

STOCHASTIC METHODS FOR ENGINEERING

EXAM 1

Exercise 1. Let $(W_t)_{t \geq 0}$ be the Brownian Motion. For every $s, t \geq 0$ compute

- i) $\mathbb{E}[W_t W_s]$
- ii) $\mathbb{E}[W_s W_t^2]$
- iii) $\mathbb{E}[W_s^2 W_t^2]$
- iv) $\mathbb{E}[W_s e^{W_t}]$.

Exercise 2. We recall that $X \sim \Gamma(\alpha, \lambda)$ for $\alpha, \lambda > 0$ if

$$f_X(x) := \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} 1_{[0, +\infty[}(x).$$

(recall that $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-\lambda x} dx$ and $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}, n \geq 1$)

i) Let ϕ_X be the characteristic function of X . Show that

$$\partial_\xi \phi_X(\xi) = -\frac{\alpha}{\xi + i\lambda} \phi_X(\xi),$$

and deduce, from this ϕ_X .

- ii) Check that if $X_j \sim \Gamma(\alpha_j, \lambda)$ for $j = 1, \dots, n$ are independent, then $X_1 + \dots + X_n \sim \Gamma(\alpha, \lambda)$ for a suitable α .
- iii) Let X and Y be i.i.d. random variables $\sim \Gamma(1, \lambda)$ and let $Z = X + Y$. Show that the following formula holds:

$$\mathbb{E}[\psi(X) | Z] = \frac{1}{Z} \int_0^Z \psi(x) dx.$$

Exercise 3. What does it mean that $X_n \xrightarrow{a.s.} X$ and $X_n \xrightarrow{\mathbb{P}} X$? What relationships exist between these two types of convergence?

Let $X_0 \sim U([0, 1])$ and (Y_n) are i.i.d. $\mathcal{N}(0, 1)$ random variables. For $n \geq 1$ define

$$X_n := \frac{X_{n-1}}{2} + Y_n.$$

- ii) Show that $X_n \xrightarrow{d} X$ where $X = \dots$
- iii) Is $X_n \xrightarrow{\mathbb{P}} X$?

EXAM 2

Exercise 4. Let (X_n) be i.i.d. random variables with exponential distribution, $X_n \sim \exp(\lambda)$. Let also N be independent of (X_n) with geometric distribution of parameter $0 < p < 1$ (namely, $\mathbb{P}(N = n) = (1 - p)^{n-1}p, n \geq 1$). Define

$$Y_n := \min\{X_1, \dots, X_n\}.$$

- i) Determine the cdf of Y_n .
- ii) Determine the cdf of Y_N .
- iii) Compute $\mathbb{E}[Y_N]$.

Exercise 5. Let U, V be i.i.d. uniformly distributed on $[0, 1]$. Derive

$$R = \sqrt{-2 \log U}, \quad \Theta := 2\pi V.$$

- i) Determine the joint distribution of (R, Θ) and the marginal distributions of R and Θ .
- ii) Set

$$X := R \cos \Theta, \quad Y := R \sin \Theta.$$

What is the joint distribution of (X, Y) ? Are X and Y independent? What are their univariate distributions?

Exercise 6. Let (X_n) be a sequence of random variables such that

$$X_0 \equiv 1, \quad X_{n+1} - X_n = \frac{1}{2} Y_n X_n,$$

where Y_n is a Bernoulli r.v. with $\mathbb{P}(Y_n = \pm 1) = \frac{1}{2}$, and Y_n is independent of Y_0, \dots, Y_{n-1} . We may interpret X_n as the amount of money an investor will have after n days if he wins or loses half of the money daily, both with probability $1/2$.

- i) Prove that $X_n \xrightarrow{a.s.} 0$ (hint: start estimating $\mathbb{P}(|X_n| \geq \frac{1}{2^m})$ for $m \in \mathbb{N}$ fixed, then use Borel-Cantelli's Lemma).
- ii) What about $\lim_n \mathbb{E}[X_n]$? Does $X_n \xrightarrow{L^1} 0$?

EXAM 3

Exercise 7. Let $X \in L^1(\Omega)$.

i) Prove the triangular inequality

$$|\mathbb{E}[X | \mathcal{G}]| \leq \mathbb{E}[|X| | \mathcal{G}].$$

ii) Show that if $X, Y \in L^1(\Omega)$ are such that $\mathbb{E}[X | Y] = 0$, then $\|X + Y\|_1 \geq \|Y\|_1$.

iii) Show that if $X, Y \in L^1(\Omega)$ are such that $\mu_{XY} = \mu_{YX}$ then

$$\mathbb{E}[X \pm Y | X \mp Y] = 0.$$

Use this to deduce $\|3X - Y\|_1 \geq \|X + Y\|_1$.

Exercise 8. Let (X_n) be independent with X_n uniformly distributed on $[-1 - \frac{1}{n}, 1 + \frac{1}{n}]$. Let

$$Y_n := \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k.$$

Discuss convergence in distribution of (Y_n) identifying also the limit (if any).

(hint: $\tilde{X}_k = \frac{k}{k+1} X_k \sim U[-1, 1]$).

Exercise 9. Let $(W_t)_{t \geq 0}$ be the Brownian Motion and define

$$B_t := \begin{cases} tW_{1/t}, & t > 0, \\ 0, & t = 0. \end{cases}$$

i) Check that (B_t) are gaussian random variables (determining their distributions), and that the increments of (B_t) , namely $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}} - B_{t_n}$ are independent random variables.

Check also that $B_t(\omega) \in \mathcal{C}([0, +\infty[)$.

ii) Check that

$$B_t \xrightarrow{\mathbb{P}} 0, \quad t \rightarrow 0+.$$

EXAM 4

Exercise 10. Let X, Y be independent random variables both with geometric distribution of parameter $p \in]0, 1[$ (that is, $\mathbb{P}(X = n) = \mathbb{P}(Y = n) = (1 - p)^{n-1}p$). Define

$$Z := \min(X, Y), \quad W := |X - Y|.$$

- i) Determine the distribution of (Z, W) .
- ii) Are Z and W independent?

Exercise 11. What does Borel–Cantelli’s Lemma state?

Let now (X_n) be i.i.d. random variables, $X_n \sim \exp(1)$. For $n \geq 2$ define

$$Y_n := \frac{X_n - \log n}{\log(\log n)}.$$

Prove that, for every $\varepsilon > 0$ fixed,

- i) $\mathbb{P}(\bigcup_N \bigcap_n \{Y_n \leq 1 + \varepsilon\}) = 1$
- ii) $\mathbb{P}(\bigcap_N \bigcup_n \{Y_n \geq 1 - \varepsilon\}) = 1$.

(it might be helpful to know that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha} < +\infty$ iff $\alpha > 1$)

Exercise 12. Let W be a BM on $(\Omega, \mathcal{F}, \mathbb{P})$. For $T > 0$ fixed, let $\mathcal{F}_T := \sigma(W_t : 0 \leq t \leq T)$ and define

$$\mathbb{Q}(E) := \mathbb{E} \left[\mathbf{1}_E e^{aW_T - \frac{a^2}{2}T} \right], \quad E \in \mathcal{F}_T.$$

- i) Check that \mathbb{Q} is a well defined probability measure on (Ω, \mathcal{F}_T) .
- ii) Check that

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E} \left[X e^{aW_T - \frac{a^2}{2}T} \right].$$

- ii) Let $B_t := W_t - at$. Check that $(B_t)_{0 \leq t \leq T}$ is a BM on $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

EXAM 5

Exercise 13. Let $0 < a < b$ and set

$$A := \{(x, y) \in \mathbb{R}^2 : |x - y| < a, |x + y| < b\}.$$

Let (X, Y) have uniform distribution in A , that is

$$f_{X,Y}(x, y) = \frac{1}{\lambda_2(A)} 1_A(x, y),$$

where $\lambda_2(A)$ stands for the Lebesgue measure of A .

- i) Determine f_Y .
- ii) Compute $\mathbb{E}[X | Y]$.
- iii) Compute the density of $\mathbb{E}[X | Y]$.

Exercise 14. Let (W_t) be a BM on $(\Omega, \mathcal{F}, \mathbb{P})$ and define

$$X_n := e^{aW_n - \frac{a^2}{2}n}, \quad n \in \mathbb{N}.$$

- i) Check that $X_n \in L^1(\Omega)$ for every $a \in \mathbb{R}$.
- ii) Show that (X_n) is a martingale w.r.t. $\mathcal{F}_n := \sigma(W_m : m \in \mathbb{N}, m \leq n)$.
- iii) Show that $\lim_n X_n = 0$ a.s.

Exercise 15. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X \in L(\Omega)$ such that $X \geq 0$ a.s.

- i) Show that, if X is \mathbb{N} valued (that is $\mathbb{P}(X \in \mathbb{N}) = 1$) then

$$\mathbb{E}[X] = \sum_{n=0}^{\infty} \mathbb{P}(X > n).$$

- ii) Show that, in general,

$$\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X > t) dt.$$

Hint for both cases: $\mathbb{P}(E) = \mathbb{E}[1_E] \dots$

Exercise 16. Let $(X, Y) \sim \mathcal{N}(0, C)$ where

$$C := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

- i) For which values of ρ matrix C is a true covariance matrix for (X, Y) ?
- ii) Imagine (X, Y) as the coordinates of a random point in the cartesian plane. Let (R, Θ) its polar coordinates ($R \geq 0$, $\Theta \in [0, 2\pi[$). Determine the joint distribution of (R, Θ) and, in particular, the distribution of Θ .
- iii) Compute $\mathbb{E}[R \mid \Theta]$ and $\mathbb{E}[R^2 \mid \Theta]$.
- iv) Under which conditions are R and Θ independent? In this case, determine also the density of R .

Exercise 17. Let X_n be i.i.d. random variables with common density

$$f_{X_n}(x) = \frac{1}{2}e^{-|x|}.$$

Prove that, $\forall \varepsilon > 0$,

- i) $\mathbb{P}\left(\frac{|X_n|}{\log n} \leq 1 + \varepsilon \text{ for all but finitely many } n\right) = 1.$
- ii) $\mathbb{P}\left(\frac{|X_n|}{\log n} \geq 1 - \varepsilon \text{ for infinitely many } n\right) = 1.$

Exercise 18. Let (W_t) be a BM. Define

$$X_t := \int_0^t \frac{W_u}{u} du.$$

- i) Explain why, for $t > 0$ fixed and for almost every ω , X_t is a well defined random variable.
 - ii) Compute $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$ (hint: $(\int_0^t f_u du)^2 = \int_0^t f_u du \int_0^t f_s ds \dots$)
 - iii) Define $B_t := W_t - X_t$. Compute $\mathbb{E}[B_t]$ and $\mathbb{V}[B_t]$.
 - iv) Let $\mathcal{F}_s := \sigma(W_r : 0 \leq r \leq s)$. Compute $\mathbb{E}[B_t \mid \mathcal{F}_s]$ for $0 \leq s < t$.
- (if needed, you are allowed to switch \mathbb{E} with \int_0^t)

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Exercise 19. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub σ -algebra.

i) Let $X \in L^2(\Omega)$. What is the conditional expectation $\mathbb{E}[X | \mathcal{G}]$? How is characterized? Prove that

$$\|\mathbb{E}[X | \mathcal{G}]\|_2 \leq \|X\|_2.$$

ii) Let $X, Y \in L^2(\Omega)$ be such that

$$\mathbb{E}[X | Y] = Y, \quad \mathbb{E}[Y | X] = X.$$

Deduce that $X = Y$ a.s. (hint: check that $\|X - Y\|_2^2 = 0$)

iii) Let $X, Y, Z \in L^2(\Omega)$ be such that

$$\mathbb{E}[X | Y] = Y, \quad \mathbb{E}[Y | Z] = Z, \quad \mathbb{E}[Z | X] = X.$$

Prove that $X = Y = Z$ a.s.

Exercise 20. Let

$$F(x) := e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

i) Check that F is a cumulative distribution function.

Let now (X_n) be i.i.d. random variables with $F_{X_n}(x) \equiv (1 - e^{-x})1_{[0, +\infty[}(x)$.

ii) Determine the cdf of $Y_n := \max\{X_1, \dots, X_n\}$.

iii) Use ii) to prove that $Y_n - \log n$ converges in distribution, determining also the limit distribution.

Exercise 21. Show that, if X and Y are absolutely continuous independent random variables with densities f_X and f_Y respectively, then $X + Y$ is also absolutely continuous and

$$f_{X+Y}(x) = f_X * f_Y(x), \quad a.e. x \in \mathbb{R}. \quad (3)$$

Let now $X_n, n \in \mathbb{N}, n \geq 1$ be i.i.d. exponential random variables, $f_{X_n}(x) = \lambda e^{-\lambda x} 1_{[0, +\infty[}(x)$.

i) Determine the distribution of $X_1 + \dots + X_n$ (hint: use FT)

ii) Let N be independent of $(X_n)_{n \in \mathbb{N}}$, with $\mathbb{P}(N = n) = (1 - p)^{n-1} p$, with $0 < p < 1$. Determine the distribution of

$$X_1 + \dots + X_N.$$

Exercise 22.

- i) What is the characteristic function of a random variable X ?
- ii) Justifying carefully the calculations, show that if $\mathbb{E}[X^2] < +\infty$ then the characteristic function ϕ_X of X is twice differentiable, and compute $\partial_\xi^2 \phi_X(0)$.
- iii) Justifying your answer, say if there exists a r.v. X such that $\phi_X(\xi) = e^{-c\xi^4}$.

Exercise 23. Let (W_t) be a BM.

- i) What are the characteristic properties of any Brownian Motion (BM)? And what are the characteristic properties of a martingale? Is the BM a martingale?
- ii) Prove that $W_t^3 - 3tW_t$ is a martingale.
- iii) Determine what terms you should add to W_t^4 in order to get a martingale.

Exercise 24. Suppose that (X_n) are i.i.d. random variables taking strictly positive values and such that $\mathbb{E}[|\log X_n|] < +\infty$. Discuss the limit of

$$Y_n := \left(\prod_{k=1}^n X_k \right)^{1/n}.$$

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Exercise 25. Let X, Y be absolutely continuous random variables with densities, resp., f_X and f_Y .

