Stochastic Methods for Engineering

Exam 1

Exercise 1. Let $(W_t)_{t\geqslant 0}$ be the Brownian Motion. For every $s,t\geqslant 0$ compute

i) $\mathbb{E}[W_t W_s]$
ii) $\mathbb{E}[W_s W_t^2]$
iii) $\mathbb{E}[W_s^2 W_t^2]$
iv) $\mathbb{E}[W_s e^{W_t}]$.

Exercise 2. We recall that $X\sim\Gamma(\alpha,\lambda)$ for $\alpha,\lambda>0$ if

$$f_X(x) := \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} 1_{[0,+\infty[}(x).$$

(recall that $\Gamma(\alpha)=\int_0^{+\infty} x^{\alpha-1}e^{-\lambda x}\,dx$ and $\Gamma(n)=(n-1)!$ for $n\in\mathbb{N}$, $n\geqslant 1$)

i) Let ϕ_X be the characteristic function of X. Show that

$$\partial_\xi \phi_X(\xi) = -\frac{\alpha}{\xi + i\lambda}\phi_X(\xi),$$

and deduce, from this ϕ_X.

ii) Check that if $X_j\sim\Gamma(\alpha_j,\lambda)$ for $j=1,\dots,n$ are independent, then $X_1+\cdots+X_n\sim\Gamma(\alpha,\lambda)$ for a suitable α.

iii) Let X and Y be i.i.d. random variables $\sim\Gamma(1,\lambda)$ and let $Z=X+Y$. Show that the following formula holds:

$$\mathbb{E}[\psi(X)\mid Z] = \frac{1}{Z}\int_0^Z \psi(x)\,dx.$$

Exercise 3. What does it mean that $X_n \xrightarrow{a.s.} X$ and $X_n \xrightarrow{\mathbb{P}} X$? What relationships exist between these two types of convergence?

Let $X_0\sim U([0,1])$ and (Y_n) are i.i.d. $\mathcal{N}(0,1)$ random variables. For $n\geqslant 1$ define

$$X_n := \frac{X_{n-1}}{2} + Y_n.$$

ii) Show that $X_n \xrightarrow{d} X$ where $X = \dots$.
iii) Is $X_n \xrightarrow{\mathbb{P}} X$?

Exam 2

Exercise 4. Let (X_n) be i.i.d. random variables with exponential distribution, $X_n \sim \exp(\lambda)$. Let also N be independent of (X_n) with geometric distribution of parameter $0 < p < 1$ (namely, $\mathbb{P}(N = n) = (1-p)^{n-1}p,\ n \geqslant 1$). Define

$$Y_n := \min\{X_1, \ldots, X_n\}.$$

i) Determine the cdf of Y_n.
ii) Determine the cdf of Y_N.
iii) Compute $\mathbb{E}[Y_N]$.

Exercise 5. Let U, V be i.i.d. uniformly distributed on $[0, 1]$. Derive

$$R = \sqrt{-2\log U}, \quad \Theta := 2\pi V.$$

i) Determine the joint distribution of (R, Θ) and the marginal distributions of R and Θ.
ii) Set

$$X := R\cos\Theta, \quad Y := R\sin\Theta.$$

WHat is the joint distribution of (X, Y)? Are X and Y independent? What are their univariate distributions?

Exercise 6. Let (X_n) be a sequence of random variables such that

$$X_0 \equiv 1, \quad X_{n+1} - X_n = \frac{1}{2}Y_n X_n,$$

where Y_n is a Bernoulli r.v. with $\mathbb{P}(Y_n = \pm 1) = \frac{1}{2}$, and Y_n is independent of $Y_0, \ldots, Y_{n-1}$. We may interpret X_n as the amount of money an investor will have after n days if he wins or loses half of the money daily, both with probability 1/2).

i) Prove that $X_n \xrightarrow{a.s.} 0$ (hint: start estimating $\mathbb{P}(|X_n| \geqslant \frac{1}{2^m})$ for $m \in \mathbb{N}$ fixed, then use Borel–Cantelli's Lemma).
ii) What about $\lim_n \mathbb{E}[X_n]$? Does $X_n \xrightarrow{L^1} 0$?

Exam 3

Exercise 7. Let $X \in L^1(\Omega)$.

i) Prove the triangular inequality

$$|\mathbb{E}[X \mid \mathscr{G}]| \leqslant \mathbb{E}[|X| \mid \mathscr{G}].$$

ii) Show that if $X, Y \in L^1(\Omega)$ are such that $\mathbb{E}[X \mid Y] = 0$, then $\|X+Y\|_1 \geqslant \|Y\|_1$.

iii) Show that if $X, Y \in L^1(\Omega)$ are such that $\mu_{XY} = \mu_{YX}$ then

$$\mathbb{E}[X \pm Y \mid X \mp Y] = 0.$$

Use this to deduce $\|3X - Y\|_1 \geqslant \|X+Y\|_1$.

Exercise 8. Let (X_n) be independent with X_n uniformly distributed on $[-1-\frac{1}{n}, 1+\frac{1}{n}]$. Let

$$Y_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k.$$

Discuss convergence in distribution of (Y_n) identifying also the limit (if any).
(hint: $\widetilde{X}_k = \frac{k}{k+1} X_k \sim U[-1,1]$).

Exercise 9. Let $(W_t)_{t \geqslant 0}$ be the Brownian Motion and define

$$B_t := \begin{cases} tW_{1/t}, & t > 0, \\ 0, & t = 0. \end{cases}$$

i) Check that (B_t) are gaussian random variables (determining their distributions), and that the increments of (B_t), namely $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}} - B_{t_n}$ are independent random variables. Check also that $B_t(\omega) \in \mathscr{C}(]0, +\infty[)$.

ii) Check that

$$B_t \xrightarrow{\mathbb{P}} 0, \; t \longrightarrow 0+.$$

Exam 4

Exercise 10. Let X, Y be independent random variables both with geometric distribution of parameter $p \in]0,1[$ (that is, $\mathbb{P}(X = n) = \mathbb{P}(Y = n) = (1-p)^{n-1}p$). Define

$$Z := \min(X, Y), \quad W := |X - Y|.$$

i) Determine the distribution of (Z, W).
ii) Are Z and W independent?

