 1\}$

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}) \qquad p \in [1, +\infty)$

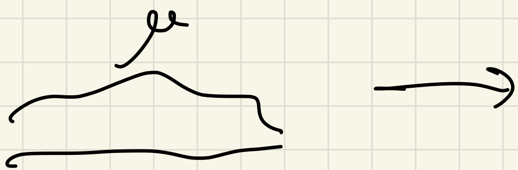
Monge OPTIMAL TRANSPORT PROBLEM:

Among all maps $\psi : \mathbb{R} \to \mathbb{R}$ which transport μ to ν
that is $\psi \# \mu = \nu$, i.e. $\nu(A) = \mu\{x \quad \psi(x) \in A\}$
find the one which has minimal cost

$$\inf_{\substack{\psi : \mathbb{R} \to \mathbb{R} \\ \psi \# \mu = \nu}} \int_{\mathbb{R}} \underbrace{|x - \psi(x)|^p}_{\parallel} \, d\mu(x)$$

cost of moving x to $\psi(x)$ is
given by $|x - \psi(x)|^p$

$$\mathbb{R} \xrightarrow{\psi} \mathbb{R} \qquad x \longrightarrow \psi(x)$$



$$\nu = \psi_\# \mu$$

$$\nu(A) = \mu\{x \in \mathbb{R} , \psi(x) \in A\}$$

$$\boxed{\nu(A) = \mu(\psi^{-1}(A))}$$

$$\text{“}\, \nu(\psi(B)) = \mu(B) \,\text{”}$$

$$\begin{tikzcd} B \arrow[r, "\psi"] & \psi(B) \\ \mu(B) \arrow[r] & \nu(\psi(B)) \end{tikzcd}$$

We may also restate the problem as follows

Let $\mu \in \mathcal{P}(\mathbb{R})$ and $\bar{\psi} : \mathbb{R} \to \mathbb{R}$ GIVEN

It is possible to find $\psi : \mathbb{R} \to \mathbb{R}$ such that

① $\psi \# \mu = \bar{\psi} \# \mu$

② $\displaystyle\int_{\mathbb{R}} |x - \psi(x)|^p \, dx \leq \int_{\mathbb{R}} |x - \bar{\psi}(x)|^p \, dx$

(Among all possible transport maps ψ such that $\psi \# \mu$ is the same $(= \bar{\psi} \# \mu = \nu)$ there is one which is associated to minimal cost?)

(1) First case in which this problem can be solved

$$\mu = \frac{1}{N}\delta_{x_1} + \frac{1}{N}\delta_{x_2} + \dots + \frac{1}{N}\delta_{x_N} \qquad N \in \mathbb{N}.$$

$$\downarrow$$

$$\nu = \frac{1}{N}\delta_{y_1} + \frac{1}{N}\delta_{y_2} + \dots + \frac{1}{N}\delta_{y_N}$$

all possible $\varphi : \mathbb{R} \to \mathbb{R}$ which satisfy $\varphi \# \mu = \nu$

are $\varphi(x_i) = y_j$ $\quad i = 1 \dots N$, $j = 1 \dots N$

(BIJECTIVE) $\quad \varphi : \{x_1, \dots, x_N\} \to \{y_1 \dots y_N\}$

Monge problem

$$\boxed{\mu = \frac{1}{N}\delta_{x_1} + \frac{1}{N}\delta_{x_2} + \dots + \frac{1}{N}\delta_{x_N}}$$

$$\inf_{\substack{\psi : \{x_1 \dots x_N\} \to \{y_1 \dots y_N\} \\ \text{BIJECTION}}} \underbrace{\int_{\mathbb{R}} |x - \psi(x)|^p \, \underline{\underline{d\mu(x)}}}$$

$$\Updownarrow \quad //$$

$$= \min_{\substack{\psi : \{x_1 \dots x_N\} \to \{y_1 \dots y_N\} \\ \text{BIJECTION}}} \left[|x_1 - \psi(x_1)|^p + |x_2 - \psi(x_2)|^p + \dots + |x_N - \psi(x_N)|^p \right]$$