- i) By using the injectivity of the L^1 FT, prove that X and Y are independent iff $\phi_{X,Y}(\xi, \eta)$ (characteristic function of random vector (X, Y)) coincides with $\phi_X(\xi)\phi_Y(\eta)$.

Let now X, Y be independent and both standard Gaussian $\mathcal{N}(0, 1)$.

- ii) Show that $X + Y$ and $X - Y$ are also independent and gaussian.
 iii) Compute the conditional expectation $\mathbb{E}[XY | X - Y]$ (hint: $(x + y)^2 - (x - y)^2 = \dots$)

Exercise 26. Let N_k be i.i.d. random variables with $\mathbb{E}[N_k] = 0$ and $\mathbb{V}[N_k^2] \equiv \mathbb{E}[(N_k - \mathbb{E}[N_k])^2] = \sigma^2$, $\forall k \in \mathbb{N}$. We define (X_k) as

$$X_0 := x_0 \in \mathbb{R}, \quad X_k = \alpha X_{k-1} + N_{k-1}, \quad k \geq 1.$$

with $|\alpha| < 1$ and $x_0 \in \mathbb{R}$ fixed.

- i) Calculate means $\mathbb{E}[X_k]$ and variance $\mathbb{V}[X_k]$.
 ii) Let $\mathcal{F}_k := \sigma(X_1, \dots, X_k)$. Is (X_k) a martingale w.r.t. \mathcal{F}_k ? Justify your answer.
 iii) Compute $\mathbb{E}[(X_{k+1} - X_k)^2]$. What can you conclude about convergence in L^2 of (X_k) ?
 iv) Assume also that $N_k \sim \mathcal{N}(0, \sigma^2)$, $\forall k \in \mathbb{N}$. Prove that X_k converges in distribution and determine the limit distribution.

Exercise 27. Let (W_t) be a Brownian Motion (BM).

- i) What are the characteristic properties of W_t ?
 ii) Let

$$X_t := W_t^3 - 3 \int_0^t W_r \, dr, \quad t \geq 0.$$

Check that X_t is a martingale w.r.t. $\mathcal{F}_t := \sigma(W_s : s \leq t)$. (if needed, you are allowed to exchange the conditional expectation with the Riemann integral $\int_0^t \dots \, dr$)

Exercise 28. Let $(X, Y) \sim \mathcal{N}(0, C)$ where

$$C = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

- i) Is C a well defined covariance matrix?
- ii) Check that Y and $2X - Y$ are independent.
- ii) Compute $\mathbb{E}[X^2Y \mid 2X - Y]$ (hint: $X = \frac{1}{2}(2X - Y) + \frac{1}{2}Y \dots$).

Exercise 29. i) What does the Borel-Cantelli Lemma says? Provide a precise statement (no proof is required).

Let now X_n be independent Bernoulli random variables with

$$\mathbb{P}(X_n = 1) = \frac{1}{\sqrt{n}}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{\sqrt{n}}.$$

- i) Let $E := \{X_n = X_{n+1} = X_{n+2} = 1, \text{ for infinitely many } n\}$. Check that E is an event and prove that $\mathbb{P}(E) = 0$.
- ii) Let $F := \{X_n = X_{n+1} = 1, \text{ for infinitely many } n\}$. Check that F is an event. What about $\mathbb{P}(F)$?

Exercise 30. Let $X_k \sim U([0, 1])$ i.i.d. random variables, and let $S_n := \sum_{k=1}^n X_k$. Define

$$N := \min\{n \geq 2 : S_n > 1\}$$

- i) Compute $\mathbb{P}(N > n)$ and $\mathbb{P}(N = n)$ for $n \in \mathbb{N}$.
- ii) Compute $\mathbb{E}[N]$ and $\mathbb{V}[N]$.
- iii) Compute $\mathbb{E}[S_N]$.

It may be helpful to know that

$$I_n^j := \int_{0 \leq x_1, \dots, x_n \leq 1, x_1 + \dots + x_n \leq 1} (x_1 + \dots + x_n)^j dx_1 \cdots dx_n = \frac{1}{(n-1)!(n+j)}.$$

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Exercise 31. Let $X, Y \sim \mathcal{N}(0, 1)$ be independent standard Gaussian random variables.

i) Let

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be invertible. Show that $(U, V) := T(X, Y)$ is Gaussian, and determine its mean vector and covariance matrix. Under which conditions on a, b, c, d are U and V independent?

ii) Compute $\mathbb{E}[X + 2Y \mid X - Y]$.

Exercise 32. Let $X \in L^2(\Omega)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra.

i) Explain how the conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$ is defined under the current assumptions, and list its main properties.

ii) Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be two sub- σ -algebras. Prove that

$$\mathbb{V}[\mathbb{E}[X \mid \mathcal{G}]] \geq \mathbb{V}[\mathbb{E}[X \mid \mathcal{H}]].$$

(Here $\mathbb{V}[Z] = \mathbb{E}[(Z - \mathbb{E}[Z])^2] = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$ denotes the variance of Z .)

Exercise 33. Let $(W_t)_{t \geq 0}$ be a Brownian motion.

i) Prove that $\mathbb{E}[W_s W_t] = \min(s, t)$ for all $s, t \geq 0$.

Define now

$$X_n := \frac{1}{2^n} \sum_{k=0}^{2^n-1} W_{k/2^n}, \quad n \in \mathbb{N}.$$

ii) Determine the characteristic function of X_n (you may assume that X_n is Gaussian).

iii) Does (X_n) converge in distribution? If so, what is the limit distribution?

SOLUTIONS.

Exercise 1. i) If $s = t$, $\mathbb{E}[W_t^2] = t$. Assume $0 \leq s < t$. Then

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2] = \mathbb{E}[W_s]\mathbb{E}[W_t - W_s] + s = s.$$

Therefore, $\mathbb{E}[W_s W_t] = s \wedge t$.

ii) If $s = t$ then $\mathbb{E}[W_t^3] = 0$. Assume $s < t$; then

$$\mathbb{E}[W_s W_t^2] = \mathbb{E}[W_s(W_t - W_s + W_s)^2] = \mathbb{E}[W_s(W_t - W_s)^2] + 2\mathbb{E}[W_s^2(W_t - W_s)] + \mathbb{E}[W_s^3].$$

By independence of increments

$$\mathbb{E}[W_s(W_t - W_s)^2] = \mathbb{E}[W_s]\mathbb{E}[(W_t - W_s)^2] = 0,$$

and, similarly, $\mathbb{E}[W_s^2(W_t - W_s)] = \mathbb{E}[W_s^2]\mathbb{E}[W_t - W_s] = 0$. We conclude that $\mathbb{E}[W_s W_t^2] = 0$. If $t < s$, then

$$\mathbb{E}[W_s W_t^2] = \mathbb{E}[(W_s - W_t + W_t)W_t^2] = 0.$$

We conclude that $\mathbb{E}[W_s W_t^2] \equiv 0$.

iii) If $s = t$ we have $\mathbb{E}[W_t^4] = 3t^2$. If $s \neq t$, so for instance $s < t$ we have

$$\mathbb{E}[W_s^2 W_t^2] = \mathbb{E}[W_s^2(W_t - W_s + W_s)^2] = \mathbb{E}[W_s^2(W_t - W_s)^2] + 2\mathbb{E}[W_s^2(W_t - W_s)] + \mathbb{E}[W_s^4]$$

The middle term equals 0 while the latter equals $3s^2$. About the first

$$\mathbb{E}[W_s^2(W_t - W_s)^2] = \mathbb{E}[W_s^2]\mathbb{E}[(W_t - W_s)^2] = s(t - s).$$

In conclusion

$$\mathbb{E}[W_s^2 W_t^2] = s(t - s) + 3s^2 = st + 2s^2.$$

iv) Arguing in a similar way, for $s = t$ we have

$$\mathbb{E}[W_s e^{W_s}] = \int_{\mathbb{R}} x e^x e^{-\frac{x^2}{2s}} \frac{dx}{\sqrt{2\pi s}} = \int_{\mathbb{R}} x e^{-\frac{x^2 - 2sx + s^2 - s^2}{2s}} \frac{dx}{\sqrt{2\pi s}} = e^{\frac{s}{2}} \int_{\mathbb{R}} x e^{-\frac{(x-s)^2}{2s}} \frac{dx}{\sqrt{2\pi s}} = s e^{s/2}.$$

If $s < t$,

$$\mathbb{E}[W_s e^{W_t}] = \mathbb{E}[W_s e^{W_t - W_s} e^{W_s}] = \mathbb{E}[W_s e^{W_s}] \mathbb{E}[e^{W_t - W_s}] = s e^{s/2} \int_{\mathbb{R}} e^x e^{-\frac{x^2}{2(t-s)}} \frac{dx}{\sqrt{2\pi(t-s)}}$$

Now,

$$\int_{\mathbb{R}} e^x e^{-\frac{x^2}{2(t-s)}} \frac{dx}{\sqrt{2\pi(t-s)}} = e^{\frac{t-s}{2}} \int_{\mathbb{R}} e^{-\frac{(x-(t-s))^2}{2(t-s)}} \frac{dx}{\sqrt{2\pi(t-s)}} = e^{\frac{t-s}{2}},$$

so,

$$\mathbb{E}[W_s e^{W_t}] = s e^{\frac{s}{2}} e^{\frac{t-s}{2}} = s e^{t/2}.$$

Finally, if $t < s$,

$$\mathbb{E}[W_s e^{W_t}] = \mathbb{E}[(W_s - W_t) e^{W_t}] + \mathbb{E}[W_t e^{W_t}] = t e^{t/2}.$$

Therefore $\mathbb{E}[W_s e^{W_t}] = (s \wedge t) e^{t/2}$. □

Exercise 2. i) We have

$$\phi_X(\xi) = \widehat{f_X}(-\xi) = \int_{\mathbb{R}} f_X(x) e^{i\xi x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} e^{i\xi x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{-(\lambda-i\xi)x} dx.$$

Now, being $x^{\alpha-1} = \partial_x \frac{x^\alpha}{\alpha}$, by integrating by parts we have

$$\int_0^{+\infty} \partial_x \left(\frac{x^\alpha}{\alpha} \right) e^{-(\lambda-i\xi)x} dx = \frac{1}{\alpha} \left(\left[x^\alpha e^{-(\lambda-i\xi)x} \right]_{x=0}^{x=+\infty} + (\lambda-i\xi) \int_0^{+\infty} x^\alpha e^{-(\lambda-i\xi)x} dx \right)$$

and since $\left[x^\alpha e^{-(\lambda-i\xi)x} \right]_{x=0}^{x=+\infty} = 0$ we get

$$\phi_X(\xi) = \frac{\lambda-i\xi}{\alpha} \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} e^{i\xi x} dx.$$

On the other hand

$$\partial_\xi \phi_X(\xi) = i \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} e^{i\xi x} dx = i \frac{\alpha}{\lambda-i\xi} \phi_X(\xi) = -\alpha \frac{1}{\xi+i\lambda} \phi_X(\xi)$$

So,

$$\log \phi_X(\xi) = -\alpha \log(\xi+i\lambda) + k,$$

from which

$$\phi_X(\xi) = K(\xi+i\lambda)^{-\alpha}.$$

Now, since $\phi_X(0) = 1$, we have $1 = K(i\lambda)^{-\alpha}$, that is $K = (i\lambda)^\alpha$, from which we obtain

$$\phi_X(\xi) = (i\lambda)^\alpha (\xi+i\lambda)^{-\alpha} = \left(1 + \frac{\xi}{i\lambda} \right)^{-\alpha} = \left(1 - i \frac{\xi}{\lambda} \right)^{-\alpha}.$$

ii) If $X_j \sim \Gamma(\alpha_j, \lambda)$ are independent, then

$$\phi_{X_1+\dots+X_n}(\xi) = \prod_{j=1}^n \phi_{X_j}(\xi) = \prod_{j=1}^n \left(1 - i \frac{\xi}{\lambda} \right)^{-\alpha_j} = \left(1 - i \frac{\xi}{\lambda} \right)^{-(\alpha_1+\dots+\alpha_n)}.$$

From this, and from the uniqueness of the FT, $X_1 + \dots + X_n \sim \Gamma(\alpha_1 + \dots + \alpha_n, \lambda)$.

iii) We have

$$\mathbb{E}[\psi(X) | Z] = \varphi(Z),$$

where

$$\varphi(z) = \int_{\mathbb{R}} \psi(x) f_{X|Z}(x|z) dx,$$

with

$$f_{X|Z}(x|z) = \frac{f_{XZ}(x, z)}{f_Z(z)},$$

(provided XZ is abs. cont.). Now, $(X, Z) = (X, X+Y) = T(X, Y)$ where $T(x, y) = (x, x+y)$ is clearly a bijection on \mathbb{R}^2 , so

$$f_{XZ}(x, z) = f_{XY}(T^{-1}(x, z)) |\det(T^{-1})'(x, z)|.$$

Now $(x, z) = (x, x+y)$ iff $(x, y) = (x, z-x)$, $\det(T^{-1})' = (\det T')^{-1} = 1$, so

$$f_{XZ}(x, z) = f_{XY}(x, z-x) \stackrel{indep}{=} f_X(x) f_Y(z-x).$$

Since $X, Y \sim \Gamma(1, \lambda)$ we have

$$f_X(x) = \lambda e^{-\lambda x} 1_{[0, +\infty[},$$

and, by ii), about f_Z we have

$$f_Z(z) = \frac{\lambda^2}{\Gamma(2)} z e^{-\lambda z} 1_{[0,+\infty[}(z) = \lambda^2 z e^{-\lambda z} 1_{[0,+\infty[}(z).$$

Therefore

$$f_{X|Z}(x|z) = \frac{\lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} 1_{[0,+\infty[}(x) 1_{[0,+\infty[}(z-x)}{\lambda^2 z e^{-\lambda z} 1_{[0,+\infty[}(z)} = \frac{1}{z} 1_{[0,+\infty[}(z) 1_{[0,z]}(x),$$

from which we obtain

$$\varphi(z) = \int_{\mathbb{R}} \psi(x) \frac{1}{z} 1_{[0,+\infty[}(z) 1_{[0,z]}(x) dx = \left(\frac{1}{z} \int_0^z \psi(x) dx \right) 1_{[0,+\infty[}(z).$$