Exercise 11. What does Borel–Cantelli's Lemma state?

Let now (X_n) be i.i.d. random variables, $X_n \sim \exp(1)$. For $n \geq 2$ define

$$Y_n := \frac{X_n - \log n}{\log(\log n)}.$$

Prove that, for every $\varepsilon > 0$ fixed,

i) $\mathbb{P}\left(\bigcup_N \bigcap_n \{Y_n \leqslant 1 + \varepsilon\}\right) = 1$
ii) $\mathbb{P}\left(\bigcap_N \bigcup_n \{Y_n \geqslant 1 - \varepsilon\}\right) = 1$.

(it might be helpful to know that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha} < +\infty$ iff $\alpha > 1$)

Exercise 12. Let W be a BM on $(\Omega, \mathscr{F}, \mathbb{P})$. For $T > 0$ fixed, let $\mathscr{F}_T := \sigma(W_t \ : \ 0 \leqslant t \leqslant T)$ and define

$$\mathbb{Q}(E) := \mathbb{E}\left[1_E e^{aW_T - \frac{a^2}{2}T}\right], \ E \in \mathscr{F}_T.$$

i) Check that $\mathbb{Q}$ is a well defined probability measure on $(\Omega, \mathscr{F}_T)$.
ii) Check that

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}\left[X e^{aW_T - \frac{a^2}{2}T}\right].$$

ii) Let $B_t := W_t - at$. Check that $(B_t)_{0 \leqslant t \leqslant T}$ is a BM on $(\Omega, \mathscr{F}_T, \mathbb{Q})$.

Exam 5

Exercise 13. Let $0 < a < b$ and set

$$A := \{(x, y) \in \mathbb{R}^2 \ :\ |x - y| < a,\ |x + y| < b\}.$$

Let (X, Y) have uniform distribution in A, that is

$$f_{X,Y}(x, y) = \frac{1}{\lambda_2(A)} 1_A(x, y),$$

where $\lambda_2(A)$ stands for the Lebesgue measure of A.

i) Determine f_Y.
ii) Compute $\mathbb{E}[X \mid Y]$.
iii) Compute the density of $\mathbb{E}[X \mid Y]$.

Exercise 14. Let (W_t) be a BM on $(\Omega, \mathscr{F}, \mathbb{P})$ and define

$$X_n := e^{aW_n - \frac{a^2}{2} n},\ n \in \mathbb{N}.$$

i) Check that $X_n \in L^1(\Omega)$ for every $a \in \mathbb{R}$.
ii) Show that (X_n) is a martingale w.r.t. $\mathscr{F}_n := \sigma(W_m \ :\ m \in \mathbb{N},\ m \leqslant n)$.
iii) Show that $\lim_n X_n = 0$ a.s.

Exercise 15. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $X \in L(\Omega)$ such that $X \geqslant 0$ a.s.

i) Show that, if X is $\mathbb{N}$ valued (that is $\mathbb{P}(X \in \mathbb{N}) = 1$) then

$$\mathbb{E}[X] = \sum_{n=0}^{\infty} \mathbb{P}(X > n).$$

ii) Show that, in general,

$$\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X > t)\, dt.$$

Hint for both cases: $\mathbb{P}(E) = \mathbb{E}[1_E] \ldots$

January 2025

Exercise 16. Let $(X, Y) \sim \mathcal{N}(0, C)$ where

$$C := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

i) For which values of ρ matrix C is a true covariance matrix for (X, Y)?
ii) Imagine (X, Y) as the coordinates of a random point in the cartesian plane. Let (R, Θ) its polar coordinates ($R \geqslant 0$, $\Theta \in [0, 2\pi[$). Determine the joint distribution of (R, Θ) and, in particular, the distribution of Θ.
iii) Compute $\mathbb{E}[R \mid \Theta]$ and $\mathbb{E}[R^2 \mid \Theta]$.
iv) Under which conditions are R and Θ independent? In this case, determine also the density of R.

Exercise 17. Let X_n be i.i.d. random variables with common density

$$f_{X_n}(x) = \frac{1}{2} e^{-|x|}.$$

Prove that, $\forall \varepsilon > 0$,

i) $\mathbb{P}\left(\frac{|X_n|}{\log n} \leqslant 1 + \varepsilon \text{ for all but finitely many } n\right) = 1.$
ii) $\mathbb{P}\left(\frac{|X_n|}{\log n} \geqslant 1 - \varepsilon \text{ for infinitely many } n\right) = 1.$

Exercise 18. Let (W_t) be a BM. Define

$$X_t := \int_0^t \frac{W_u}{u}\, du.$$

i) Explain why, for $t > 0$ fixed and for almost every ω, X_t is a well defined random variable.
ii) Compute $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$ (hint: $(\int_0^t f_u\, du)^2 = \int_0^t f_u\, du \int_0^t f_s\, ds \ldots$)
iii) Define $B_t := W_t - X_t$. Compute $\mathbb{E}[B_t]$ and $\mathbb{V}[B_t]$.
iv) Let $\mathscr{F}_s := \sigma(W_r \ : \ 0 \leqslant r \leqslant s)$. Compute $\mathbb{E}[B_t \mid \mathscr{F}_s]$ for $0 \leqslant s < t$.

(if needed, you are allowed to switch $\mathbb{E}$ with $\int_0^t$)

February 2025

Exercise 19. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathscr{G} \subset \mathscr{F}$ a sub $\sigma-$algebra.

i) Let $X \in L^2(\Omega)$. What is the conditional expectation $\mathbb{E}[X \mid \mathscr{G}]$? How is characterized? Prove that
$$\|\mathbb{E}[X \mid \mathscr{G}]\|_2 \leqslant \|X\|_2.$$

ii) Let $X, Y \in L^2(\Omega)$ be such that
$$\mathbb{E}[X \mid Y] = Y, \quad \mathbb{E}[Y \mid X] = X.$$
Deduce that $X = Y$ a.s. (hint: check that $\|X - Y\|_2^2 = 0$)

iii) Let $X, Y, Z \in L^2(\Omega)$ be such that
$$\mathbb{E}[X \mid Y] = Y, \ \ \mathbb{E}[Y \mid Z] = Z, \ \ \mathbb{E}[Z \mid X] = X.$$
Prove that $X = Y = Z$ a.s.