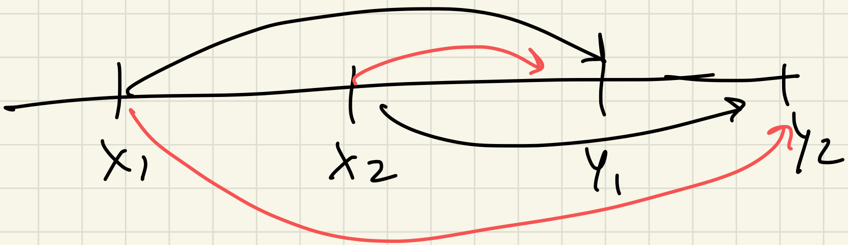
↓ the solution EXISTS since I am minimizing on a finite set (the set of all bijections $\{x_1 \dots x_N\} \to \{y_1 \dots y_N\}$ is FINITE)

UNIQUENESS.

example $\{x_1, x_2\} \longrightarrow \{y_1, y_2\}$

$\psi_1 : \begin{array}{l} x_1 \longrightarrow y_1 \\ x_2 \longrightarrow y_2 \end{array}$

$\psi_2 : \begin{array}{l} x_1 \longrightarrow y_2 \\ x_2 \longrightarrow y_1 \end{array}$



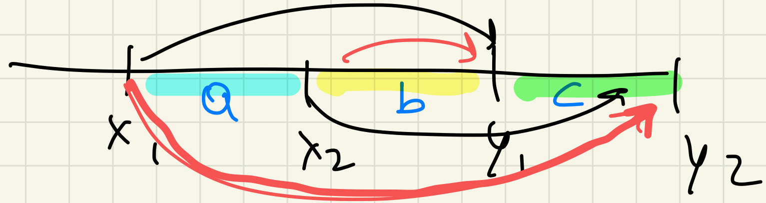
Cost of ψ_1 $\quad |x_1 - y_1|^p + |x_2 - y_2|^p$

cost of ψ_2 $\quad |x_1 - y_2|^p + |x_2 - y_1|^p$

$\boxed{p = 1}$ Cost of $\psi_1 =$ cost of ψ_2.

$$|x_1 - y_1| + |x_2 - y_2| = |x_1 - y_2| + |x_2 - y_1|$$

$p = 2$



$$\text{cost of } \psi_1 = |x_1 - y_1|^2 + |x_2 - y_2|^2 =$$

$$= (a+b)^2 + (b+c)^2 = a^2 + 2b^2 + c^2 + 2ab + 2bc$$

$$\text{cost of } \psi_2 = |x_1 - y_2|^2 + |x_2 - y_1|^2 =$$

$$= (a+b+c)^2 + b^2 = a^2 + 2b^2 + c^2 + 2ab + 2bc + \underline{\underline{2ac}}$$

$$\text{cost of } \psi_1 < \text{cost of } \psi_2$$

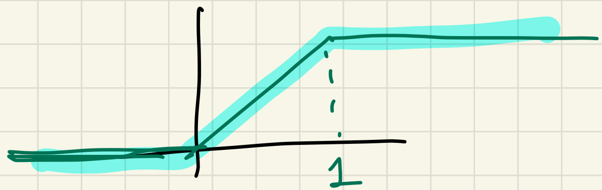
ψ_1 is the MONOTONE TRANSPORT MAP $\quad x_1 < x_2 \quad \psi(x_1) < \psi(x_2)$

Ex (BOOK SHIFT) $\qquad \psi_1 \# \mu = \nu \qquad \psi_2 \# \mu = \nu$

$\mu = \mathcal{L}_{[0,1]}$ (Lebesgue measure restricted to [0,1]

$\downarrow$

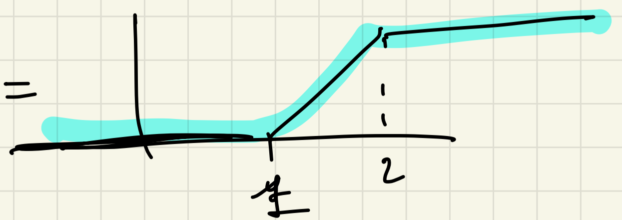
F_μ



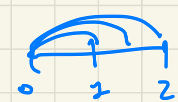
$$F_\mu(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$\nu = \mathcal{L}_{[1,2]}$ Lebesgue measure restricted to [1,2]