So, since $Z \geq 0$ with probability 1,

$$\varphi(Z) = \frac{1}{Z} \int_0^Z \psi(x) dx. \quad \square$$

Exercise 3. See notes for definitions and relations between convergence in distribution and in probability.

i) We use characteristic functions. We notice that

$$X_n = \frac{1}{2} X_{n-1} + Y_n = \frac{1}{2} \left(\frac{1}{2} X_{n-2} + Y_{n-1} \right) + Y_n = \frac{1}{2^2} X_{n-2} + \sum_{k=0}^1 \frac{1}{2^k} Y_{n-k}$$

Iterating, after n steps we arrive at formula

$$X_n = \frac{1}{2^n} X_0 + \sum_{k=0}^{n-1} \frac{1}{2^k} Y_{n-k}$$

Because of the assumptions on independence

$$\phi_{X_n}(\xi) = \phi_{\frac{1}{2^n} X_0}(\xi) \prod_{k=0}^{n-1} \phi_{\frac{1}{2^k} Y_{n-k}}(\xi) = \phi_{X_0} \left(\frac{\xi}{2^n} \right) \prod_{k=0}^{n-1} e^{-\frac{1}{2} \left(\frac{\xi}{2^k} \right)^2} = \phi_{X_0} \left(\frac{\xi}{2^n} \right) e^{-\frac{1}{2} \left(\sum_{k=0}^{n-1} \frac{1}{4^k} \right) \xi^2}$$

Letting $n \rightarrow +\infty$ we have

$$\phi_{X_n}(\xi) \longrightarrow \phi_{X_0}(0) e^{-\frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{1}{4^k} \right) \xi^2} = 1 \cdot e^{-\frac{1}{2} \cdot \frac{4}{3} \xi^2} = \phi_{\mathcal{N}(0, \frac{4}{3})}(\xi).$$

We conclude that $X_n \xrightarrow{d} X \sim \mathcal{N}(0, \frac{4}{3})$.

ii) Since convergence in probability is stronger than convergence in distribution, if (X_n) converges in probability then it converges also in distribution to the same limit. Thus, the unique possibility is $X_n \xrightarrow{\mathbb{P}} X$. But then also $X_{n-1} \xrightarrow{\mathbb{P}} \mathbb{P}$, whence, being

$$X_n = \frac{X_{n-1}}{2} + Y_n, \implies Y_n = X_n - \frac{X_{n-1}}{2} \xrightarrow{\mathbb{P}} \frac{X}{2}, \implies Y_n \xrightarrow{d} \frac{X}{2} \neq X. \quad \square$$

Exercise 4. i) Let

$$Y_n := \min\{X_1, \dots, X_n\}.$$

We have

$$\{Y_n \leq y\} = \{\min(X_1, \dots, X_n) \leq y\} = \bigcap_{k=1}^n \{X_1 > y, \dots, X_{k-1} > y, X_k \leq y\}$$

so

$$\mathbb{P}(Y_n \leq y) = \sum_{k=1}^n \mathbb{P}(X_1 > y, \dots, X_{k-1} > y, X_k \leq y) = \sum_{k=1}^n \prod_{j=1}^{k-1} \mathbb{P}(X_j > y) \cdot \mathbb{P}(X_k \leq y),$$

with the agreement that $\prod_{j=1}^0 = 1$. Since $X_j \sim \exp(\lambda)$, we have

$$F_{X_j}(x) = \mathbb{P}(X_j \leq x) = (1 - e^{-\lambda x})1_{[0, +\infty[}(x),$$

for $y \geq 0$ we have

$$\begin{aligned} F_{Y_n}(y) = \mathbb{P}(Y_n \leq y) &= \sum_{k=1}^n (e^{-\lambda y})^{k-1} (1 - e^{-\lambda y}) = \sum_{k=0}^{n-1} (e^{-\lambda y})^k (1 - e^{-\lambda y}) \\ &= (1 - e^{-\lambda y}) \frac{1 - (e^{-\lambda y})^n}{1 - e^{-\lambda y}} = 1 - e^{-n\lambda y}. \end{aligned}$$

Clearly, $F_{Y_n}(y) = 0$ for $y \leq 0$.

ii) We notice that

$$\{Y_N \leq y\} = \bigcap_{n=1}^{\infty} \{Y_N \leq y, N = n\} = \bigcap_{n=1}^{\infty} \{Y_n \leq y, N = n\}.$$

Thus,

$$F_{Y_N}(y) = \mathbb{P}(Y_N \leq y) = \sum_{n=1}^{\infty} \mathbb{P}(Y_n \leq y, N = n).$$

Since N is independent of the (X_n) , N is independent of Y_n for every n , so we have

$$\begin{aligned} \mathbb{P}(Y_n \leq y, N = n) &= \mathbb{P}(Y_n \leq y, N = n) = \mathbb{P}(Y_n \leq y) \mathbb{P}(N = n) = (1 - p)^{n-1} p \mathbb{P}(Y_n \leq y) \\ &= p(1 - p)^{n-1} (1 - e^{-n\lambda y}). \end{aligned}$$

Therefore,

$$\begin{aligned} F_Y(y) &= \sum_{n=1}^{\infty} p(1 - p)^{n-1} (1 - e^{-n\lambda y}) = \underbrace{\sum_{n=1}^{\infty} p(1 - p)^{n-1}}_{=1} - p e^{-\lambda y} \sum_{n=1}^{\infty} \left((1 - p) e^{-\lambda y} \right)^{n-1} \\ &= 1 - p e^{-\lambda y} \frac{1}{1 - (1 - p) e^{-\lambda y}} = \frac{1 - e^{-\lambda y}}{1 - (1 - p) e^{-\lambda y}}. \end{aligned}$$

For $y < 0$ clearly $F_Y(y) = 0$.

iii) We have

$$\mathbb{E}[Y_N] = \int_0^{+\infty} y f_{Y_N}(y) dy,$$

where

$$f_{Y_N}(y) = \partial_y F_{Y_N}(y) = -\partial_y(1 - F_{Y_N}(y)).$$

Integrating by parts,

$$\mathbb{E}[Y_N] = \underbrace{[-y(1 - F_{Y_N}(y))]_{y=0}^{y=+\infty}}_{=0} + \int_0^{+\infty} 1 - F_{Y_N}(y) dy = \int_0^{+\infty} \frac{pe^{-\lambda y}}{1 - (1-p)e^{-\lambda y}} dy.$$

Setting $u = e^{-\lambda y}$ (that is $y = -\frac{1}{\lambda} \log u$) we obtain

$$\mathbb{E}[Y_N] = \frac{1}{\lambda} \int_0^1 \frac{pu}{1 - (1-p)u} \frac{du}{u} = -\frac{p}{\lambda(1-p)} [\log(1 - (1-p)u)]_{u=0}^{u=1} = -\frac{p \log p}{\lambda(1-p)}. \quad \square$$

Exercise 5. i) Notice that $(R, \Theta) = \Psi(U, V)$ where $\Psi(u, v) = (\sqrt{-2 \log u}, 2\pi v)$. Since $(U, V) \in [0, 1]^2$, and since $\mathbb{P}(U = 0) = 0$ and same for V , actually $(U, V) \in]0, 1]^2$ with probability 1, we consider $\Psi :]0, 1]^2 \rightarrow \Psi(]0, 1]^2) \subset [0, +\infty[\times]0, 2\pi]$. Ψ is invertible and $(r, \theta) = \Psi(u, v)$ iff $r = \sqrt{-2 \log u}$, $\theta = 2\pi v$, that is $u = e^{-\frac{r^2}{2}}$ and $v = \frac{\theta}{2\pi}$, so $\Psi^{-1}(r, \theta) = (e^{-\frac{r^2}{2}}, \theta/2\pi)$. According to the change of variable formula, we have

$$\begin{aligned} f_{R\Theta}(r, \theta) &= f_{UV}(\Psi^{-1}(r, \theta)) |\det(\Psi^{-1})'(r, \theta)| = 1_{[0,1]}(e^{-\frac{r^2}{2}}) 1_{[0,1]}(\frac{\theta}{2\pi}) \left| \det \begin{bmatrix} -re^{-\frac{r^2}{2}} & 0 \\ 0 & \frac{1}{2\pi} \end{bmatrix} \right| \\ &= |r|e^{-\frac{r^2}{2}} \cdot \frac{1}{2\pi} 1_{[0,2\pi]}(\theta) \end{aligned}$$

In particular,

$$f_R(r) = |r|e^{-\frac{r^2}{2}}, \quad f_\Theta(\theta) = \frac{1}{2\pi} 1_{[0,2\pi]}(\theta).$$

ii) $(X, Y) = \Psi(R, \Theta)$ where $\Psi(r, \theta) = (r \cos \theta, r \sin \theta)$ is the usual polar coordinate map. We have

$$f_{XY}(x, y) = f_{R\Theta}(\Psi^{-1}(x, y)) |\det(\Psi^{-1})'(x, y)|.$$

Since $\det \Psi'(\rho, \theta) = \rho$ and

$$\Psi^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \theta(x, y) \right), \quad \text{where } \theta(x, y) = \begin{cases} \arccos \frac{x}{\sqrt{x^2 + y^2}}, & y > 0, \\ \arccos \frac{x}{\sqrt{x^2 + y^2}} + \pi, & y < 0 \end{cases}$$

we deduce

$$f_{XY}(x, y) = |\sqrt{x^2 + y^2}| e^{-\frac{x^2 + y^2}{2}} \frac{1}{2\pi} \frac{1}{|\sqrt{x^2 + y^2}|} = \frac{1}{2\pi} e^{-\frac{x^2 + y^2}{2}}$$

from which $(X, Y) \sim \mathcal{N}(0, \mathbb{I})$ (where \mathbb{I} is the identity matrix). Clearly X, Y are independent, each distributed as a standard Gaussian. \square

Exercise 6. i) We recall that $X_n \xrightarrow{a.s.} 0$ iff

$$\forall \varepsilon > 0, \mathbb{P} \left(\bigcap_N \bigcup_{n \geq N} |X_n| \geq \varepsilon \right) = 0.$$

According to Borel–Cantelli’s Lemma, a sufficient condition for this happens is

$$\sum_n \mathbb{P}(|X_n| \geq \varepsilon) < +\infty.$$

We notice that,

$$X_n = \left(1 + \frac{Y_n}{2}\right) X_{n-1} = \left(1 + \frac{Y_n}{2}\right) \left(1 + \frac{Y_{n-1}}{2}\right) X_{n-2} = \dots = \prod_{k=0}^n \left(1 + \frac{Y_k}{2}\right) X_0 = \prod_{k=0}^n \left(1 + \frac{Y_k}{2}\right) \geq 0.$$

For convenience, let $\varepsilon = \frac{1}{2^m}$. So

$$\{|X_n| \geq \varepsilon\} = \{X_n \geq \varepsilon\} = \left\{ \prod_{k=0}^n \left(1 + \frac{Y_k}{2}\right) \geq \frac{1}{2^m} \right\}$$

Notice that, for $n > m$,

$$X_n = \frac{1}{2^n} \prod_{k=0}^n (2 + Y_k) \geq \frac{1}{2^m}, \iff \prod_{k=0}^n (2 + Y_k) \geq 2^{n-m},$$

and this happens iff at least $n - m$ of the $Y_k = 1$. Because of independence

$$\mathbb{P}(Y_{k_1} = 1, \dots, Y_{k_{n-m}} = 1) = \frac{1}{2^{n-m}},$$

so

$$\mathbb{P} \left(X_n \geq \frac{1}{2^m} \right) \leq \binom{n}{n-m} \frac{1}{2^{n-m}} = \frac{n!}{(n-m)! m!} \frac{1}{2^n} 2^m$$

and

$$\sum_n \mathbb{P} \left(X_n \geq \frac{1}{2^m} \right) \leq \frac{2^m}{m!} \sum_{n \geq m} \frac{n!}{2^n (n-m)!}.$$

Now the last series converges by root test because, if $a_n := \frac{n!}{2^n (n-m)!}$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n+1} (n+1-m)!} \frac{2^n (n-m)!}{n!} = \frac{1}{2} \frac{n+1}{n+1-m} \rightarrow \frac{1}{2} < 1.$$

Thus, we conclude that

$$\sum_n \mathbb{P} \left(X_n \geq \frac{1}{2^m} \right) < +\infty,$$

and Borel–Cantelli’s Lemma applies.

ii) Since

$$X_n = \prod_{k=0}^n \left(1 + \frac{Y_k}{2}\right),$$

by the independence of Y_k we have

$$\mathbb{E}[X_n] = \prod_{k=0}^n \mathbb{E} \left[1 + \frac{Y_k}{2} \right] = \prod_{k=0}^n \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{2} \right) = \prod_{k=0}^n 1 = 1,$$

so $\lim_n \mathbb{E}[X_n] = 1$. We conclude that $X_n \not\rightarrow 0$ in L^1 . Indeed, if this would happen,

$$1 = |\mathbb{E}[X_n]| \leq \mathbb{E}[|X_n|] = \|X_n\|_1 \rightarrow 0,$$

which is impossible. □

Exercise 7. i) Recalling of the monotonicity property of the conditional expectation, we have

$$-|X| \leq X \leq |X|, \text{ a.s.} \implies -\mathbb{E}[|X| \mid \mathcal{G}] \leq \mathbb{E}[X \mid \mathcal{G}] \leq \mathbb{E}[|X| \mid \mathcal{G}],$$

from which

$$|\mathbb{E}[X \mid \mathcal{G}]| \leq \mathbb{E}[|X| \mid \mathcal{G}].$$

ii) We notice that

$$\mathbb{E}[X + Y \mid Y] = \mathbb{E}[X \mid Y] + \mathbb{E}[Y \mid Y] = Y.$$

So,

$$|Y| = |\mathbb{E}[X + Y \mid Y]| \leq \mathbb{E}[|X + Y| \mid Y], \text{ a.s.},$$

and, by taking expectations,

$$\mathbb{E}[|Y|] \leq \mathbb{E}[\mathbb{E}[|X + Y| \mid Y]] = \mathbb{E}[|X + Y|],$$

which is the conclusion.

iii) We notice that if $A := \mathbb{E}[X - Y \mid X + Y]$ then A is characterized by

$$\begin{aligned} \mathbb{E}[A\varphi(X + Y)] &= \mathbb{E}[(X - Y)\varphi(X + Y)] = \int_{\mathbb{R}^2} (x - y)\varphi(x + y) d\mu_{XY}(x, y) \\ &= \int_{\mathbb{R}^2} (x - y)\varphi(x + y) d\mu_{YX}(x, y) = \mathbb{E}[(Y - X)\varphi(Y + X)] \\ &= -\mathbb{E}[A\varphi(X + Y)], \end{aligned}$$

from which

$$2\mathbb{E}[A\varphi(X + Y)] = 0, \forall \varphi(X + Y) \in L_{X+Y}^\infty.$$

In particular, since $A \in L_{X+Y}^1$ we conclude that $A = 0$ a.s., as desired.