Exercise 20. Let
$$F(x) := e^{-e^{-x}}, \ x \in \mathbb{R}.$$

i) Check that F is a cumulative distribution function.

Let now (X_n) be i.i.d. random variables with $F_{X_n}(x) \equiv (1 - e^{-x})1_{[0,+\infty[}(x)$.

ii) Determine the cdf of $Y_n := \max\{X_1, \dots, X_n\}$.

iii) Use ii) to prove that $Y_n - \log n$ converges in distribution, determining also the limit distribution.

Exercise 21. Show that, if X and Y are absolutely continuous independent random variables with densities f_X and f_Y respectively, then $X + Y$ is also absolutely continuous and
$$f_{X+Y}(x) = f_X * f_Y(x), \ a.e. \ x \in \mathbb{R}. \ \ (3)$$
Let now X_n, $n \in \mathbb{N}$, $n \geqslant 1$ be i.i.d. exponential random variables, $f_{X_n}(x) = \lambda e^{-\lambda x} 1_{[0,+\infty[}(x)$.

i) Determine the distribution of $X_1 + \cdots + X_n$ (hint: use FT)

ii) Let N be independent of $(X_n)_{n\in\mathbb{N}}$, with $\mathbb{P}(N = n) = (1 - p)^{n-1}p$, with $0 < p < 1$. Determine the distribution of
$$X_1 + \cdots + X_N.$$

July 2025

Exercise 22.

i) What is the characteristic function of a random variable X?
ii) Justifying carefully the calculations, show that if $\mathbb{E}[X^2] < +\infty$ then the characteristic function ϕ_X of X is twice differentiable, and compute $\partial^2_\xi \phi_X(0)$.
iii) Justifying your answer, say if there exists a r.v. X such that $\phi_X(\xi) = e^{-c\xi^4}$.

Exercise 23. Let (W_t) be a BM.

i) What are the characteristic properties of any Brownian Motion (BM)? And what are the characteristic properties of a martingale? Is the BM a martingale?
ii) Prove that $W_t^3 - 3tW_t$ is a martingale.
iii) Determine what terms you should add to W_t^4 in order to get a martingale.

Exercise 24. Suppose that (X_n) are i.i.d. random variables taking strictly positive values and such that $\mathbb{E}[|\log X_n|] < +\infty$. Discuss the limit of

$$Y_n := \left(\prod_{k=1}^{n} X_k\right)^{1/k}.$$

September 2025

Exercise 25. Let X, Y be absolutely continuous random variables with densities, resp., f_X and f_Y.

i) By using the injectivity of the L^1 FT, prove that X and Y are independent iff $\phi_{X,Y}(\xi, \eta)$ (characteristic function of random vector (X, Y)) coincides with $\phi_X(\xi)\phi_Y(\eta)$.

Let now X, Y be independent and both standard Gaussian $\mathcal{N}(0, 1)$.

ii) Show that $X + Y$ and $X - Y$ are also independent and gaussian.
iii) Compute the conditional expectation $\mathbb{E}[XY \mid X - Y]$ (hint: $(x + y)^2 - (x - y)^2 = \ldots$)

Exercise 26. Let N_k be i.i.d. random variables with $\mathbb{E}[N_k] = 0$ and $\mathbb{V}[N_k^2] \equiv \mathbb{E}[(N_k - \mathbb{E}[N_k])^2] = \sigma^2$, $\forall k \in \mathbb{N}$. We define (X_k) as

$$X_0 := x_0 \in \mathbb{R},\ X_k = \alpha X_{k-1} + N_{k-1},\ k \geqslant 1.$$

with $|\alpha| < 1$ and $x_0 \in \mathbb{R}$ fixed.

i) Calculate means $\mathbb{E}[X_k]$ and variance $\mathbb{V}[X_k]$.
ii) Let $\mathscr{F}_k := \sigma(X_1, \ldots, X_k)$. Is (X_k) a martingale w.r.t. $\mathscr{F}_k$? Justyify your answer.
iii) Compute $\mathbb{E}[(X_{k+1} - X_k)^2]$. What can you conclude about convergence in L^2 of (X_k)?
iv) Assume also that $N_k \sim \mathcal{N}(0, \sigma^2)$, $\forall k \in \mathbb{N}$. Prove that X_k converges in distribution and determine the limit distribution.

Exercise 27. Let (W_t) be a Brownian Motion (BM).

i) What are the characteristic properties of W_t?
ii) Let

$$X_t := W_t^3 - 3\int_0^t W_r\, dr,\ t \geqslant 0.$$

Check that X_t is a martingale w.r.t. $\mathscr{F}_t := \sigma(W_s\ :\ s \leqslant t)$. (if needed, you are allowed to exchange the conditional expectation with the Riemann integral $\int_0^t \ldots dr$)

Exercise 28. Let $(X, Y) \sim \mathcal{N}(0, C)$ where

$$C = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

i) Is C a well defined covariance matrix?
ii) Check that Y and $2X - Y$ are independent.
ii) Compute $\mathbb{E}[X^2 Y \mid 2X - Y]$ (hint: $X = \frac{1}{2}(2X - Y) + \frac{1}{2}Y \ldots$).

Exercise 29. i) What does the Borel-Cantelli Lemma says? Provide a precise statement (no proof is required).

Let now X_n be independent Bernoulli random variables with

$$\mathbb{P}(X_n = 1) = \frac{1}{\sqrt{n}}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{\sqrt{n}}.$$

i) Let $E := \{X_n = X_{n+1} = X_{n+2} = 1, \text{ for infinitely many } n\}$. Check that E is an event and prove that $\mathbb{P}(E) = 0$.
ii) Let $F := \{X_n = X_{n+1} = 1, \text{ for infinitely many } n\}$. Check that F is an event. What about $\mathbb{P}(F)$?