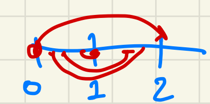
$F_\nu =$



$\psi_1 : x \longrightarrow 1 + x$



$\psi_2 : x \longrightarrow 2 - x$



Cost of φ_1

$p=1$

$\mu = \mathcal{L}_{[0,1]}$

$\mu = \chi_{[0,1]}\,dx$

$$\int_{\mathbb{R}} |x-\varphi_1(x)|^p \, d\mu(x) =$$

$$= \int_0^1 |x-\varphi_1(x)|^p \, dx =$$

$\varphi_1(x) = x+1$

$$= \int_0^1 |x-x-1|^p \, dx = 1$$

cost of φ_2

$\varphi_2(x) = 2-x$

$$\int_{\mathbb{R}} |x-\varphi_2(x)| \, d\mu = \int_0^1 |x-\varphi_2(x)| \, dx =$$

$$= \int_0^1 |x-2+x| \, dx = \int_0^1 \underbrace{2-2x}\, dx$$

$$= 2 - \left[x^2\right]_0^1 = 2-1 = 1$$

$p=1$ cost φ_2 = cost φ_1

$p = 2$ $\quad \text{cost } \psi_1 = 1$

$$\text{cost } \psi_2 = \int_0^1 (2-2x)^2 \, dx =$$

$$= \int_0^1 4 - 4x + 4x^2 \, dx = 4 - 2\left[x^2\right]_0^1 + \frac{4}{3}\left[x^3\right]_0^1 =$$

$$= 4 - 2 + \frac{4}{3} > 1$$

~ . ~

So a solution to the Monge problem may not be unique

Moreover if μ, ν are given

$$\{ \psi : \mathbb{R} \to \mathbb{R} \quad \psi \# \mu = \nu \}$$ COULD ALSO BE EMPTY

ex $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \quad \nu = \mathcal{L}_{[0,1]}$

WE RESTATE THE PROBLEM IN A MORE GENERAL SETTING

$\psi \# \mu = \nu \implies (id, \psi) \# \mu = \pi$

$\nu(B) = \mu\{x \in \mathbb{R} : \psi(x) \in B\}$

π is a probability measure on $\mathbb{R} \times \mathbb{R}$

$\mu \rightsquigarrow \pi$

$$(id, \psi) : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$$
$$x \longmapsto (x, \psi(x))$$

$\pi(A \times B) = (id, \psi) \# \mu (A \times B) = \mu\{x \in \mathbb{R} \quad (x, \psi(x)) \in A \times B\}$

$= \mu\{x, \quad x \in A, \psi(x) \in B\}$

$\pi(A \times \mathbb{R}) = \mu\{x \ , x \in A \ , \underline{\psi(x) \in \mathbb{R}}\} = \mu(A)$

$\pi(\mathbb{R} \times B) = \mu\{x, \quad x \in \mathbb{R} \ \psi(x) \in B\} = \nu(B)$

So $\psi: \mathbb{R} \to \mathbb{R}$ such that $\psi \# \mu = \nu$ is associated to a measure π on $\mathbb{R} \times \mathbb{R}$ such that $\pi(A \times \mathbb{R}) = \mu(A)$
$\pi(\mathbb{R} \times B) = \nu(B)$

Instead of considering $\{\psi: \mathbb{R} \to \mathbb{R} \quad \psi \# \mu = \nu\}$ (set which can be empty) I consider the set

$\{\pi : \mathcal{B}(\mathbb{R} \times \mathbb{R}) \to [0, +\infty] \quad \pi(A \times \mathbb{R}) = \mu(A) \quad \pi(\mathbb{R} \times B) = \nu(B)\}$
COUPLING BETWEEN μ, ν

$\{\psi: \mathbb{R} \to \mathbb{R} \quad \psi \# \mu = \nu\} \subseteq \{\pi$ coupling between $\mu, \nu\}$

(indeed $\pi = (id, \psi) \# \mu$ is a coupling.

π coupling between μ, and ν is also called a TRANSPORT PLAN (generalization of the transport map $\psi: \mathbb{R} \to \mathbb{R}$)

$$\pi(A\times B) \overset{\text{" "}}{=} \int_A \text{"}\pi(\{x\}\times B)\text{"}\, d\mu(x) = \int_B \pi(A\times\{y\})\, d\nu(y)$$

$$A\times B = \bigcup_{x\in A} (\{x\}\times B)$$

("DISINTEGRATION OF MEASURES")