Now, setting $Z = X + Y$ and W in such a way that $3X - Y = Z + W$, thus $W = (3X - Y) - (X + Y) = 2(X - Y)$, being $\mathbb{E}[Z \mid W] = \mathbb{E}[X + Y \mid X - Y] = 0$, by ii) we get

$$\|W\|_1 \leq \|Z + W\|_1, \iff 2\|X - Y\|_1 \leq \|3X - Y\|_1,$$

from which the conclusion follows. □

Exercise 8. Let $\tilde{X}_k := \frac{k}{k+1}X_k \sim U[-1, 1]$. Then

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n \left(1 + \frac{1}{k} \right) \tilde{X}_k.$$

So

$$\phi_{Y_n}(\xi) = \mathbb{E} \left[e^{i \frac{\xi}{\sqrt{n}} \sum_{k=1}^n \left(1 + \frac{1}{k}\right) \tilde{X}_k} \right] \stackrel{\text{indep}}{=} \prod_{k=1}^n \phi \left(\frac{\xi}{\sqrt{n}} \left(1 + \frac{1}{k}\right) \right),$$

where

$$\phi(\eta) = \phi_{\tilde{X}}(\eta) = \frac{\sin \eta}{\eta}.$$

Now, since for ξ fixed $\frac{\xi}{\sqrt{n}} \left(1 + \frac{1}{k}\right) \rightarrow 0$, recalling that

$$\phi(\eta) = \phi(0) + \phi'(0)\eta + \frac{1}{2}\phi''(0)\eta^2 + o(\eta^2),$$

and being,

$$\phi(\eta) = \frac{\sin \eta}{\eta},$$

$$\phi(0) = 1,$$

$$\phi'(\eta) = \frac{\eta \cos \eta - \sin \eta}{\eta^2},$$

$$\phi'(0) = \lim_{\eta \rightarrow 0} \phi'(\eta) \stackrel{H}{=} \lim_{\eta \rightarrow 0} \frac{\cos \eta - \eta \sin \eta - \cos \eta}{2\eta} = 0,$$

$$\phi''(\eta) = \frac{-\eta^2 \sin \eta - 2(\eta \cos \eta - \sin \eta)}{\eta^3} \quad \phi''(0) = \lim_{\eta \rightarrow 0} \frac{-\eta^2(\eta + o(\eta)) - 2\left(\eta\left(1 - \frac{\eta^2}{2} + o(\eta^2)\right) - \left(\eta - \frac{\eta^3}{6} + o(\eta^3)\right)\right)}{\eta^3} = -\frac{1}{3}$$

we have

$$\phi(\eta) = 1 - \frac{\eta^2}{6} + o(\eta^2).$$

Therefore,

$$\phi_{Y_n}(\xi) = \prod_{k=1}^n \left(1 - \frac{1}{6} \frac{\xi^2}{n} \left(1 + \frac{1}{k}\right)^2 + o\left(\frac{1}{n}\right) \right) \sim \left(1 - \frac{1}{6} \frac{\xi^2}{n} \right)^n \rightarrow e^{-\frac{1}{6} \xi^2}$$

Therefore

$$Y_n \xrightarrow{d} \mathcal{N}(0, 1/3).$$

Alternative solution. We may notice that

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n \left(1 + \frac{1}{k}\right) \tilde{X}_k = \frac{1}{\sqrt{n}} \sum_{k=1}^n \tilde{X}_k + \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{1}{k} \tilde{X}_k.$$

Now, by the CLT

$$W_n := \frac{1}{\sqrt{n}} \sum_{k=1}^n \tilde{X}_k \xrightarrow{d} \mathcal{N}(0, 1/3).$$

We claim that

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{1}{k} \tilde{X}_k \xrightarrow{a.s.} 0.$$

If this is true we have $Y_n = W_n + Z_n \xrightarrow{d} \mathcal{N}(0, 1/3)$. Indeed

$$\phi_{Y_n}(\xi) = \mathbb{E}[e^{i\xi W_n} e^{i\xi Z_n}] = \mathbb{E} \left[\left(e^{i\xi Z_n} - 1 \right) e^{i\xi W_n} \right] + \mathbb{E}[e^{i\xi W_n}].$$

By dominated convergence the first expectation goes to 0 while the second goes to $e^{-\frac{1}{6}\xi^2}$.

To prove that $Z_n \xrightarrow{a.s.} 0$ we notice that, being $|\widetilde{X}_n| \leq 1$ with probability 1,

$$|Z_n| \leq \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{1}{n} \leq \frac{1}{\sqrt{n}} \left(1 + \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x} dx \right) = \frac{1}{\sqrt{n}} (1 + \log n) \longrightarrow 0. \quad \square$$

Exercise 9. i) We have

$$\phi_{B_t}(\xi) = \mathbb{E}[e^{i\xi t W_{1/t}}] = e^{-\frac{1}{2} \frac{1}{t} (t\xi)^2} = e^{-\frac{1}{2} t \xi^2} \implies B_t \sim \mathcal{N}(0, t).$$

If $0 < s < t$, then $0 < \frac{1}{t} < \frac{1}{s}$ and

$$\begin{aligned} (B_s, B_t - B_s) &= (sW_{1/s}, tW_{1/t} - sW_{1/s}) = (s(W_{1/s} - W_{1/t}) + sW_{1/t}, (t-s)W_{1/t} - s(W_{1/s} - W_{1/t})) \\ &= s(1, -1)(W_{1/s} - W_{1/t}) + (s, t-s)W_{1/t}. \end{aligned}$$

Being $W_{1/t}$ independent of $W_{1/s} - W_{1/t}$ we have that

$$\begin{aligned} \mathbb{E} \left[e^{i(\xi, \eta)(B_s, B_t - B_s)} \right] &= \mathbb{E} \left[e^{is(\xi - \eta)(W_{1/s} - W_{1/t})} e^{i(s\xi + (t-s)\eta)W_{1/t}} \right] \\ &= \mathbb{E} \left[e^{is(\xi - \eta)(W_{1/s} - W_{1/t})} \right] \mathbb{E} \left[e^{i(s\xi + (t-s)\eta)W_{1/t}} \right] \\ &= e^{-\frac{1}{2} \left(\frac{1}{s} - \frac{1}{t} \right) s^2 (\xi - \eta)^2} e^{-\frac{1}{2} \frac{1}{t} (t\xi + (t-s)\eta)^2} \\ &= e^{-\frac{1}{2} \frac{t-s}{t} s (\xi^2 + \eta^2 - 2\xi\eta)} e^{-\frac{1}{2} \frac{1}{t} (t^2 \eta^2 + s^2 (\xi^2 + \eta^2 - 2\xi\eta) + 2st\eta(\xi - \eta))} \\ &= e^{-\frac{1}{2} [s(\xi^2 + \eta^2 - 2\xi\eta) + t\eta^2 + 2s\eta(\xi - \eta)]} \\ &= e^{-\frac{1}{2} (s\xi^2 + (t-s)\eta^2)} = e^{-\frac{1}{2} s \xi^2} e^{-\frac{1}{2} (t-s)\eta^2}, \end{aligned}$$

that, at once, says $B_s \sim \mathcal{N}(0, s)$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$ and they are independent. Finally, $B_t \in \mathcal{C}(]0, +\infty[)$ being $W_t \in \mathcal{C}([0, +\infty[)$.

ii) We have

$$\begin{aligned} \mathbb{P}(|B_t| \geq \varepsilon) &= \mathbb{P}(|tW_{1/t}| \geq \varepsilon) = \mathbb{P}(|W_{1/t}| \geq \frac{\varepsilon}{t}) = 2 \cdot \int_{\varepsilon/t}^{+\infty} e^{-\frac{x^2}{2(1/t)}} \frac{dx}{\sqrt{2\pi \frac{1}{t}}} \\ &\stackrel{y=\sqrt{t}x}{=} 2 \int_{\varepsilon/\sqrt{t}}^{+\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \longrightarrow 0, \quad t \rightarrow 0+. \quad \square \end{aligned}$$

Exercise 10. i) Notice that $Z, W \in \mathbb{N}$. So, let $z, w \in \mathbb{N}$ and let's compute

$$\mathbb{P}(Z = z, W = w) = \mathbb{P}(\min(X, Y) = z, |X - Y| = w).$$

We distinguish $w = 0$ from $w \geq 1$. If $w = 0$,

$$\mathbb{P}(Z = z, W = 0) = \mathbb{P}(\min(X, Y) = z, |X - Y| = 0) = \mathbb{P}(X = Y, X = z) = \mathbb{P}(X = z, Y = z)$$

$$\stackrel{\text{indep}}{=} \mathbb{P}(X = z)\mathbb{P}(Y = z) = ((1 - p)^{z-1}p)^2 = (1 - p)^{2z-2}p^2.$$

If $w \geq 1$, noticed that $|X - Y| = w$ iff $X = Y \pm w$ we have

$$\begin{aligned} \mathbb{P}(Z = z, W = w) &= \mathbb{P}(\min(X, Y) = z, X = Y \pm w) \\ &= \mathbb{P}(X = Y - w, X = z) + \mathbb{P}(X = Y + w, Y = z) \\ &= \mathbb{P}(X = z, Y = z + w) + \mathbb{P}(Y = z, X = z + w) \\ &\stackrel{\text{indep}}{=} \mathbb{P}(X = z)\mathbb{P}(Y = z + w) + \mathbb{P}(Y = z)\mathbb{P}(X = z + w) \\ &= 2(1 - p)^{z-1}p(1 - p)^{z+w-1}p \\ &= 2(1 - p)^{2z+w-2}p^2. \end{aligned}$$

ii) Z and W are independent iff

$$\mathbb{P}(Z = z, W = w) = \mathbb{P}(Z = z)\mathbb{P}(W = w).$$

From previous calculation we have

$$\begin{aligned} \mathbb{P}(Z = z) &= \mathbb{P}\left(\bigsqcup_w Z = z, W = w\right) = \sum_{w=0}^{\infty} \mathbb{P}(Z = z, W = w) \\ &= (1 - p)^{2z-2}p^2 + \sum_{w=1}^{\infty} 2(1 - p)^{2z+w-2}p^2 = p^2(1 - p)^{2z-2} \left(1 + 2 \sum_{w=1}^{\infty} (1 - p)^w\right) \\ &= p^2(1 - p)^{2z-2} \left(1 + 2 \left(\frac{1}{1 - (1 - p)} - 1\right)\right) \\ &= p^2(1 - p)^{2z-2} \frac{p + 2(1 - p)}{p} = p(1 - p)^{2z-2}(2 - p). \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{P}(W = w) &= \mathbb{P}\left(\bigsqcup_{z \geq 1} Z = z, W = w\right) = \sum_{z=1}^{\infty} \mathbb{P}(Z = z, W = w) \\ &= \begin{cases} w = 0, & \sum_{z=1}^{\infty} (1 - p)^{2z-2}p^2 = p^2 \frac{1}{1 - (1 - p)^2} = \frac{p}{2 - p}, \\ w \geq 1 & 2p^2(1 - p)^w \sum_{z=1}^{\infty} (1 - p)^{2z-2} = 2(1 - p)^w \frac{p}{2 - p} \end{cases} \end{aligned}$$

So

$$\begin{aligned} \mathbb{P}(Z = z)\mathbb{P}(W = w) &= \begin{cases} w = 0, & p(1-p)^{2z-2}(2-p) \cdot \frac{p}{2-p} = p^2(1-p)^{2z-2}, \\ w \geq 1, & p(1-p)^{2z-2}(2-p) \cdot 2(1-p)^w \frac{p}{2-p} = 2p^2(1-p)^{2z+w-2} \end{cases} \\ &= \mathbb{P}(Z = z, W = w). \end{aligned}$$

We conclude that Z and W are independent. □

Exercise 11. See notes for Borel–Cantelli’s Lemma.

i) We prove that

$$\mathbb{P}\left(\bigcap_N \bigcup_n \{Y_n \geq 1 + \varepsilon\}\right) = 0.$$

To this aim we apply first Borel–Cantelli’s Lemma proving that

$$\sum_n \mathbb{P}(Y_n \geq 1 + \varepsilon) < +\infty.$$

We have

$$\mathbb{P}(Y_n \geq 1 + \varepsilon) = \mathbb{P}\left(X_n \geq \log n + (1 + \varepsilon) \log(\log n) = \log\left(n(\log n)^{1+\varepsilon}\right)\right).$$

Being $X_n \sim \exp(1)$, we have

$$\mathbb{P}(X_n \geq \alpha) = e^{-\alpha},$$

so

$$\mathbb{P}(Y_n \geq 1 + \varepsilon) = e^{-\log(n(\log n)^{1+\varepsilon})} = \frac{1}{n(\log n)^{1+\varepsilon}},$$

from which

$$\sum_n \mathbb{P}(Y_n \geq 1 + \varepsilon) = \sum_n \frac{1}{n(\log n)^{1+\varepsilon}} < +\infty.$$

ii) According to second Borel–Cantelli’s Lemma, if the events E_n are independent and

$$\sum_n \mathbb{P}(E_n) = +\infty, \implies \mathbb{P}\left(\bigcap_N \bigcup_n E_n\right) = 1.$$