Exercise 30. Let $X_k \sim U([0,1])$ i.i.d. random variables, and let $S_n := \sum_{k=1}^n X_k$. Define

$$N := \min\{n \geqslant 2 \ : \ S_n > 1\}$$

i) Compute $\mathbb{P}(N > n)$ and $\mathbb{P}(N = n)$ for $n \in \mathbb{N}$.
ii) Compute $\mathbb{E}[N]$ and $\mathbb{V}[N]$.
iii) Compute $\mathbb{E}[S_N]$.

It may be helpful to know that

$$I_n^j := \int_{0 \leqslant x_1, \ldots, x_n \leqslant 1, \ x_1 + \cdots + x_n \leqslant 1} (x_1 + \cdots + x_n)^j \, dx_1 \cdots dx_n = \frac{1}{(n-1)!(n+j)}.$$

Solutions.

Exercise 2. i) We have

$$\phi_X(\xi) = \widehat{f_X}(-\xi) = \int_{\mathbb{R}} f_X(x) e^{i\xi x}\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} e^{i\xi x}\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{-(\lambda - i\xi)x}\, dx.$$

Now, being $x^{\alpha-1} = \partial_x \frac{x^\alpha}{\alpha}$, by integrating by parts we have

$$\int_0^{+\infty} \partial_x \left(\frac{x^\alpha}{\alpha}\right) e^{-(\lambda-i\xi)x}\, dx = \frac{1}{\alpha}\left(\left[x^\alpha e^{-(\lambda-i\xi)x}\right]_{x=0}^{x=+\infty} + (\lambda - i\xi)\int_0^{+\infty} x^\alpha e^{-(\lambda-i\xi)x}\, dx\right)$$

and since $\left[x^\alpha e^{-(\lambda-i\xi)x}\right]_{x=0}^{x=+\infty} = 0$ we get

$$\phi_X(\xi) = \frac{\lambda - i\xi}{\alpha} \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} e^{i\xi x}\, dx.$$

On the other hand

$$\partial_\xi \phi_X(\xi) = i \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} e^{i\xi x}\, dx = i \frac{\alpha}{\lambda - i\xi} \phi_X(\xi) = -\alpha \frac{1}{\xi + i\lambda} \phi_X(\xi)$$

So,

$$\log \phi_X(\xi) = -\alpha \log(\xi + i\lambda) + k,$$

from which

$$\phi_X(\xi) = K(\xi + i\lambda)^{-\alpha}.$$

Now, since $\phi_X(0) = 1$, we have $1 = K(i\lambda)^{-\alpha}$, that is $K = (i\lambda)^\alpha$, from which we obtain

$$\phi_X(\xi) = (i\lambda)^\alpha (\xi + i\lambda)^{-\alpha} = \left(1 + \frac{\xi}{i\lambda}\right)^{-\alpha} = \left(1 - i\frac{\xi}{\lambda}\right)^{-\alpha}.$$

ii) If $X_j \sim \Gamma(\alpha_j, \lambda)$ are independent, then

$$\phi_{X_1+\cdots+X_n}(\xi) = \prod_{j=1}^n \phi_{X_j}(\xi) = \prod_{j=1}^n \left(1 - i\frac{\xi}{\lambda}\right)^{-\alpha_j} = \left(1 - i\frac{\xi}{\lambda}\right)^{-(\alpha_1+\cdots+\alpha_n)}.$$

From this, and from the uniqueness of the FT, $X_1 + \cdots + X_n \sim \Gamma(\alpha_1 + \cdots + \alpha_n, \lambda)$.

iii) We have

$$\mathbb{E}[\psi(X) \mid Z] = \varphi(Z),$$

where

$$\varphi(z) = \int_{\mathbb{R}} \psi(x) f_{X|Z}(x|z)\, dx,$$

with

$$f_{X|Z}(x|z) = \frac{f_{XZ}(x,z)}{f_Z(z)},$$

(provided XZ is abs. cont.). Now, $(X, Z) = (X, X + Y) = T(X, Y)$ where $T(x, y) = (x, x + y)$ is clearly a bijection on $\mathbb{R}^2$, so

$$f_{XZ}(x,z) = f_{XY}(T^{-1}(x,z))|\det(T^{-1})'(x,z)|.$$

Now $(x, z) = (x, x + y)$ iff $(x, y) = (x, z - x)$, $\det(T^{-1})' = (\det T')^{-1} = 1$, so

$$f_{XZ}(x, z) = f_{XY}(x, z - x) \overset{indep}{=} f_X(x) f_Y(z - x).$$

Since $X, Y \sim \Gamma(1, \lambda)$ we have

$$f_X(x) = \lambda e^{-\lambda x} 1_{[0,+\infty[},$$

and, by ii), about f_Z we have

$$f_Z(z) = \frac{\lambda^2}{\Gamma(2)} z e^{-\lambda z} 1_{[0,+\infty[}(z) = \lambda^2 z e^{-\lambda z} 1_{[0,+\infty[}(z).$$

Therefore

$$f_{X|Z}(x|z) = \frac{\lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} 1_{[0,+\infty[}(x) 1_{[0,+\infty[}(z - x)}{\lambda^2 z e^{-\lambda z} 1_{[0,+\infty[}(z)} = \frac{1}{z} 1_{[0,+\infty[}(z) 1_{[0,z]}(x),$$

from which we obtain

$$\varphi(z) = \int_{\mathbb{R}} \psi(x) \frac{1}{z} 1_{[0,+\infty[}(z) 1_{[0,z]}(x)\ dx = \left(\frac{1}{z} \int_0^z \psi(x)\ dx \right) 1_{[0,+\infty[}(z).$$

So, since $Z \geq 0$ with probability 1,

$$\varphi(Z) = \frac{1}{Z} \int_0^Z \psi(x)\ dx. \qquad \square$$

Exercise 3. See notes for definitions and relations between convergence in distribution and in probability.