$\sim\circ\sim$

MONGE PROBLEM

KANTOROVICH PROBLEM

$$\underbrace{\inf_{\substack{\psi:\mathbb{R}\to\mathbb{R}\\ \psi\#\mu=\nu}} \int_{\mathbb{R}} |x-\psi(x)|^p\, d\mu}_{\pi = (id,\psi)\#\mu} \geq \underbrace{\inf_{\substack{\pi \text{ COUPLING}\\ \pi(A\times\mathbb{R})=\mu(A)\\ \pi(\mathbb{R}\times B)=\nu(B)}} \int_{\mathbb{R}} |x-y|^p\, d\pi(x,y)}$$

$$\pi = (id,\psi)\#\mu$$

the set of all coupling is always NOT EMPTY

$\pi(A \times B) = \mu(A)\,\nu(B)$ is always a possible coupling

$$\inf_{\substack{\pi \text{ coupling} \\ \text{between} \\ \mu,\nu}} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, d\pi(x,y) = \inf_{\substack{\Omega \text{ prob. space} \\ \mathcal{L}_X = \mu \\ \mathcal{L}_Y = \nu \\ X,Y \text{ random variables}}} \mathbb{E}\,|X-Y|^p$$

$\pi(A \times \mathbb{R}) = \mu(A)$

$\pi(\mathbb{R} \times B) = \nu(B)$

π coupling between μ,ν $\leadsto$ π joint law of some X,Y
$\mathcal{L}_X = \mu \quad \mathcal{L}_Y = \mu$

$\psi : \mathbb{R} \to \mathbb{R}$ transport map $\psi \# \mu = \nu \rightarrow \exists X, Y \quad \mathcal{L}_X = \mu \quad \mathcal{L}_Y = \nu \quad Y = \psi(X)$

DEFINITION

Let $\mathcal{P}_1(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) \quad \int_{\mathbb{R}} |x| d\mu < +\infty\}$

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$

$W_1(\mu,\nu) =$ WASSERSTEIN DISTANCE BETWEEN μ, ν $= \inf_{\substack{\pi \\ \text{COUPLING} \\ \text{Between } \mu,\nu}} \int_{\mathbb{R}\times\mathbb{R}} |x-y| \, d\pi(x,y)$

$(\mathbb{P}_1(\mathbb{R}), W_1)$ is a complete METRIC SPACE

Note that $W_1(\mu,\nu) = \inf_{\substack{\Omega \\ \mathcal{L}_X = \mu \\ \mathcal{L}_Y = \nu}} \underbrace{\mathbb{E}|X-Y|}$ → distance in $M^1 = \{X : \Omega \to \mathbb{R} \quad \mathbb{E}|X| < +\infty\}$

(distance between μ, ν is the minimal distance in M^1 sense between random variables which have laws μ and ν).

Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$

$$W_2(\mu,\nu) = \inf_{\substack{\pi \text{ coupling} \\ \pi(A\times\mathbb{R}) = \mu(A) \\ \pi(\mathbb{R}\times B) = \nu(B)}} \left[\int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y) \right]^{1/2}$$

$\downarrow$

distance $\quad (\mathcal{P}_2, W_2)$ is a COMPLETE metric space

$$= \inf_{\substack{\Omega \\ \mathcal{L}_X = \mu \\ \mathcal{L}_Y = \nu}} \underbrace{\left[\mathbb{E}\,|X-Y|^2 \right]^{1/2}}_{\substack{\text{distance} \\ \text{between} \\ X \text{ and } Y \text{ in } M^2}}$$

$W_2(\mu,\nu)$ = infimum of all possible distances in M^2 sense of random variables X, Y, $\mathcal{L}_X = \mu$, $\mathcal{L}_Y = \nu$

Theorem (BRENIER)

1) $\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R})$ $\exists$ always at LEAST one $\bar{\pi}$ coupling such that
$$\int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, d\bar{\pi}(x,y) = \inf_{\pi \text{ coupling}} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, d\pi(x,y)$$
(for every $p \geq 1$)

($\exists$ $\bar{X}, \bar{Y}$ random variables $\mathcal{L}_{(\bar{X},\bar{Y})} = \bar{\pi}$ $\mathcal{L}_{\bar{X}} = \mu$ $\mathcal{L}_{\bar{Y}} = \nu$

$$W_p(\mu,\nu) = \left[\mathbb{E}|\bar{X}-\bar{Y}|^p\right]^{1/p}.$$

2) $\bar{\pi}$ is NOT UNIQUE for $p=1$

$\bar{\pi}$ is UNIQUE for $p=2$ ($p>1$) either if $\mu \ll \mathcal{L}$

or if μ, ν are both supported in some COMPACT

3) if $\mu \ll \mathcal{L}$ the optimal $\overline{\pi}$ is a wholly associated to a transport map $\overline{\varphi}$

$$\left(\nu = \overline{\varphi} \# \mu \right) \qquad \overline{Y} = \varphi(\overline{X})$$