So, since the Y_n are independent, to prove that

$$\mathbb{P}\left(\bigcap_N \bigcup_n \{Y_n \geq 1 - \varepsilon\}\right) = 1$$

we just need to verify that

$$\sum_n \mathbb{P}(\{Y_n \geq 1 - \varepsilon\}) = +\infty.$$

By a calculation similar to that of i),

$$\mathbb{P}(Y_n \geq 1 - \varepsilon) = \frac{1}{n(\log n)^{1-\varepsilon}}, \implies \sum_n \mathbb{P}(Y_n \geq 1 - \varepsilon) = \sum_n \frac{1}{n(\log n)^{1-\varepsilon}} = +\infty,$$

and the conclusion follows. □

Exercise 13. i) We have

$$\lambda_2(A) = \int_{|x-y|<a, |x+y|<b} dx dy \stackrel{u=x-y, v=x+y}{=} \frac{1}{2} \int_{|u|<a, |v|<b} dudv = 2ab,$$

so

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f_{X,Y}(x, y) dx = \frac{1}{2ab} \int_{\mathbb{R}} 1_{y-a \leq x \leq y+a, -y-b \leq x \leq -y+b}(x) dx \\ &= \frac{1}{2ab} \int_{\mathbb{R}} 1_{\max(y-a, -y-b) \leq x \leq \min(y+a, -y+b)} dx = \frac{M(y) - m(y)}{2ab} 1_{m(y) \leq M(y)} \end{aligned}$$

where

$$m(y) = \max(y-a, -y-b) = \begin{cases} -y-b, & y < -\frac{b-a}{2}, \\ y-a, & y > -\frac{b-a}{2}. \end{cases} \quad M(y) = \min(y+a, -y+b) = \begin{cases} y+a, & y < \frac{b-a}{2}, \\ -y+b, & y > \frac{b-a}{2}. \end{cases}$$

Returning to $f_Y(y)$ we have

$$f_Y(y) = \begin{cases} y < -\frac{b-a}{2}, & = \frac{1}{2ab} \int_{\mathbb{R}} 1_{[-y-b, y+a]}(x) dx, \\ -\frac{b-a}{2} \leq y < \frac{b-a}{2}, & = \frac{1}{2ab} \int_{\mathbb{R}} 1_{[y-a, y+a]}(x) dx = \frac{1}{b}, \\ y \geq \frac{b-a}{2}, & = \frac{1}{2ab} \int_{\mathbb{R}} 1_{[y-a, -y+b]}(x) dx. \end{cases}$$

Now, $-y-b < y+a$ iff $y > -\frac{a+b}{2}$, and since $-\frac{a+b}{2} < -\frac{b-a}{2}$ we have

$$f_Y(y) = \begin{cases} 0, & y < -\frac{a+b}{2}, \\ \frac{1}{2ab}(2y+a+b) = \frac{1}{ab} \left(y + \frac{a+b}{2} \right), & -\frac{a+b}{2} \leq y < -\frac{b-a}{2}. \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} 0, & y > \frac{a+b}{2}, \\ \frac{1}{ab} \left(\frac{a+b}{2} - y \right), & \frac{b-a}{2} \leq y < \frac{b+a}{2}. \end{cases}$$

We conclude that

$$f_Y(y) = \begin{cases} 0, & y < -\frac{a+b}{2}, \\ \frac{1}{2ab}(2y+a+b) = \frac{1}{ab} \left(y + \frac{a+b}{2} \right), & -\frac{a+b}{2} \leq y < -\frac{b-a}{2}, \\ \frac{1}{b}, & -\frac{b-a}{2} \leq y < \frac{b-a}{2}, \\ \frac{1}{ab} \left(\frac{a+b}{2} - y \right), & \frac{b-a}{2} \leq y < \frac{b+a}{2}, \\ 0, & y > \frac{a+b}{2}. \end{cases}$$

ii) We have

$$\mathbb{E}[X | Y = y] = 0, \quad |y| > \frac{a+b}{2},$$

while, for $|y| \leq \frac{a+b}{2}$,

$$\mathbb{E}[X | Y = y] = \int_{\mathbb{R}} x f_{X|Y}(x | y) dx = \frac{1}{f_Y(y)} \frac{1}{2ab} \int_{\mathbb{R}} x 1_{\max(y-a, -y-b) \leq x \leq \min(y+a, -y+b)} dx.$$

Denoting by $m(y) := \max(y-a, -y-b)$ and $M(y) := \min(y+a, -y+b)$, we have

$$\frac{1}{M(y) - m(y)} \int_{\mathbb{R}} x 1_{\max(y-a, -y-b) \leq x \leq \min(y+a, -y+b)} dx = \frac{m(y) + M(y)}{2},$$

and since $f_Y(y) = \frac{1}{2ab}(M(y) - m(y))$, we have

$$\mathbb{E}[X | Y = y] = \frac{m(y) + M(y)}{2}.$$

Here we have the following cases:

- case $a = b$. Then $m(y) = \max(y-a, -y-a) = |y| - a$, $M(y) = \min(y+a, -y+a) = -|y| + a$ so

$$\mathbb{E}[X | Y = y] = 0, \quad |y| \leq a, \implies \mathbb{E}[X | Y = y] \equiv 0.$$

- case $0 \leq a < b$: if we have

$$\mathbb{E}[X | Y = y] = \begin{cases} \frac{-y-b+y+a}{2} = -\frac{b-a}{2}, & -\frac{a+b}{2} \leq y \leq -\frac{b-a}{2}, \\ \frac{y-a+y+a}{2} = y & -\frac{b-a}{2} \leq y \leq \frac{b-a}{2}, \\ \frac{y-a+(-y+b)}{2} = \frac{b-a}{2}, & \frac{b-a}{2} \leq y \leq \frac{b+a}{2} \end{cases}$$

- case $0 \leq b < a$: if we have

$$\mathbb{E}[X | Y = y] = \begin{cases} \frac{y-a+(-y+b)}{2} = \frac{b-a}{2}, & -\frac{a+b}{2} \leq y \leq -\frac{b-a}{2}, \\ \frac{-y-b+(-y+b)}{2} = -y & -\frac{b-a}{2} \leq y \leq \frac{b-a}{2}, \\ \frac{-y-b+(y+a)}{2} = -\frac{b-a}{2}, & \frac{b-a}{2} \leq y \leq \frac{b+a}{2} \end{cases}$$

iii) Since in all cases take constant values with positive probability, $\mathbb{E}[X | Y]$ is not absolutely continuous, so there is no density for it. \square

Exercise 15. i) Let $X \geq 0$ natural valued. By monotone convergence,

$$\sum_{k=0}^{\infty} \mathbb{P}(X > k) = \sum_{k=0}^{\infty} \mathbb{E}[1_{X>k}] = \mathbb{E} \left[\sum_{k=0}^{\infty} 1_{X>k} \right].$$

Now, assume $X(\Omega) = n$. Then

$$\sum_{k=0}^{\infty} 1_{X(\omega) > k} = \sum_{k=0}^{n-1} 1 = n = X(\omega),$$

From this the conclusion follows.

ii) We have

$$\int_0^{+\infty} \mathbb{P}(X > t) dt = \int_0^{+\infty} \mathbb{E}[1_{X>t}] dt = \mathbb{E} \left[\int_0^{+\infty} \underbrace{1_{X>t}}_{=1_{[0, X](t)}} dt \right] = \mathbb{E} \left[\int_0^X dt \right] = \mathbb{E}[X]. \quad \square$$

Exercise 16. i) To be a true covariance matrix C must be symmetric and positive definite. Clearly, C is symmetric. To be positive definite we need that

$$C \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} > 0, \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0.$$

A quick condition for positivity, if you remind, is that all the $k \times k$ sub-determinants on the diagonal are positive. These are 1 and $1 - \rho^2$, so the condition is $1 - \rho^2 > 0$, that is $-1 < \rho < 1$.

Equivalently,

$$C \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \rho y \\ \rho x + y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (x + \rho y)x + (\rho x + y)y = x^2 + y^2 + 2\rho xy$$

If $x = \pm y$ we have $x^2 + x^2 \pm 2\rho x^2 = 2x^2(1 \pm \rho) > 0$ iff $1 \pm \rho > 0$, that is $-1 < \rho < 1$. For such ρ ,

$$x^2 + y^2 + 2\rho xy > x^2 + y^2 - 2|x||y| = (|x| + |y|)^2 > 0, \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0.$$

ii) We know that $(X, Y) = (R \cos \Theta, R \sin \Theta) = \Phi(R, \Theta)$, so

$$f_{R, \Theta}(r, \theta) = f_{X, Y}(\Phi(r, \theta)) |\det \Phi'(r, \theta)|$$

where

$$\begin{aligned} f_{X, Y}(x, y) &= \frac{1}{\sqrt{(2\pi)^2 \det C}} e^{-\frac{1}{2} C^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}} = \frac{1}{\sqrt{(2\pi)^2 (1 - \rho^2)}} e^{-\frac{1}{2} \frac{1}{1 - \rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}} \\ &= \frac{1}{\sqrt{(2\pi)^2 (1 - \rho^2)}} e^{-\frac{1}{2(1 - \rho^2)} (x^2 + y^2 - 2\rho xy)}. \end{aligned}$$

Thus,

$$f_{R, \Theta}(r, \theta) = \frac{1}{\sqrt{(2\pi)^2 (1 - \rho^2)}} e^{-\frac{r^2}{2(1 - \rho^2)} (1 - 2\rho \cos \theta \sin \theta)} \cdot r 1_{[0, +\infty[}(r) 1_{[0, 2\pi]}(\theta).$$

For the distribution of Θ we have

$$\begin{aligned} f_{\Theta}(\theta) &= \int_{\mathbb{R}} f_{R,\Theta}(r, \theta) dr = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{+\infty} r e^{-\frac{1-\rho \sin(2\theta)}{2(1-\rho^2)} r^2} dr \cdot 1_{[0,2\pi]}(\theta) \\ &= -\frac{1}{2\pi\sqrt{1-\rho^2}} \frac{2(1-\rho^2)}{1-\rho \sin(2\theta)} \left[e^{-\frac{1-\rho \sin(2\theta)}{2(1-\rho^2)} r^2} \right]_{r=0}^{r=+\infty} 1_{[0,2\pi]}(\theta) \\ &= \frac{1}{2\pi} \frac{\sqrt{1-\rho^2}}{1-\rho \sin(2\theta)} 1_{[0,2\pi]}(\theta) \end{aligned}$$

iii) R and Θ are independent iff $f_{R,\Theta} = f_R f_{\Theta}$. It is evident that this can happen iff $\rho = 0$, that is iff X and Y are independent. In this case we have

$$f_{\Theta}(\theta) = \frac{1}{2\pi} 1_{[0,2\pi]}(\theta),$$

and

$$f_R(r) = \frac{f_{R,\Theta}(r, \theta)}{f_{\Theta}(\theta)} = r e^{-\frac{r^2}{2}} 1_{[0,+\infty]}(r).$$

iv) We have

$$\begin{aligned} \mathbb{E}[R \mid \Theta = \theta] &= \int_{\mathbb{R}} r f_{R|\Theta}(r|\theta) dr = \frac{2\pi(1-\rho \sin(2\theta))}{\sqrt{1-\rho^2}} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{+\infty} r^2 e^{-\frac{1-\rho \sin(2\theta)}{2(1-\rho^2)} r^2} dr \\ &= \sqrt{\frac{1-\rho^2}{1-\rho \sin(2\theta)}} \int_0^{+\infty} u^2 e^{-\frac{u^2}{2}} du = \sqrt{\frac{\pi(1-\rho^2)}{2(1-\rho \sin(2\theta))}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}[R^2 \mid \Theta = \theta] &= \int_{\mathbb{R}} r^2 f_{R|\Theta}(r|\theta) dr = \frac{1-\rho \sin(2\theta)}{1-\rho^2} \int_0^{+\infty} r^3 e^{-\frac{1-\rho \sin(2\theta)}{2(1-\rho^2)} r^2} dr \\ &= \frac{1-\rho^2}{1-\rho \sin(2\theta)} \int_0^{+\infty} u^3 e^{-\frac{u^2}{2}} du. \end{aligned}$$

Noticed that

$$\int_0^{+\infty} u^3 e^{-\frac{u^2}{2}} du = \left[u^2 (-e^{-\frac{u^2}{2}}) \right]_{r=0}^{r=+\infty} + \int_0^{+\infty} 2ue^{-\frac{u^2}{2}} du = 2$$

we deduce

$$\mathbb{R}[R^2 \mid \Theta = \theta] = 2 \frac{1-\rho^2}{1-\rho \sin(2\theta)}. \quad \square$$

Exercise 17. i) We have to show that

$$\mathbb{P}\left(\frac{|X_n|}{\log n} \leq 1 + \varepsilon \text{ for all but finitely many } n\right) = \mathbb{P}\left(\bigcup_N \bigcap_{n \geq N} \frac{|X_n|}{\log n} \leq 1 + \varepsilon\right) = 1,$$

that is, taking the complementaries,

$$\mathbb{P}\left(\bigcap_N \bigcup_{n \geq N} \left\{ \frac{|X_n|}{\log n} \geq 1 + \varepsilon \right\}\right) = 0.$$

We apply first Borel-Cantelli's Lemma. Notice first that

$$\mathbb{P}\left(\frac{|X_n|}{\log n} \geq 1 + \varepsilon\right) = \mathbb{P}(|X_n| \geq (1 + \varepsilon) \log n).$$

Now, for $\alpha > 0$,

$$\mathbb{P}(|X_n| \geq \alpha) = \int_{]-\infty, -\alpha] \cup [\alpha, +\infty[} \frac{1}{2} e^{-|x|} dx = \int_{\alpha}^{+\infty} e^{-x} dx = e^{-\alpha},$$

so

$$\mathbb{P}(|X_n| \geq (1 + \varepsilon) \log n) = e^{-(1+\varepsilon) \log n} = \frac{1}{n^{1+\varepsilon}}.$$