i) We use characteristic functions. We notice that

$$X_n = \frac{1}{2} X_{n-1} + Y_n = \frac{1}{2} \left(\frac{1}{2} X_{n-2} + Y_{n-1} \right) + Y_n = \frac{1}{2^2} X_{n-2} + \sum_{k=0}^{1} \frac{1}{2^k} Y_{n-k}$$

Iterating, after n steps we arrive at formula

$$X_n = \frac{1}{2^n} X_0 + \sum_{k=0}^{n-1} \frac{1}{2^k} Y_{n-k}$$

Because of the assumptions on independence

$$\phi_{X_n}(\xi) = \phi_{\frac{1}{2^n} X_0}(\xi) \prod_{k=0}^{n-1} \phi_{\frac{1}{2^k} Y_{n-k}}(\xi) = \phi_{X_0}\left(\frac{\xi}{2^n} \right) \prod_{k=0}^{n-1} e^{-\frac{1}{2}\left(\frac{\xi}{2^k}\right)^2} = \phi_{X_0}\left(\frac{\xi}{2^n} \right) e^{-\frac{1}{2}\left(\sum_{k=0}^{n-1} \frac{1}{4^k}\right)\xi^2}$$

Letting $n \longrightarrow +\infty$ we have

$$\phi_{X_n}(\xi) \longrightarrow \phi_{X_0}(0) e^{-\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{1}{4^k}\right)\xi^2} = 1 \cdot e^{-\frac{1}{2} \cdot \frac{4}{3} \xi^2} = \phi_{\mathcal{N}(0, \frac{4}{3})}(\xi).$$

We conclude that $X_n \overset{d}{\longrightarrow} X \sim \mathcal{N}(0, \frac{4}{3})$.

ii) Since convergence in probability is stronger than convergence in distribution, if (X_n) converges in probability then it converges also in distribution to the same limit. Thus, the unique possibility is $X_n \xrightarrow{\mathbb{P}} X$. But then also $X_{n-1} \xrightarrow{\mathbb{P}} \mathbb{P}$, whence, being

$$X_n = \frac{X_{n-1}}{2} + Y_n, \implies Y_n = X_n - \frac{X_{n-1}}{2} \xrightarrow{\mathbb{P}} \frac{X}{2}, \implies Y_n \xrightarrow{d} \frac{X}{2} \neq X. \quad \square$$

Exercise 4. i) Let

$$Y_n := \min\{X_1, \dots, X_n\}.$$

We have

$$\{Y_n \leqslant y\} = \{\min(X_1, \dots, X_n) \leqslant y\} = \bigsqcup_{k=1}^{n} \{X_1 > y, \dots, X_{k-1} > y, X_k \leqslant y\}$$

so

$$\mathbb{P}(Y_n \leqslant y) = \sum_{k=1}^{n} \mathbb{P}(X_1 > y, \dots, X_{k-1} > y, X_k \leqslant y) = \sum_{k=1}^{n} \prod_{j=1}^{k-1} \mathbb{P}(X_j > y) \cdot \mathbb{P}(X_k \leqslant y),$$

with the agreement that $\prod_{j=1}^{0} = 1$. Since $X_j \sim \exp(\lambda)$, we have

$$F_{X_j}(x) = \mathbb{P}(X_j \leqslant x) = (1 - e^{-\lambda x}) 1_{[0,+\infty[}(x),$$

for $y \geqslant 0$ we have

$$\begin{aligned} F_{Y_n}(y) = \mathbb{P}(Y_n \leqslant y) \quad &= \sum_{k=1}^{n} (e^{-\lambda y})^{k-1} (1 - e^{-\lambda y}) = \sum_{k=0}^{n-1} (e^{-\lambda y})^{k} (1 - e^{-\lambda y}) \\ &= (1 - e^{-\lambda y}) \frac{1 - (e^{-\lambda y})^n}{1 - e^{-\lambda y}} = 1 - e^{-n\lambda y}. \end{aligned}$$

Clearly, $F_{Y_n}(y) = 0$ for $y \leqslant 0$.

ii) We notice that

$$\{Y_N \leqslant y\} = \bigsqcup_{n=1}^{\infty} \{Y_N \leqslant y,\ N = n\} = \bigsqcup_{n=1}^{\infty} \{Y_n \leqslant y,\ N = n\}.$$

Thus,

$$F_{Y_N}(y) = \mathbb{P}(Y_N \leqslant y) = \sum_{n=1}^{\infty} \mathbb{P}(Y_n \leqslant y,\ N = n).$$

Since N is independent of the (X_n), N is independent of Y_n for every n, so we have

$$\begin{aligned} \mathbb{P}(Y_n \leqslant y,\ N = n) \quad &= \mathbb{P}(Y_n \leqslant y,\ N = n) = \mathbb{P}(Y_n \leqslant y)\mathbb{P}(N = n) = (1-p)^{n-1} p \mathbb{P}(Y_n \leqslant y) \\ &= p(1-p)^{n-1}(1 - e^{-n\lambda y}). \end{aligned}$$

Therefore,

$$F_Y(y) = \sum_{n=1}^{\infty} p(1-p)^{n-1}(1-e^{-n\lambda y}) = \underbrace{\sum_{n=1}^{\infty} p(1-p)^{n-1}}_{=1} - pe^{-\lambda y}\sum_{n=1}^{\infty}\Big((1-p)e^{-\lambda y}\Big)^{n-1}$$

$$= 1 - pe^{-\lambda y}\frac{1}{1-(1-p)e^{-\lambda y}} = \frac{1-e^{-\lambda y}}{1-(1-p)e^{-\lambda y}}.$$

For $y < 0$ clearly $F_Y(y) = 0$.