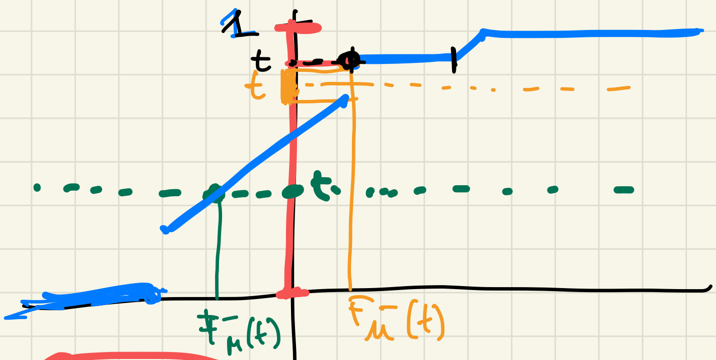
$$W_2(\mu, \nu) = \left(\int_{\mathbb{R}} |x - \overline{\varphi}(x)|^2 \, d\mu(x) \right)^{1/2} = \left(\mathbb{E} |\overline{X} - \varphi(\overline{X})|^2 \right)^{1/2}$$

where $\boxed{\overline{\varphi} \text{ is MONOTONE}}$

In dim 1 everything can be computed EXPLICITLY.
by using cumulative DISTRIBUTION FUNCTIONS

$$F_\mu : \mathbb{R} \to [0,1]$$

$\mu \in \mathcal{P}(\mathbb{R})$ $\quad$ $F_\mu(x) = \mu(-\infty, x]$ $\quad$ Cumulative DISTR. FUNCTION



F_μ is monotone, right cont.

$0 \le F_\mu(x) \le 1$

$F_\mu(-\infty) = 0$ $\quad$ $\underbrace{F_\mu(\text{"}+\infty\text{"}) = 1}_{\mu(-\infty,+\infty)=1}$

$t \in [0,1]$ (PSEUDO INVERSE)

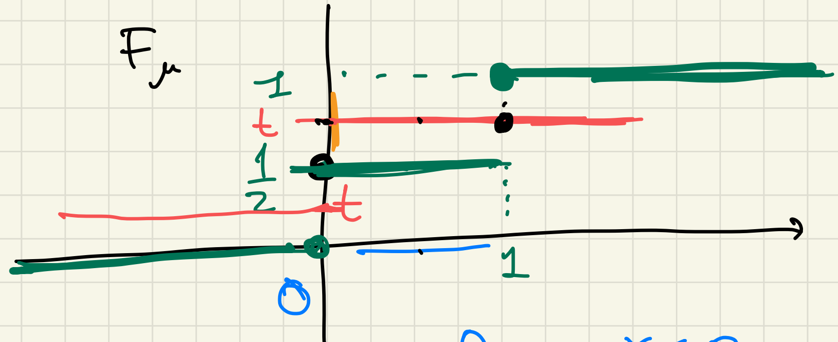
$$F_\mu^-(t) = \inf \{x \in \mathbb{R} \quad F_\mu(x) \ge t\}$$

Ex 1

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

$$\mu(A) = \begin{cases} 0 & 0,1 \notin A \\ \frac{1}{2} & 0 \in A \text{ or } 1 \in A \\ 1 & 0,1 \in A \end{cases}$$

$$F_\mu(x) = \mu(-\infty, x]$$



$$\mu(-\infty, x] = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

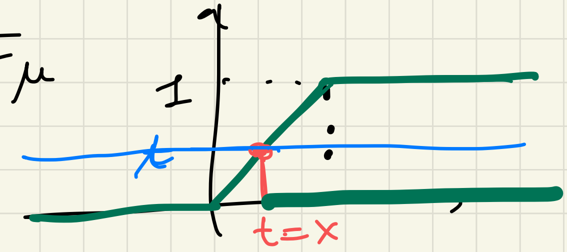
$$F_\mu^-(t) = \inf \{ x \quad F_\mu(x) \ge t \}$$

$$F_\mu^-(t) = \begin{cases} 0 & 0 < t \le \frac{1}{2} & F_\mu(x) \ge t > 0 \\ 1 & \frac{1}{2} < t \le 1 & F_\mu(x) \ge t > \frac{1}{2} \end{cases}$$