Therefore

$$\sum_n \mathbb{P}\left(\frac{|X_n|}{\log n} \geq 1 + \varepsilon\right) = \sum_n \frac{1}{n^{1+\varepsilon}} < +\infty, \forall \varepsilon > 0,$$

so the conclusion follows.

ii) We now have to prove that

$$\mathbb{P}\left(\frac{|X_n|}{\log n} \geq 1 - \varepsilon \text{ for infinitely many } n\right) = \mathbb{P}\left(\bigcap_N \bigcup_{n \geq N} \left\{ \frac{|X_n|}{\log n} \geq 1 - \varepsilon \right\}\right) = 1.$$

Here we can apply the second Borel-Cantelli Lemma: since the X_n are independent, the conclusion will follow once we prove

$$\sum_n \mathbb{P}\left(\left\{ \frac{|X_n|}{\log n} \geq 1 - \varepsilon \right\}\right) = +\infty.$$

As above,

$$\mathbb{P}\left(\left\{ \frac{|X_n|}{\log n} \geq 1 - \varepsilon \right\}\right) = \mathbb{P}(|X_n| \geq (1 - \varepsilon) \log n) = e^{-(1-\varepsilon) \log n} = \frac{1}{n^{1-\varepsilon}},$$

so

$$\sum_n \mathbb{P}\left(\left\{ \frac{|X_n|}{\log n} \geq 1 - \varepsilon \right\}\right) = \sum_n \frac{1}{n^{1-\varepsilon}} = +\infty. \quad \square$$

Exercise 19. In L^2 , conditional expectation is the orthogonal projection ΠX . In particular $X - \Pi X \perp \Pi X$ so, by Pythagorean's theorem,

$$\|X\|_2^2 = \|\Pi X\|_2^2 + \|X - \Pi X\|_2^2 \geq \|\Pi X\|_2^2,$$

that is

$$\|\mathbb{E}[X | \mathcal{G}]\|_2^2 \leq \|X\|_2^2,$$

from which the conclusion follows.

i) By the Pythagorean theorem,

$$\|X\|_2^2 = \|\mathbb{E}[X | Y]\|_2^2 + \|X - \mathbb{E}[X | Y]\|_2^2 = \|Y\|_2^2 + \|X - Y\|_2^2, \implies \|X - Y\|_2^2 = \|X\|_2^2 - \|Y\|_2^2.$$

For the same argument, from $\mathbb{E}[Y | X] = X$ we have

$$\|Y - X\|_2^2 = \|Y\|_2^2 - \|X\|_2^2,$$

and summing these identities we get

$$2\|X - Y\|_2^2 = 0, \implies X = Y \text{ a.s.}$$

ii) Arguing as in i) we get

$$\left\{ \begin{array}{l} \|X - Y\|_2^2 = \|X\|_2^2 - \|Y\|_2^2, \\ \|Y - Z\|_2^2 = \|Y\|_2^2 - \|Z\|_2^2, \\ \|Z - X\|_2^2 = \|Z\|_2^2 - \|X\|_2^2, \end{array} \right. \implies \|X - Y\|_2^2 + \|Y - Z\|_2^2 + \|Z - X\|_2^2 = 0, \implies X = Y = Z \text{ a.s.} \quad \square$$

Exercise 20. i) Clearly F is well defined, increasing, $F(-\infty) = e^{-(+\infty)} = 0$ and $F(+\infty) = e^{-0} = 1$. Moreover $F \in \mathcal{C}(\mathbb{R})$, so F is a cdf.

ii) We have

$$\begin{aligned} F_{Y_n}(y) &= \mathbb{P}(Y_n \leq y) = \mathbb{P}(\max\{X_1, \dots, X_n\} \leq y) \\ &= \mathbb{P}(X_1 \leq y, \dots, X_n \leq y) = \prod_{k=1}^n \mathbb{P}(X_k \leq y) \\ &= \begin{cases} 0, & y < 0, \\ (1 - e^{-y})^n, & y > 0. \end{cases} \end{aligned}$$

iii) Let $Z_n := Y_n - \log n$. Then

$$F_{Z_n}(z) = \mathbb{P}(Z_n \leq z) = \mathbb{P}(Y_n \leq z + \log n) = F_{Y_n}(z + \log n).$$

For $z \in \mathbb{R}$ fixed and n large enough, $z + \log n > 0$ so

$$F_{Z_n}(z) = \left(1 - e^{-(z+\log n)}\right)^n = \left(1 - \frac{e^{-z}}{n}\right)^n \longrightarrow e^{-e^{-z}} = F(z), \quad \forall z \in \mathbb{R}.$$

Since F is a continuous cdf, we conclude that $Z_n \xrightarrow{d} Z$ where $F_Z = F$. □

Exercise 21. We notice that

$$\begin{aligned}
 \mathbb{P}(X+Y \in E) &= \int_{\mathbb{R}^2} 1_E(x+y) f_{X,Y}(x,y) \, dx dy = \int_{\mathbb{R}^2} 1_E(x+y) f_X(x) f_Y(y) \, dx dy \\
 &\stackrel{u=x+y, v=x}{=} \int_{\mathbb{R}^2} 1_E(u) f_X(v) f_Y(u-v) \, dudv \\
 &= \int_E \left(\int_{\mathbb{R}} f_X(v) f_Y(u-v) \, dv \right) du \\
 &= \int_E f_X * f_Y(u) \, du.
 \end{aligned}$$

This says that $f_{X+Y}(u) = f_X * f_Y(u)$.

i) By induction,

$$f_{X_1+\dots+X_n} = f_{X_1} * \dots * f_{X_n}.$$

Applying the FT,

$$\widehat{f_{X_1+\dots+X_n}}(\xi) = \widehat{f_{X_1}}(\xi) \cdots \widehat{f_{X_n}}(\xi).$$

Notice that

$$\begin{aligned}
 \widehat{f_{X_k}}(\xi) &= \int_{\mathbb{R}} f_{X_k}(x) e^{-i\xi x} \, dx = \int_0^{+\infty} \lambda e^{-\lambda x} e^{-i\xi x} \, dx = \lambda \int_0^{+\infty} e^{-(\lambda+i\xi)x} \, dx \\
 &= \lambda \left[\frac{e^{-(\lambda+i\xi)x}}{-(\lambda+i\xi)} \right]_{x=0}^{x=+\infty} = \lambda \left(0 - \frac{1}{-(\lambda+i\xi)} \right) = \frac{\lambda}{\lambda+i\xi}.
 \end{aligned}$$

So,

$$\widehat{f_{X_1+\dots+X_n}}(\xi) = \left(\frac{\lambda}{\lambda+i\xi} \right)^n = \frac{\lambda^n}{(-i)^{n-1} (n-1)!} \partial_{\xi}^{n-1} (\lambda+i\xi)^{-1}.$$

Applying the FT,

$$\begin{aligned}
 2\pi f_{X_1+\dots+X_n}(-x) &= \widehat{\widehat{f_{X_1+\dots+X_n}}}(x) = \frac{\lambda^n}{(-i)^{n-1} (n-1)!} \partial_{\xi}^{n-1} (\lambda+i\xi)^{-1}(x) \\
 &= (-1)^{n-1} \frac{\lambda^n}{(n-1)!} x^{n-1} (\lambda+i\xi)^{-1}(x).
 \end{aligned}$$

Now, since

$$\lambda(\lambda+i\xi)^{-1} = \lambda e^{-\lambda\#} 1_{[0,+\infty[}(\xi),$$

we have

$$\lambda (\lambda+i\#)^{-1}(x) = \lambda e^{-\lambda\#} 1_{[0,+\infty[}(x) = 2\pi \lambda e^{\lambda x} 1_{[0,+\infty[}(-x),$$

from which, finally,

$$f_{X_1+\dots+X_n}(x) = \lambda \frac{(-\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} 1_{[0,+\infty[}(x).$$

ii) Let $S_n := X_1 + \dots + X_n$. We compute the cdf of S_N : for $x \geq 0$,

$$\begin{aligned} \mathbb{P}(S_N \leq x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq x, N = n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq x)(1-p)^{n-1}p \\ &= \lambda p \sum_{n=1}^{\infty} \int_0^x \frac{(-\lambda y)^{n-1}}{(n-1)!} e^{-\lambda y} (1-p)^{n-1} dy \\ &\stackrel{\text{mon.conv.}}{=} \lambda p \int_0^x \sum_{n=1}^{\infty} \frac{(-(1-p)\lambda y)^{n-1}}{(n-1)!} e^{-\lambda y} dy = \lambda p \int_0^x e^{-(1-p)\lambda y} e^{-\lambda y} dy \end{aligned}$$

that is

$$F_{S_N}(x) = \lambda p \int_0^x e^{-\lambda p y} dy, \implies f_{S_N}(x) = \lambda p e^{-\lambda p x} 1_{[0, +\infty[}(x). \quad \square$$

Exercise 22. i) See notes.

ii) Let $\phi_X(\xi) := \mathbb{E}[e^{i\xi X}]$. To show that ϕ_X is differentiable we apply the differentiability under integral theorem. Formally

$$\partial_{\xi} \phi_X(\xi) = \mathbb{E}[iX e^{i\xi X}].$$

Notice that $|iX e^{i\xi X}| = |X| \in L^1(\Omega)$ being, by Cauchy-Schwarz inequality, $\mathbb{E}[|X|] \leq \mathbb{E}[X^2]^{1/2} < +\infty$. Thus differentiability theorem applies. Then,

$$\partial_{\xi}^2 \phi_X(\xi) = \mathbb{E}[(iX)^2 e^{i\xi X}].$$

Again, being $|(iX)^2 e^{i\xi X}| = X^2 \in L^1(\Omega)$, because $\mathbb{E}[X^2] < +\infty$. In particular,

$$\partial_{\xi}^2 \phi_X(0) = \mathbb{E}[(iX)^2] = -\mathbb{E}[X^2].$$

iii) Notice that $\partial_{\xi} \phi_X(\xi) = -c4\xi^3 e^{-c\xi^4}$ and $\partial_{\xi}^2 \phi_X(\xi) = e^{-c\xi^4}(16c^2\xi^6 - 12c\xi^2)$, so, in particular, $\partial_{\xi}^2 \phi_X(0) = 0$. By ii), $\mathbb{E}[X^2] = 0$, thus $X = 0$ a.s., from which $\phi_X(\xi) = \mathbb{E}[1] = 1 = e^{-c\xi^4}$ iff $c = 0$. \square

Exercise 23. i) See notes.

ii) Let $\mathcal{F}_t := \sigma(W_s : s \leq t)$. We have to verify

$$\mathbb{E}[W_t^3 - 3tW_t \mid \mathcal{F}_s] = W_s^3 - 3sW_s, \quad \forall t \geq s \geq 0.$$

We have

$$W_t^3 = (W_t - W_s + W_s)^3 = (W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3,$$

so, recalling that $W_t - W_s$ independent of \mathcal{F}_s and W_s is \mathcal{F}_s , by the properties of the conditional expectation we have

$$\begin{aligned} \mathbb{E}[W_t^3 \mid \mathcal{F}_s] &= \mathbb{E}[(W_t - W_s)^3] + 3W_s \mathbb{E}[(W_t - W_s)^2 \mid \mathcal{F}_s] + 3W_s^2 \mathbb{E}[W_t - W_s \mid \mathcal{F}_s] + W_s^3 \\ &= 0 + 3W_s \mathbb{E}[(W_t - W_s)^2] + 3W_s^2 \mathbb{E}[W_t - W_s] + W_s^3 \\ &= 3W_s(t-s) + W_s^3 \end{aligned}$$

so

$$\begin{aligned}\mathbb{E}[W_t^3 - 3tW_t \mid \mathcal{F}_s] &= 3W_s(t-s) + W_s^3 - 3t\mathbb{E}[W_t \mid \mathcal{F}_s] = 3W_s(t-s) + W_s^3 - 3tW_s \\ &= W_s^3 - 3sW_s,\end{aligned}$$

which is the conclusion.

iii) We start noticing that

$$W_t^4 = (W_t - W_s + W_s)^4 = (W_t - W_s)^4 + 4(W_t - W_s)^3W_s + 6(W_t - W_s)^2W_s^2 + 4(W_t - W_s)W_s^3 + W_s^4,$$

so

$$\mathbb{E}[W_t^4 \mid \mathcal{F}_s] = 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4$$

Now, notice also that

$$\mathbb{E}[W_t^2 \mid \mathcal{F}_s] = t-s + W_s^2, \iff \mathbb{E}[W_t^2 - t \mid \mathcal{F}_s] = W_s^2 - s,$$

so

$$6tW_s^2 = 6t(\mathbb{E}[W_t^2 - t \mid \mathcal{F}_s] + s) = \mathbb{E}[6tW_t^2 \mid \mathcal{F}_s] - 6t(t-s),$$

from which

$$\begin{aligned}\mathbb{E}[W_t^4 - 6tW_t^2 \mid \mathcal{F}_s] &= W_s^4 - 6sW_s^2 + 3(t-s)^2 - 6t(t-s) = \\ &= W_s^4 - 6sW_s^2 - 3(t^2 - s^2),\end{aligned}$$

so, finally,

$$\mathbb{E}[W_t^4 - 6tW_t^2 + 3t^2 \mid \mathcal{F}_s] = W_s^4 - 6sW_s^2 + 3s^2,$$

that is, $W_t^4 - 6tW_t^2 + 3t^2$ is a martingale. \square

Exercise 25 i) Assume (X, Y) are independent, so $f_{X,Y} = f_X f_Y$. Then

$$\begin{aligned}\phi_{X,Y}(\xi, \eta) &= \mathbb{E}[e^{i(\xi, \eta) \cdot (X, Y)}] = \int_{\mathbb{R}^2} e^{i(\xi, \eta) \cdot (x, y)} f_{X,Y}(x, y) dx dy = \int_{\mathbb{R}^2} e^{i(\xi, \eta) \cdot (x, y)} f_X(x) f_Y(y) dx dy \\ &= \int_{\mathbb{R}} e^{i\xi x} f_X(x) dx \int_{\mathbb{R}} e^{i\eta y} f_Y(y) dy = \mathbb{E}[e^{i\xi X}] \mathbb{E}[e^{i\eta Y}] = \phi_X(\xi) \phi_Y(\eta)\end{aligned}$$

this for every $(\xi, \eta) \in \mathbb{R}^2$.