iii) We have

$$\mathbb{E}[Y_N] = \int_0^{+\infty} y f_{Y_N}(y)\, dy,$$

where

$$f_{Y_N}(y) = \partial_y F_{Y_N}(y) = -\partial_y(1 - F_{Y_N}(y)).$$

Integrating by parts,

$$\mathbb{E}[Y_N] = \underbrace{\big[-y(1-F_{Y_N}(y)\big]_{y=0}^{y=+\infty}}_{=0} + \int_0^{+\infty} 1 - F_{Y_N}(y)\, dy = \int_0^{+\infty} \frac{pe^{-\lambda y}}{1-(1-p)e^{-\lambda y}}\, dy.$$

Setting $u = e^{-\lambda y}$ (that is $y = -\frac{1}{\lambda}\log u$) we obtain

$$\mathbb{E}[Y_N] = \frac{1}{\lambda}\int_0^1 \frac{pu}{1-(1-p)u}\frac{du}{u} = -\frac{p}{\lambda(1-p)}\left[\log(1-(1-p)u)\right]_{u=0}^{u=1} = -\frac{p\log p}{\lambda(1-p)}. \qquad \square$$

Exercise 5. i) Notice that $(R,\Theta) = \Psi(U,V)$ where $\Psi(u,v) = (\sqrt{-2\log u}, 2\pi v)$. Since $(U,V) \in [0,1]^2$, and since $\mathbb{P}(U=0) = 0$ and same for V, actually $(U,V) \in]0,1]^2$ with probability 1, we consider $\Psi :]0,1]^2 \longrightarrow \Psi(]0,1]^2) \subset [0,+\infty[\times]0,2\pi]$. Ψ is invertible and $(r,\theta) = \Psi(u,v)$ iff $r = \sqrt{-2\log u}$, $\theta = 2\pi v$, that is $u = e^{-\frac{r^2}{2}}$ and $v = \frac{\theta}{2\pi}$, so $\Psi^{-1}(r,\theta) = (e^{-\frac{r^2}{2}}, \theta/2\pi)$. According to the change of variable formula, we have

$$f_{R\Theta}(r,\theta) = f_{UV}(\Psi^{-1}(r,\theta))|\det(\Psi^{-1})'(r,\theta)| = 1_{[0,1]}(e^{-\frac{r^2}{2}})1_{[0,1]}(\tfrac{\theta}{2\pi})\left|\det\begin{bmatrix} -re^{-\frac{r^2}{2}} & 0 \\ 0 & \frac{1}{2\pi}\end{bmatrix}\right|$$

$$= |r|e^{-\frac{r^2}{2}} \cdot \tfrac{1}{2\pi}1_{[0,2\pi]}(\theta)$$

In particular,

$$f_R(r) = |r|e^{-\frac{r^2}{2}}, \quad f_\Theta(\theta) = \frac{1}{2\pi}1_{[0,2\pi]}(\theta).$$

ii) $(X,Y) = \Psi(R,\Theta)$ where $\Psi(r,\theta) = (r\cos\theta, r\sin\theta)$ is the usual polar coordinate map. We have

$$f_{XY}(x,y) = f_{R\Theta}(\Psi^{-1}(x,y))|\det(\Psi^{-1})'(x,y)|.$$

Since $\det \Psi'(\rho,\theta) = \rho$ and

$$\Psi^{-1}(x,y) = \left(\sqrt{x^2+y^2}, \theta(x,y)\right), \quad \text{where } \theta(x,y) = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}}, & y > 0, \\ \arccos \frac{x}{\sqrt{x^2+y^2}} + \pi, & y < 0 \end{cases}$$

we deduce

$$f_{XY}(x,y) = |\sqrt{x^2+y^2}| e^{-\frac{x^2+y^2}{2}} \frac{1}{2\pi} \frac{1}{|\sqrt{x^2+y^2}|} = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

from which $(X,Y) \sim \mathcal{N}(0,\mathbb{I})$ (where $\mathbb{I}$ is the identity matrix). Clearly X,Y are independent, each distributed as a standard Gaussian. □

Exercise 6. i) We recall that $X_n \xrightarrow{a.s.} 0$ iff

$$\forall \varepsilon > 0, \ \mathbb{P}\left(\bigcap_N \bigcup_{n \geqslant N} |X_n| \geqslant \varepsilon\right) = 0.$$

According to Borel–Cantelli's Lemma, a sufficient condition for this happens is

$$\sum_n \mathbb{P}(|X_n| \geqslant \varepsilon) < +\infty.$$

We notice that,

$$X_n = \left(1 + \frac{Y_n}{2}\right) X_{n-1} = \left(1 + \frac{Y_n}{2}\right)\left(1 + \frac{Y_{n-1}}{2}\right) X_{n-2} = \ldots = \prod_{k=0}^{n} \left(1 + \frac{Y_k}{2}\right) X_0 = \prod_{k=0}^{n} \left(1 + \frac{Y_k}{2}\right) \geqslant 0.$$

For convenience, let $\varepsilon = \frac{1}{2^m}$. So

$$\{|X_n| \geqslant \varepsilon\} = \{X_n \geqslant \varepsilon\} = \left\{\prod_{k=0}^{n} \left(1 + \frac{Y_k}{2}\right) \geqslant \frac{1}{2^m}\right\}$$

Notice that, for $n > m$,

$$X_n = \frac{1}{2^n} \prod_{k=0}^{n} (2 + Y_k) \geqslant \frac{1}{2^m}, \quad \Longleftrightarrow \quad \prod_{k=0}^{n} (2 + Y_k) \geqslant 2^{n-m},$$

and this happens iff at least $n-m$ of the $Y_k = 1$. Because of independence

$$\mathbb{P}(Y_{k_1} = 1, \ldots, Y_{k_{n-m}} = 1) = \frac{1}{2^{n-m}},$$

so

$$\mathbb{P}\left(X_n \geqslant \frac{1}{2^m}\right) \leqslant \binom{n}{n-m} \frac{1}{2^{n-m}} = \frac{n!}{(n-m)!m!} \frac{1}{2^n} 2^m$$

and

$$\sum_n \mathbb{P}\left(X_n \geqslant \frac{1}{2^m}\right) \leqslant \frac{2^m}{m!} \sum_{n \geqslant m} \frac{n!}{2^n (n-m)!}.$$