Ex 2

$\mu = \chi_{[0,1]}\, dx = \mathcal{L}_{[0,1]}$

F_μ



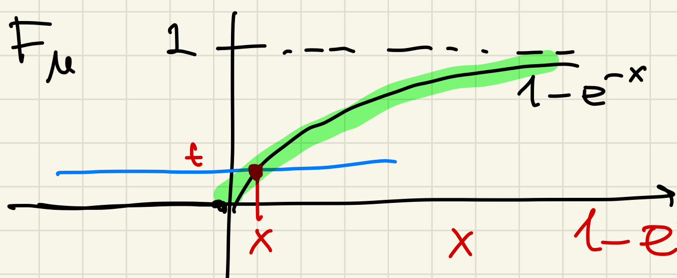
$F_\mu^{-1}(t) = t = \inf\{x \;\; F_\mu(x) \geq t\}$

Ex 3

$\mu = \underbrace{e^{-x} \chi_{[0,+\infty)}}\, dx \qquad F_\mu = \begin{cases} 0 & x < 0 \\ \underbrace{1 - e^{-x}} & x \geq 0 \end{cases}$

$\mu(-\infty, x] = \int_{-\infty}^{x} e^{-y} \chi_{[0,+\infty)}(y)\, dy = \int_0^x e^{-y}\, dy$

F_μ



$F_\mu^{-1}(t) = \log\left(\frac{1}{1-t}\right) \quad t \in [0,1)$

$1 - e^{-x} = t \qquad 1 - t = e^{-x} \qquad -x = \log(1-t)$

Proposition $\mu = F_\mu^- \# \mathcal{L}_{[0,1]}$

(observe $\mathcal{L}_{[0,1]} = \chi_{[0,1]}\,dx$ Lebesgue measure restricted to $[0,1]$

$F_\mu^- : [0,1] \longrightarrow \mathbb{R}$

$\mathcal{L}|_{[0,1]} \rightsquigarrow F_\mu^- \# \mathcal{L}_{[0,1]}$

Proof. First of all recall that

$F_\mu^-(t) = \inf \{ x \;\; F_\mu(x) \geq t \}$

then ① $F_\mu^-(t) \leq a \iff \inf\{x \; F_\mu(x) \geq t\} \leq a \iff \forall x \;\; F_\mu(x) \leq t, \Rightarrow x \leq a$

$\Rightarrow F_\mu(a) \geq t$

② $F_\mu^-(t) > a \iff \inf\{x \; F_\mu(x) \geq t\} > a \iff \forall x \;\; F_\mu(x) \leq t$ it hold $x > a$

$\Rightarrow F_\mu(a) < t$

Let us fix $x \in \mathbb{R}$

$F_\mu^- \# \mathcal{L}_{[0,1]} (-\infty, x] = \mathcal{L}\{ t \in [0,1] \;\; F_\mu^-(t) \in (-\infty, x] \}$

$= \mathcal{L}\{ t \in [0,1] \;\; F_\mu^-(t) \leq x \} =$

$= \mathcal{L}\{ t \in [0,1] \;\; F_\mu(x) \geq t \} = \mathcal{L}\,[0, F_\mu(x)]$

$$F_\mu^{-}{}_\# \mathcal{L}_{[0,1]} (-\infty, x] = \underbrace{\mathcal{L}([0, F_\mu(x)])}_{} = F_\mu(x) = \mu(-\infty, x]$$

LENGHT OF THE INTERVAL $\mathcal{L}[0,a] = a$

therefore $\forall x \in \mathbb{R}$ $\quad F_\mu^{-}{}_\# \mathcal{L}_{[0,1]}(-\infty, x] = \mu(-\infty, x]$

$$\Downarrow$$

$$F_\mu^{-}{}_\# \mathcal{L}_{[0,1]} = \mu .$$

<u>Proposition</u> Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that F_μ is continuous (e.g. $\mu \ll \mathcal{L}$) then it holds