Vice versa: assume $\phi_{X,Y} \equiv \phi_X \phi_Y$. The previous calculation shows that

$$\widehat{f_{X,Y}} = \widehat{f_X f_Y},$$

and since both $f_{X,Y}, f_X f_Y \in L^1(\mathbb{R}^2)$ because they are probability densities, by the injectivity of L^1 FT we conclude that $f_{X,Y} \stackrel{a.s.}{=} f_X f_Y$ as claimed.

ii) We can use the characteristic functions:

$$\begin{aligned}\phi_{X+Y, X-Y}(\xi, \eta) &= \mathbb{E}[e^{i(X+Y, X-Y) \cdot (\xi, \eta)}] = \mathbb{E}[e^{(\xi+\eta)X + (\xi-\eta)Y}] = \mathbb{E}[e^{i(\xi+\eta)X}] \mathbb{E}[e^{i(\xi-\eta)Y}] \\ &= e^{-\frac{1}{2}(\xi+\eta)^2} e^{-\frac{1}{2}(\xi-\eta)^2} = e^{-\frac{1}{2}(2\xi^2 + 2\eta^2)} = e^{-\xi^2} e^{-\eta^2}.\end{aligned}$$

On the other hand

$$\phi_{X \pm Y}(t) = \mathbb{E} \left[e^{i(X \pm Y)t} \right] = \mathbb{E} \left[e^{itX} \right] \mathbb{E} \left[e^{\pm itY} \right] = e^{-\frac{t^2}{2}} e^{\frac{-t^2}{2}} = e^{-t^2}.$$

We can conclude that $\phi_{X+Y, X-Y} = \phi_{X+Y} \phi_{X-Y}$, thus $X+Y, X-Y$ are independent and that they are both Gaussian $\mathcal{N}(0, 2)$.

iii) As suggested, $(x+y)^2 - (x-y)^2 = 4xy$, so

$$\mathbb{E}[XY | X - Y] = \frac{1}{4} \mathbb{E} \left[(X+Y)^2 - (X-Y)^2 | X - Y \right]$$

By independence,

$$\mathbb{E} \left[(X+Y)^2 | X - Y \right] = \mathbb{E} \left[(X+Y)^2 \right] = 2,$$

while

$$\mathbb{E} \left[(X-Y)^2 | X - Y \right] = (X-Y)^2.$$

Therefore

$$\mathbb{E}[XY | X - Y] = \frac{1}{4} \left(2 - (X - Y)^2 \right). \quad \square$$

Exercise 26 i) We start noticing that

$$X_k = \alpha X_{k-1} + N_{k-1} = \alpha(\alpha X_{k-2} + N_{k-2}) + N_{k-1} = \alpha^2 X_{k-2} + N_{k-1} + \alpha N_{k-2}.$$

Iterating this we get

$$X_k = \alpha^k X_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j} = \alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j}.$$

Clearly,

$$\mathbb{E}[X_k] = \alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j \mathbb{E}[N_{k-1-j}] = \alpha^k x_0.$$

About the variance we have

$$\mathbb{V}[X_k] = \mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2 = \mathbb{E} \left[\left(\alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j} \right)^2 \right] - \alpha^{2k} x_0^2.$$

Noticed that $\mathbb{E}[\alpha^k x_0 \alpha^j N_{k-1-j}] = 0$, and being the N_k independent, we have

$$\mathbb{V}[X_k] = \alpha^{2k} x_0^2 + \sum_{j=0}^{k-1} \alpha^{2j} \mathbb{E}[N_{k-1-j}^2] - \alpha^{2k} x_0^2 = \sigma^2 \sum_{j=0}^{k-1} (\alpha^2)^j = \sigma^2 \frac{1 - \alpha^{2k}}{1 - \alpha^2},$$

because of the formula $\sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}$.

ii) We have

$$\mathbb{E}[X_{k+1} | \mathcal{F}_k] = \mathbb{E}[\alpha X_k + N_k | \mathcal{F}_k] = \alpha X_k + \mathbb{E}[N_k | \mathcal{F}_k].$$

Since $\mathcal{F}_k = \sigma(X_1, \dots, X_k) = \sigma(N_0, \dots, N_{k-1})$ and N_k being independent of N_j for $j < k$, we have

$$\mathbb{E}[N_k | \mathcal{F}_k] = \mathbb{E}[N_k] = 0.$$

Therefore, $\mathbb{E}[X_{k+1} | \mathcal{F}_k] = X_k$, so (X_k) is a martingale w.r.t. \mathcal{F}_k iff $\alpha = 1$.

iii) We have $X_{k+1} - X_k = (1 - \alpha)X_k + N_k$. If $\alpha = 1$,

$$\|X_{k+1} - X_k\|_2^2 = \mathbb{E}[(X_{k+1} - X_k)^2] = \mathbb{E}[N_k^2] = \sigma^2,$$

whereas, if $\alpha \neq 1$,

$$\begin{aligned} \|X_{k+1} - X_k\|_2^2 &= \mathbb{E}[(X_{k+1} - X_k)^2] = \mathbb{E}[(1 - \alpha)^2 X_k^2 + N_k^2 + 2(1 - \alpha)X_k N_k] \\ &= (1 - \alpha)^2 \frac{1 - \alpha^{2k}}{1 - \alpha} \sigma^2 + \sigma^2 = \left((1 - \alpha)(1 - \alpha^{2k}) + 1 \right) \sigma^2. \end{aligned}$$

In each case, $\|X_{k+1} - X_k\|_2 \not\rightarrow 0$ when $k \rightarrow +\infty$, and this must happen to have L^2 convergence. Thus, X_k is not convergent in L^2 .

iv) We have

$$\phi_{X_k}(\xi) = \mathbb{E}[e^{i\xi X_k}] = \mathbb{E}\left[e^{i\xi(\alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j})}\right] = e^{i\xi \alpha^k x_0} \prod_{j=0}^{k-1} \mathbb{E}\left[e^{i\xi \alpha^j N_{k-1-j}}\right].$$

Let ϕ be the common characteristic function of the N_j . We have $\phi(\xi) = e^{-\frac{1}{2}\sigma^2 \xi^2}$, so

$$\phi_{X_k}(\xi) = e^{i\xi \alpha^k x_0} \prod_{j=0}^{k-1} e^{-\frac{1}{2}\sigma^2 (\alpha^j \xi)^2} = e^{i\xi \alpha^k x_0 - \frac{1}{2}\sigma^2 \xi^2 \sum_{j=0}^{k-1} \alpha^{2j}} = e^{i\xi \alpha^k x_0 - \frac{1}{2}\sigma^2 \frac{1 - \alpha^{2k}}{1 - \alpha^2} \xi^2}.$$

Letting $k \rightarrow +\infty$, $\alpha^k \rightarrow 0$ being $|\alpha| < 1$, so

$$\phi_{X_k}(\xi) \rightarrow e^{-\frac{1}{2}\frac{\sigma^2}{1 - \alpha^2} \xi^2},$$

that is

$$X_k \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{1 - \alpha^2}\right). \quad \square$$

Exercise 27. i) See notes for the definitions.

ii) We start noticing that

$$\begin{aligned} X_t &= W_t^3 - 3 \int_0^t W_r dr = (W_t - W_s + W_s)^3 - 3 \int_0^s W_r dr - 3 \int_s^t W_r dr \\ &= (W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s) W_s^2 + \underbrace{W_s^3 - 3 \int_0^s W_r dr}_{=X_s} \\ &\quad - 3 \int_s^t (W_r - W_s) dr - 3W_s(t - s). \end{aligned}$$

We now apply the conditional expectation w.r.t \mathcal{F}_s . Notice first that $X_s \in \mathcal{F}_s$. Then

$$\begin{aligned} \mathbb{E}[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 \mid \mathcal{F}_s] &= \\ &= \underbrace{\mathbb{E}[(W_t - W_s)^3]}_{=0} + 3W_s \underbrace{\mathbb{E}[(W_t - W_s)^2]}_{t-s} + 3W_s^2 \underbrace{\mathbb{E}[W_t - W_s]}_{=0} \\ &= 3W_s(t - s). \end{aligned}$$

Finally,

$$\mathbb{E} \left[\int_s^t (W_r - W_s) dr \mid \mathcal{F}_s \right] = \int_s^t \underbrace{\mathbb{E}[W_r - W_s \mid \mathcal{F}_s]}_{=\mathbb{E}[W_r - W_s]=0} dr = 0.$$

Therefore, in conclusion,

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s + 3W_s(t - s) - 3W_s(t - s) = X_s,$$

that is, X_t is an \mathcal{F}_t martingale. □

Exercise 28. i) To be a well defined covariance matrix, C must be symmetric (evident) and strictly positive definite. This last follows from positivity of $k \times k$ ($k = 1, 2$) sub-determinants that are 1 and $1 - \frac{1}{4} = \frac{3}{4}$.

ii) We use the characteristic function:

$$\begin{aligned} \phi_{Y, 2X-Y}(\xi, \eta) &= \mathbb{E} \left[e^{i(\xi, \eta) \cdot (Y, 2X-Y)} \right] = \mathbb{E} \left[e^{i(\xi Y + \eta(2X-Y))} \right] = \mathbb{E} \left[e^{i(2\eta, \xi - \eta) \cdot (X, Y)} \right] \\ &= e^{-\frac{1}{2}C(2\eta, \xi - \eta)(2\eta, \xi - \eta)} = e^{-\frac{1}{2}(4\eta^2 + 2\frac{1}{2}2\eta(\xi - \eta) + (\xi - \eta)^2)} = e^{-\frac{1}{2}(3\eta^2 + \xi^2)} \\ &= \phi_{\mathcal{N}(0,1)}(\xi) \phi_{\mathcal{N}(0,3)}(\eta) \end{aligned}$$

and since $Y \sim \mathcal{N}(0, 1)$ and $2X - Y \sim \mathcal{N}(0, 3)$ we have that

$$\phi_{Y, 2X-Y}(\xi, \eta) \equiv \phi_Y(\xi) \phi_{2X-Y}(\eta), \quad \forall (\xi, \eta) \in \mathbb{R}^2,$$

from which we deduce the independence.

iii) Since $X = \frac{1}{2}(2X - Y) + Y$, we have

$$X^2 Y = \left(\frac{1}{2}(2X - Y) + \frac{1}{2}Y \right)^2 Y = \frac{1}{4} \left((2X - Y)^2 Y + 2(2X - Y)Y^2 + Y^3 \right)$$

and since Y and $2X - Y$ are independent

$$\begin{aligned}\mathbb{E}[X^2Y \mid 2X - Y] &= \frac{1}{4} \left(\mathbb{E}[(2X - Y)^2Y \mid 2X - Y] + 2\mathbb{E}[(2X - Y)Y^2 \mid 2X - Y] + \mathbb{E}[Y^3 \mid 2X - Y] \right) \\ &= \frac{1}{4} \left((2X - Y)^2\mathbb{E}[Y \mid 2X - Y] + 2(2X - Y)\mathbb{E}[Y^2 \mid 2X - Y] + \mathbb{E}[Y^3] \right) \\ &= \frac{1}{4} \left((2X - Y)^2\mathbb{E}[Y] + 2(2X - Y)\mathbb{E}[Y^2] + \mathbb{E}[Y^3] \right) \\ &= \frac{2X - Y}{2}. \quad \square\end{aligned}$$

Exercise 29. i) See LN.

ii) Let $E_k := \{X_k = X_{k+1} = X_{k+2} = 1\} \in \mathcal{F}$ (X_k are random variables). Then,

$$E = \bigcap_n \bigcup_{k \geq n} E_k = \limsup E_k \in \mathcal{F}.$$

Moreover, since

$$\mathbb{P}(E_k) \stackrel{\text{indep}}{=} \mathbb{P}(X_k = 1)\mathbb{P}(X_{k+1} = 1)\mathbb{P}(X_{k+2} = 1) = \frac{1}{\sqrt{k(k+1)(k+2)}},$$

and

$$\sum_k \mathbb{P}(E_k) = \sum_k \frac{1}{\sqrt{k(k+1)(k+2)}} \sim \sum_k \frac{1}{k^{3/2}} < \infty,$$

by the first Borel-Cantelli lemma we deduce that

$$\mathbb{P}(E) = \mathbb{P}(\limsup_k E_k) = 0.$$

iii) As in ii), let $F_k := \{X_k = X_{k+1} = 1\} \in \mathcal{F}$ and

$$F = \bigcap_n \bigcup_{k \geq n} F_k = \limsup F_k \in \mathcal{F}.$$

Notice that the events F_k are not independent, while F_{2k} are independent. Since,

$$F \supset \bigcap_n \bigcup_{k \geq n} F_{2k} = \limsup F_{2k},$$

we are led to assess $\mathbb{P}(\limsup_k F_{2k})$. As in ii),

$$\mathbb{P}(F_k) \stackrel{\text{indep}}{=} \mathbb{P}(X_k = 1)\mathbb{P}(X_{k+1} = 1) = \frac{1}{\sqrt{k(k+1)}}, \implies \mathbb{P}(F_{2k}) = \frac{1}{\sqrt{2k(2k+1)}} \geq \frac{1}{\sqrt{4k^2}} = \frac{1}{2k},$$

so

$$\sum_k \mathbb{P}(F_{2k}) \geq \sum_k \frac{1}{2k} = +\infty.$$

By the second Borel-Cantelli lemma we conclude that $\mathbb{P}(\limsup_k F_{2k}) = 1$, so $\mathbb{P}(F) \geq 1$ from which $\mathbb{P}(F) = 1$. \square