Now the last series converges by root test because, if $a_n := \frac{n!}{2^n(n-m)!}$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n+1}(n+1-m)!}\frac{2^n(n-m)!}{n!} = \frac{1}{2}\frac{n+1}{n+1-m} \longrightarrow \frac{1}{2} < 1.$$

Thus, we conclude that

$$\sum_n \mathbb{P}\left(X_n \geqslant \frac{1}{2^m}\right) < +\infty,$$

and Borel-Cantelli's Lemma applies.

ii) Since

$$X_n = \prod_{k=0}^{n}\left(1+\frac{Y_k}{2}\right),$$

by the independence of Y_k we have

$$\mathbb{E}[X_n] = \prod_{k=0}^{n}\mathbb{E}\left[1+\frac{Y_k}{2}\right] = \prod_{k=0}^{n}\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{2}\right) = \prod_{k=0}^{n} 1 = 1,$$

so $\lim_n \mathbb{E}[X_n] = 1$. We conclude that $X_n \not\longrightarrow 0$ in L^1. Indeed, if this would happen,

$$1 = |\mathbb{E}[X_n]| \leqslant \mathbb{E}[|X_n|] = \|X_n\|_1 \longrightarrow 0,$$

which is impossible. □

Exercise 7. i) Recalling of the monotonicity property of the conditional expectation, we have

$$-|X| \leqslant X \leqslant |X|,\ a.s. \implies -\mathbb{E}[|X| \mid \mathscr{G}] \leqslant \mathbb{E}[X \mid \mathscr{G}] \leqslant \mathbb{E}[|X| \mid \mathscr{G}],$$

from which

$$|\mathbb{E}[X \mid \mathscr{G}]| \leqslant \mathbb{E}[|X| \mid \mathscr{G}].$$

ii) We notice that

$$\mathbb{E}[X+Y\ |Y] = \mathbb{E}[X \mid Y] + \mathbb{E}[Y \mid Y] = Y.$$

So,

$$|Y| = |\mathbb{E}[X+Y\ |Y]| \leqslant \mathbb{E}[|X+Y| \mid Y],\ a.s.,$$

and, by taking expectations,

$$\mathbb{E}[|Y|] \leqslant \mathbb{E}\left[\mathbb{E}[|X+Y| \mid Y]\right] = \mathbb{E}[|X+Y|],$$

which is the conclusion.

iii) We notice that if $A := \mathbb{E}[X-Y \mid X+Y]$ then A is characterized by

$$\begin{aligned}\mathbb{E}[A\varphi(X+Y)] &= \mathbb{E}[(X-Y)\varphi(X+Y)] = \int_{\mathbb{R}^2}(x-y)\varphi(x+y)\ d\mu_{XY}(x,y)\\ &= \int_{\mathbb{R}^2}(x-y)\varphi(x+y)\ d\mu_{YX}(x,y) = \mathbb{E}[(Y-X)\varphi(Y+X)]\\ &= -\mathbb{E}[A\varphi(X+Y)],\end{aligned}$$

from which

$$2\mathbb{E}[A\varphi(X+Y)] = 0,\ \forall\varphi(X+Y) \in L^\infty_{X+Y}.$$

In particular, since $A \in L^1_{X+Y}$ we conclude that $A = 0$ a.s., as desired.

Now, setting $Z = X+Y$ and W in such a way that $3X-Y = Z+W$, thus $W = (3X-Y)-(X+Y) = 2(X-Y)$, being $\mathbb{E}[Z \mid W] = \mathbb{E}[X+Y \mid X-Y] = 0$, by ii) we get

$$\|W\|_1 \leqslant \|Z+W\|_1, \quad \Longleftrightarrow \quad 2\|X-Y\|_1 \leqslant \|3X-Y\|_1,$$

from which the conclusion follows. □

Exercise 8. Let $\widetilde{X}_k := \frac{k}{k+1} X_k \sim U[-1,1]$. Then

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(1 + \frac{1}{k}\right) \widetilde{X}_k.$$

So

$$\phi_{Y_n}(\xi) = \mathbb{E}\left[e^{i \frac{\xi}{\sqrt{n}} \sum_k \left(1+\frac{1}{k}\right)\widetilde{X}_k}\right] \overset{indep}{=} \prod_{k=1}^{n} \phi\left(\frac{\xi}{\sqrt{n}}\left(1 + \frac{1}{k}\right)\right),$$

where

$$\phi(\eta) = \phi_{\widetilde{X}}(\eta) = \frac{\sin \eta}{\eta}.$$

Now, since for ξ fixed $\frac{\xi}{\sqrt{n}}\left(1+\frac{1}{k}\right) \longrightarrow 0$, recalling that

$$\phi(\eta) = \phi(0) + \phi'(0)\eta + \frac{1}{2}\phi''(0)\eta^2 + o(\eta^2),$$

and being,

$\phi(\eta) = \frac{\sin \eta}{\eta}$, $\qquad \phi(0) = 1$,

$\phi'(\eta) = \frac{\eta \cos \eta - \sin \eta}{\eta^2}$, $\qquad \phi'(0) = \lim_{\eta \to 0} \phi'(\eta) \overset{H}{=} \lim_{\eta \to 0} \frac{\cos \eta - \eta \sin \eta - \cos \eta}{2\eta} = 0$,

$\phi''(\eta) = \frac{-\eta^2 \sin \eta - 2(\eta \cos \eta - \sin \eta)}{\eta^3}$ $\qquad \phi''(0) = \lim_{\eta \to 0} \frac{-\eta^2(\eta + o(\eta)) - 2\left(\eta\left(1 - \frac{\eta^2}{2} + o(\eta^2)\right) - \left(\eta - \frac{\eta^3}{6} + o(\eta^3)\right)\right)}{\eta^3} = -\frac{1}{3}$

we have

$$\phi(\eta) = 1 - \frac{\eta^2}{6} + o(\eta^2).$$

Therefore,

$$\phi_{Y_n}(\xi) = \prod_{k=1}^{n} \left(1 - \frac{1}{6}\frac{\xi^2}{n}\left(1 + \frac{1}{k}\right)^2 + o\left(\frac{1}{n}\right)\right) \sim \left(1 - \frac{1}{6}\frac{\xi^2}{n}\right)^n \longrightarrow e^{-\frac{1}{6}\xi^2}$$