$$(F_\nu^{-} \circ F_\mu)_\# \mu = \nu$$

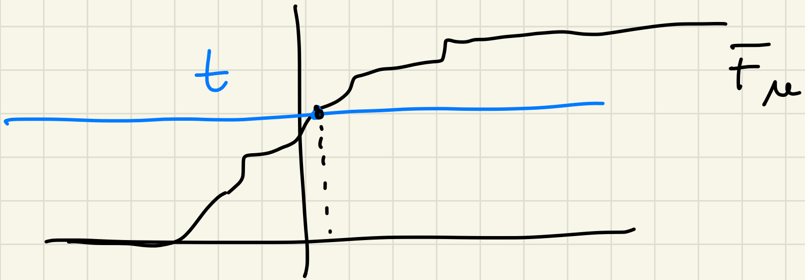
and the map

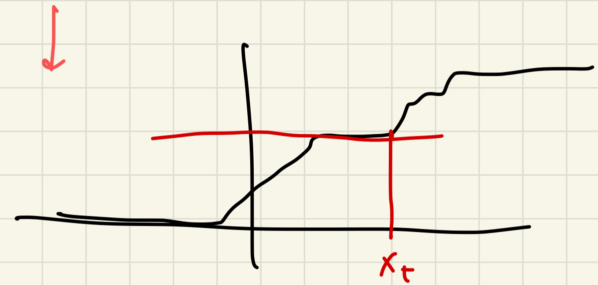
$$F_\nu^{-} \circ F_\mu : \mathbb{R} \to \mathbb{R}$$

is MONOTONE.

proof observation: if F_μ is continuous it holds

$\forall t \in [0,1]$ $\quad \underbrace{\{x,\ F_\mu(x) \le t\} = (-\infty, x_t]}$ $\quad$ where $F_\mu(x_t) = t$





F_μ is continuous $\to$ it has no jumps and by the INTERMEDIATE VALUE THEOREM for continuous functions

$\forall t \in [0,1]$ $\exists$ at least ONE x_t $F_\mu(x_t) = t$

(may be more than one!)

$\downarrow$

in any case $\{x\ F_\mu(x) \le t\} = (-\infty, x_t]$ $\quad$ x_t maximal x with $F_\mu(x_t) = t$

fix $x \in \mathbb{R}$

$F_\nu^- \circ F_\mu \quad \mathbb{R} \xrightarrow{\mu} \mathbb{R}$

$$(F_\nu^- \circ F_\mu)\#\mu(-\infty, x] = \mu\{y \in \mathbb{R} \quad F_\nu^- \circ F_\mu(y) \in (-\infty, x]\}$$

$$= \mu\{y \in \mathbb{R} \quad F_\nu^-(F_\mu(y)) \leq x\} =$$

$$\left[\begin{array}{c} F_\nu^-(a) \leq t \\ \Updownarrow \\ F_\nu(t) \geq a \end{array}\right]$$

$$= \mu\{y \in \mathbb{R} \quad F_\nu(x) \geq F_\mu(y)\}$$

by continuity of F_μ $\quad \{y \quad F_\mu(y) \leq t\} = (-\infty, y_t] \qquad F_\mu(y_t) = t$

$$= \mu(-\infty, y_{F_\nu(x)}] \qquad \text{where } F_\mu(y_{F_\nu(x)}) = F_\nu(x).$$

$$= F_\mu(y_{F_\nu(x)}) = F_\nu(x) = \nu(-\infty, x]$$

$\forall x \quad (F_\nu^- \circ F_\mu)\#\mu(-\infty, x] = \nu(-\infty, x] \Rightarrow \boxed{(F_\nu^- \circ F_\mu)\#\mu = \nu}$

COROLLARY of the BRENIER THEOREM.

If $\mu << \mathcal{L}$ then $\forall \nu \in \mathcal{P}(\mathbb{R})$, $\forall p \geq 1$

$$\inf_{\substack{\pi \text{ coupling} \\ \text{between } \mu,\nu}} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, d\pi(x,y) = \int_{\mathbb{R}} |x - \underbrace{F_\nu^{-} \circ F_\mu}(x)|^p \, d\mu(x)$$

(this means that the TRANSPORT MAP — which is MONOTONE — $\psi = F_\nu^{-} \circ F_\mu : \mathbb{R} \to \mathbb{R}$ has minimal cost among all possible transport plans between μ, ν (also among all possible couplings).

For $p = 1$ it is not the UNIQUE one.
For $p > 1$ it is the UNIQUE ONE.