Exercise 30. i) Since $0 \leq X_k \leq 1$ with $\mathbb{P} = 1$, we have that $S_{n+1} \geq S_n$, so $N > n$ iff $S_1 \leq 1, \dots, S_n \leq 1$, that is $S_n \leq 1$. In other words

$$\{N > n\} = \{S_n \leq 1\}.$$

Therefore,

$$\mathbb{P}(N > n) = \mathbb{P}(S_n \leq 1) = \mathbb{P}(X_1 + \dots + X_n \leq 1) = \int_{x_1 + \dots + x_n \leq 1} \prod_{k=1}^n 1_{[0,1]}(x_k) dx_1 \cdots dx_n = \frac{1}{n!}.$$

From this we have

$$\mathbb{P}(N = n) = \mathbb{P}(\{N > n-1\} \setminus \{N > n\}) = \mathbb{P}(N > n-1) - \mathbb{P}(N > n) = \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n-1}{n!}.$$

ii) We have

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} n \mathbb{P}(N = n) = \sum_{n=1}^{\infty} n \frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Recalling of the exponential series $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ we have

$$\mathbb{E}[N] = e.$$

iii) We have

$$\mathbb{E}[S_N] = \sum_{n=2}^{\infty} \mathbb{E}[S_n 1_{N=n}].$$

We notice that

$$\{N = n\} = \{S_{n-1} \leq 1, S_n > 1\}$$

so

$$\begin{aligned} \mathbb{E}[S_n 1_{N=n}] &= \mathbb{E}[S_n 1_{S_{n-1} \leq 1} 1_{S_n > 1}] = \mathbb{E}[\mathbb{E}[S_n 1_{S_{n-1} \leq 1} 1_{S_n > 1} \mid S_{n-1}]] \\ &= \mathbb{E}[1_{S_{n-1} \leq 1} \mathbb{E}[(S_{n-1} + X_n) 1_{X_n > 1 - S_{n-1}} \mid S_{n-1}]] \end{aligned}$$

Now, if $s = S_{n-1} < 1$,

$$\mathbb{E}[(S_{n-1} + X_n) 1_{X_n > 1 - S_{n-1}} \mid S_{n-1} = s] = s \mathbb{E}[1_{X_n > 1-s} \mid S_{n-1} = s] + \mathbb{E}[X_n 1_{X_n > 1-s} \mid S_{n-1}],$$

and by the independence of X_n from $S_{n-1} = X_1 + \dots + X_{n-1}$, we have

$$\begin{aligned} \mathbb{E}[(S_{n-1} + X_n) 1_{X_n > 1 - S_{n-1}} \mid S_{n-1} = s] &= s \mathbb{E}[1_{X_n > 1-s}] + \mathbb{E}[X_n 1_{X_n > 1-s}] \\ &= s(1 - (1-s)) + \int_{1-s}^1 x dx \\ &= s^2 + \left[\frac{x^2}{2} \right]_{x=1-s}^1 = s^2 + \frac{1}{2}(1 - (1-s)^2) \\ &= \frac{s^2}{2} + s. \end{aligned}$$

Therefore

$$\mathbb{E}[S_n 1_{N=n}] = \mathbb{E} \left[\left(\frac{1}{2} S_{n-1}^2 + S_{n-1} \right) 1_{S_{n-1} \leq 1} \right] = \frac{1}{2} I_{n-1}^2 + I_{n-1}^1 = \frac{1}{2} \frac{1}{(n-2)!(n+1)} + \frac{1}{(n-2)!n}.$$

From this

$$\mathbb{E}[S_N] = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{(n-2)!(n+1)} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!n}$$

To compute the exact value of the sum we notice that

$$\sum_{n=2}^{\infty} \frac{1}{(n-2)!n} = \sum_{n=2}^{\infty} \frac{n-1}{n!} = \sum_{n=2}^{\infty} \left(\frac{1}{(n-1)!} - \frac{1}{n!} \right) = e - 1 - (e - 2) = 1,$$

while

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{(n-2)!(n+1)} &= \sum_{n=2}^{\infty} \frac{n(n-1)}{(n+1)!} = \sum_{n=2}^{\infty} \left(\frac{(n+1)(n-1)}{(n+1)!} - \frac{n+1}{(n+1)!} + \frac{2}{(n+1)!} \right) \\ &= \sum_{n=2}^{\infty} \left(\frac{n-1}{n!} - \frac{1}{n!} + \frac{2}{(n+1)!} \right) = \sum_{n=2}^{\infty} \left(\frac{1}{(n-1)!} - \frac{2}{n!} + \frac{2}{(n+1)!} \right) \\ &= e - 1 - 2 \sum_{n=2}^{\infty} \frac{1}{n!} + 2 \sum_{n=3}^{\infty} \frac{1}{n!} = e - 1 - 2 \frac{1}{2} = e - 2. \end{aligned}$$

Thus,

$$\mathbb{E}[S_N] = \frac{1}{2}(e - 2) + 1 = \frac{e}{2}. \quad \square$$

Exercise 31. i) The characteristic function of (U, V) is

$$\phi_{U,V}(\xi, \eta) = \mathbb{E} \left[e^{i(\xi, \eta) \cdot T(X,Y)} \right] = \mathbb{E} \left[e^{iT^*(\xi, \eta) \cdot (X,Y)} \right] = e^{-\frac{1}{2}T^*(\xi, \eta) \cdot TT^*(\xi, \eta)} = e^{-\frac{1}{2}TT^*(\xi, \eta) \cdot (\xi, \eta)},$$

from which it follows that $(U, V) \sim \mathcal{N}(\vec{0}, TT^*)$. Now, U and V are independent iff

$$\phi_{U,V}(\xi, \eta) = \phi_U(\xi)\phi_V(\eta).$$

If $TT^* = [m_{ij}]$, then

$$TT^*(\xi, \eta) \cdot (\xi, \eta) = m_{11}\xi^2 + 2m_{12}\xi\eta + m_{22}\eta^2,$$

so independence holds iff $m_{12} = ac + bd = 0$.

ii) We have

$$\mathbb{E}[X + 2Y \mid X - Y] = \mathbb{E}[X + Y \mid X - Y] + \mathbb{E}[Y \mid X - Y].$$

Noticed that

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} X + Y \\ X - Y \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

we see that U, V are independent. So

$$\mathbb{E}[X + Y \mid X - Y] = \mathbb{E}[X + Y] = 0.$$

Setting

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} Y \\ X - Y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} =: T \begin{pmatrix} X \\ Y \end{pmatrix},$$

we have

$$\mathbb{E}[U \mid V = v] = \int_{\mathbb{R}} u \frac{f_{U,V}(u, v)}{f_V(v)} du.$$

By i),

$$\begin{aligned}
f_{U,V}(u,v) &= \frac{1}{\sqrt{(2\pi)^2 \det(TT^*)}} e^{-\frac{1}{2}(TT^*)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}} = \frac{1}{2\pi |\det T|} e^{-\frac{1}{2}(TT^*)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}} = \frac{1}{2\pi} e^{-(u^2 + uv + \frac{v^2}{2})} \\
&= \frac{1}{2\pi} e^{-\frac{v^2}{4}} e^{-(u + \frac{v}{2})^2}
\end{aligned}$$

About U we have

$$\phi_V(\xi) = \mathbb{E}[e^{i\xi(X-Y)}] = \mathbb{E}[e^{i\xi X}] \mathbb{E}[e^{-i\xi Y}] = e^{-\frac{1}{2}\xi^2} e^{-\frac{1}{2}(-\xi)^2} = e^{-\xi^2},$$

that is $V \sim \mathcal{N}(0, 2)$, from which $f_V(v) = \frac{1}{\sqrt{4\pi}} e^{-\frac{v^2}{4}}$. Therefore

$$\begin{aligned}
\mathbb{E}[U | V = v] &= \int_{\mathbb{R}} u \frac{1}{2\pi} e^{-\frac{v^2}{4}} e^{-(u + \frac{v}{2})^2} \sqrt{4\pi} e^{\frac{v^2}{4}} du = \sqrt{2} \int_{\mathbb{R}} u e^{-(u + \frac{v}{2})^2} \frac{du}{\sqrt{2\pi}} \\
&\stackrel{t = u + \frac{v}{2}}{=} \sqrt{2} \int_{\mathbb{R}} (t - \frac{v}{2}) e^{-t^2} \frac{dt}{\sqrt{2\pi}} = -\frac{1}{\sqrt{2}} v \int_{\mathbb{R}} e^{-t^2} \frac{dt}{\sqrt{2\pi}} = -\frac{1}{2} v.
\end{aligned}$$

Conclusion:

$$\mathbb{E}[X + 2Y | X - Y] = \mathbb{E}[Y | X - Y] = -\frac{1}{2}(X - Y).$$

□

Exercise 32. i) See LN.

ii) We have

$$\mathbb{V}[\mathbb{E}[X | \mathcal{G}]] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}]^2] - \mathbb{E}[\mathbb{E}[X | \mathcal{G}]]^2 = \mathbb{E}[\mathbb{E}[X | \mathcal{G}]^2] - \mathbb{E}[X]^2,$$

and, similarly,

$$\mathbb{V}[\mathbb{E}[X | \mathcal{H}]] = \mathbb{E}[\mathbb{E}[X | \mathcal{H}]^2] - \mathbb{E}[X]^2.$$

So the conclusion follows once we prove that

$$\mathbb{E}[\mathbb{E}[X | \mathcal{G}]^2] \geq \mathbb{E}[\mathbb{E}[X | \mathcal{H}]^2]$$

We remind that, if $X \in L^2$, $\mathbb{E}[X | \mathcal{G}]$ is the orthogonal projection on $L^2(\Omega, \mathcal{G}, \mathbb{P})$ and similarly for $\mathbb{E}[X | \mathcal{H}]$. Since $\mathcal{H} \subset \mathcal{G}$ we have that

$$L^2(\Omega, \mathcal{H}, \mathbb{P}) \subset L^2(\Omega, \mathcal{G}, \mathbb{P}),$$

and they are both closed subspaces of $L^2(\Omega)$. Now, let $Y := \mathbb{E}[X | \mathcal{G}]$. Then

$$Y - \mathbb{E}[Y | \mathcal{H}] \perp \mathbb{E}[Y | \mathcal{H}] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}] | \mathcal{H}] = \mathbb{E}[X | \mathcal{H}],$$

by sub-conditioning. By the Pythagorean theorem

$$\|Y\|_{L^2}^2 = \|\mathbb{E}[Y | \mathcal{H}]\|_{L^2}^2 + \|Y - \mathbb{E}[Y | \mathcal{H}]\|_{L^2}^2 \geq \|\mathbb{E}[Y | \mathcal{H}]\|_{L^2}^2,$$

that is

$$\mathbb{E}[\mathbb{E}[X | \mathcal{G}]^2] \geq \mathbb{E}[\mathbb{E}[Y | \mathcal{H}]^2] = \mathbb{E}[\mathbb{E}[\mathbb{E}[X | \mathcal{G}] | \mathcal{H}]^2] = \mathbb{E}[\mathbb{E}[X | \mathcal{H}]^2],$$

which is the conclusion. □

Exercise 33. i) For $s = t$, $\mathbb{E}[W_s W_t] = \mathbb{E}[W_s^2] = s$. For $s < t$ we have

$$\mathbb{E}[W_t W_s] = \mathbb{E}[(W_t - W_s)W_s] + \mathbb{E}[W_s]^2 = s,$$

from which $\mathbb{E}[W_t W_s] = \min(s, t)$.

ii) Since X_n is Gaussian,

$$\phi_{X_n}(\xi) = e^{im\xi - \frac{1}{2}\sigma^2 \xi^2},$$

where

$$m = \mathbb{E}[X_n] = \frac{1}{2^n} \sum_{k=0}^{2^n} \mathbb{E}[W_{k/2^n}] = 0,$$

and

$$\sigma^2 = \mathbb{V}[X_n] = \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 = \mathbb{E}\left[\frac{1}{2^{2n}} \left(\sum_{k=0}^{2^n} W_{k/2^n}\right)^2\right] = \frac{1}{2^{2n}} \sum_{k,h=0}^{2^n} \mathbb{E}[W_{k/2^n} W_{h/2^n}].$$

and, by i),

$$\mathbb{E}[W_{k/2^n} W_{h/2^n}] = \begin{cases} (k = h) & = \mathbb{E}[W_{k/2^n}^2] = \frac{k}{2^n}, \\ (k \neq h) & = \frac{\min(k,h)}{2^n}. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{k,h=0}^{2^n} \mathbb{E}[W_{k/2^n} W_{h/2^n}] &= \sum_{k=0}^{2^n} \mathbb{E}[W_{k/2^n}^2] + 2 \sum_{k < h} \mathbb{E}[W_{k/2^n} W_{h/2^n}] = \sum_{k=0}^{2^n} \frac{k}{2^n} + 2 \sum_{k=0}^{2^n-1} \sum_{h=k+1}^{2^n} \frac{k}{2^n} \\ &= \frac{1}{2^n} \frac{2^n(2^n+1)}{2} + \frac{1}{2^{n-1}} \sum_{k=0}^{2^n-1} k(2^n - k) \\ &= \frac{2^n+1}{2} + \frac{1}{2^{n-1}} \left(2^n \frac{(2^n-1)2^n}{2} - \frac{(2^n-1)2^n(2^{n+1}-1)}{6} \right) \\ &= 2^{n-1} + \frac{1}{2} + \frac{1}{2^n} (2^n - 1) \left(2^{2n} - \frac{2^n(2^{n+1}-1)}{3} \right) \\ &= \frac{2^n}{2} + \frac{1}{6} + \frac{2^{2n}}{3}. \end{aligned}$$

Therefore

$$\sigma^2 = \frac{1}{2^{2n}} \left(\frac{2^n}{2} + \frac{1}{6} + \frac{2^{2n}}{3} \right) = \frac{1}{2^{n+1}} + \frac{1}{6 \cdot 2^n} + \frac{1}{3}.$$

iii) From ii),

$$\phi_{X_n}(\xi) = e^{-\frac{1}{2} \left(\frac{1}{2^{n+1}} + \frac{1}{6 \cdot 2^n} + \frac{1}{3} \right) \xi^2} \longrightarrow e^{-\frac{1}{2} \cdot \frac{1}{6} \xi^2}$$

and this implies that $X_n \xrightarrow{d} \mathcal{N}(0, 1/6)$. □