$\mathcal{E}x \quad \mu = e^{-x}\chi_{[0,+\infty)}\,dx$ (EXPONENTIAL DISTRIBUTION)

$$F_\mu(x) = \mu(-\infty, x] \overset{x \geq 0}{=} \int_0^x e^{-t}\,dt = 1 - e^{-x}$$

$$\underbrace{F_\mu(x)}_{\text{CONTINUOUS!}} = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases}$$

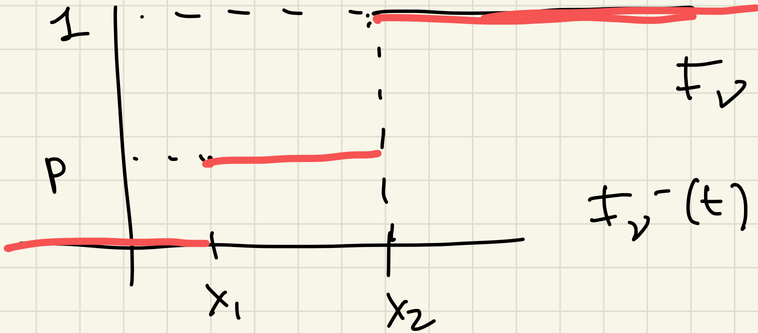
$\lambda \in [0,1]$

$\underline{x_1 < x_2}$

$\nu = \lambda\,\delta_{x_1} + (1-\lambda)\,\delta_{x_2}$ (BERNOULLI DISTRIBUTION)

$$\nu(A) = \begin{cases} 0 & x_1, x_2 \notin A \\ \lambda & x_1 \in A \quad x_2 \notin A \\ 1-\lambda & x_1 \notin A \quad x_2 \in A \\ 1 & x_1, x_2 \in A \end{cases}$$

$$F_\nu(x) = \begin{cases} 0 & x < x_1 \\ \lambda & x_1 \leq x < x_2 \\ 1 & x_2 \leq x \end{cases}$$



$$F_\nu^-(t) = \inf \{ x \mid F_\mu(x) \geq t \}$$

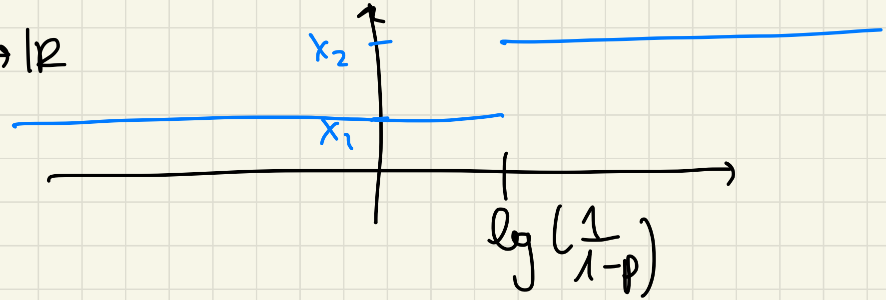
$$F_\nu^-(t) = \begin{cases} x_1 & 0 < t \leq p & F_\mu(x) \geq t > 0 \\ x_2 & p < t \leq 1 & F_\mu(x) \geq t > p \end{cases}$$

$$F_\nu^- \circ F_\mu(x) = \begin{cases} x_1 & 0 < F_\mu(x) \leq p \implies & x \leq \lg\left(\frac{1}{1-p}\right) \\ x_2 & p < F_\mu(x) \implies & x > \lg\left(\frac{1}{1-p}\right) \end{cases}$$

$$F_\mu(x) = 1 - e^{-x} \leq p \implies 1 - p \leq e^{-x} \implies \lg(1-p) \leq -x$$

$$\implies -\lg\left(\frac{1}{1-p}\right) \leq -x \implies \lg\left(\frac{1}{1-p}\right) \geq x$$

$F_\nu^- \circ F_\mu : \mathbb{R} \to \mathbb{R}$



$(F_\nu^- \circ F_\mu) \# \mu = \nu$

$$W_2(\mu,\nu)^2 = \inf_{\substack{\pi \\ \text{coupling}}} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y) = \int_{\mathbb{R}} |x - F_\nu^- \circ F_\mu x|^2 \, \underline{\underline{d\mu(x)}}$$

$\mu = \chi_{(0,+\infty)} e^{-x} dx$

$$= \int_0^{+\infty} |x - F_\nu^- \circ F_\mu(x)|^2 e^{-x} \, dx$$

$$= \int_0^{\log\left(\frac{1}{1-p}\right)} |x - x_1|^2 e^{-x} \, dx + \int_{\log\left(\frac{1}{1-p}\right)}^{+\infty} |x - x_2|^2 e^{-x} \, dx = \dots$$

= (can be computed explicitly)

Note that $W_2(\mu,\nu) = W_2(\nu,\mu)$ since W_2 is a distance!

(so it is also the optimal cost to transport ν to μ
- even if the transport is a transport plan not a transport map!)