Therefore

$$Y_n \overset{d}{\longrightarrow} \mathcal{N}(0, 1/3).$$

Alternative solution. We may notice that

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(1 + \frac{1}{k}\right) \widetilde{X}_k = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \widetilde{X}_k + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k} \widetilde{X}_k.$$

Now, by the CLT

$$W_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \widetilde{X}_k \xrightarrow{d} \mathcal{N}(0, 1/3).$$

We claim that

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k} \widetilde{X}_k \xrightarrow{a.s.} 0.$$

If this is true we have $Y_n = W_n + Z_n \xrightarrow{d} \mathcal{N}(0, 1/3)$. Indeed

$$\phi_{Y_n}(\xi) = \mathbb{E}[e^{i\xi W_n} e^{i\xi Z_n}] = \mathbb{E}\left[\left(e^{i\xi Z_n} - 1\right) e^{i\xi W_n}\right] + \mathbb{E}[e^{i\xi W_n}].$$

By dominated convergence the first expectation goes to 0 while the second goes to $e^{-\frac{1}{6}\xi^2}$.

To prove that $Z_n \xrightarrow{a.s.} 0$ we notice that, being $|\widetilde{X}_n| \leqslant 1$ with probability 1,

$$|Z_n| \leqslant \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{n} \leqslant \frac{1}{\sqrt{n}} \left(1 + \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}\, dx\right) = \frac{1}{\sqrt{n}} (1 + \log n) \longrightarrow 0. \quad \square$$

Exercise 9. i) We have

$$\phi_{B_t}(\xi) = \mathbb{E}[e^{i\xi t W_{1/t}}] = e^{-\frac{1}{2}\frac{1}{t}(t\xi)^2} = e^{-\frac{1}{2}t\xi^2} \implies B_t \sim \mathcal{N}(0, t).$$

If $0 < s < t$, then $0 < \frac{1}{t} < \frac{1}{s}$ and

$$\begin{aligned}
(B_s, B_t - B_s) &= \left(sW_{1/s}, tW_{1/t} - sW_{1/s}\right) = \left(s\left(W_{1/s} - W_{1/t}\right) + sW_{1/t}, (t-s)W_{1/t} - s(W_{1/s} - W_{1/t})\right) \\
&= s(1, -1)(W_{1/s} - W_{1/t}) + (s, t-s)W_{1/t}.
\end{aligned}$$

Being $W_{1/t}$ independent of $W_{1/s} - W_{1/t}$ we have that

$$\begin{aligned}
\mathbb{E}\left[e^{i(\xi,\eta)(B_s, B_t - B_s)}\right] &= \mathbb{E}\left[e^{is(\xi-\eta)(W_{1/s} - W_{1/t})} e^{i(s\xi + (t-s)\eta)W_{1/t}}\right] \\
&= \mathbb{E}\left[e^{is(\xi-\eta)(W_{1/s} - W_{1/t})}\right] \mathbb{E}\left[e^{i(s\xi + (t-s)\eta)W_{1/t}}\right] \\
&= e^{-\frac{1}{2}\left(\frac{1}{s} - \frac{1}{t}\right)s^2(\xi-\eta)^2} e^{-\frac{1}{2}\frac{1}{t}(t\eta + s(\xi-\eta))^2} \\
&= e^{-\frac{1}{2}\frac{t-s}{t}s(\xi^2+\eta^2-2\xi\eta)} e^{-\frac{1}{2}\frac{1}{t}(t^2\eta^2 + s^2(\xi^2+\eta^2-2\xi\eta) + 2st\eta(\xi-\eta)} \\
&= e^{-\frac{1}{2}\left[s(\xi^2+\eta^2-2\xi\eta) + t\eta^2 + 2s\eta(\xi-\eta)\right]} \\
&= e^{-\frac{1}{2}\left(s\xi^2 + (t-s)\eta^2\right)} = e^{-\frac{1}{2}s\xi^2} e^{-\frac{1}{2}(t-s)\eta^2},
\end{aligned}$$

that, at once, says $B_s \sim \mathcal{N}(0, s)$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$ and they are independent. Finally, $B_t \in \mathscr{C}(]0, +\infty[)$ being $W_t \in \mathscr{C}([0, +\infty[)$.

ii) We have

$$\begin{aligned}\mathbb{P}\left(|B_t| \geqslant \varepsilon\right) &= \mathbb{P}\left(\left|tW_{1/t}\right| \geqslant \varepsilon\right) = \mathbb{P}\left(|W_{1/t}| \geqslant \tfrac{\varepsilon}{t}\right) = 2 \cdot \int_{\varepsilon/t}^{+\infty} e^{-\frac{x^2}{2(1/t)}} \frac{dx}{\sqrt{2\pi\frac{1}{t}}} \\ &\overset{y=\sqrt{t}x}{=} 2\int_{\varepsilon/\sqrt{t}}^{+\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \longrightarrow 0,\ t \to 0+. \quad \square\end{aligned}$$

Exercise 10. i) Notice that $Z, W \in \mathbb{N}$. So, let $z, w \in \mathbb{N}$ and let's compute

$$\mathbb{P}(Z = z,\ W = w) = \mathbb{P}\left(\min(X, Y) = z,\ |X - Y| = w\right).$$

We distinguish $w = 0$ from $w \geqslant 1$. If $w = 0$,

$$\begin{aligned}\mathbb{P}(Z = z,\ W = 0) &= \mathbb{P}\left(\min(X, Y) = z,\ |X - Y| = 0\right) = \mathbb{P}(X = Y, X = z) = \mathbb{P}(X = z,\ Y = z) \\ &\overset{indep}{=} \mathbb{P}(X = z)\mathbb{P}(Y = z) = \left((1-p)^{z-1}p\right)^2 = (1-p)^{2z-2}p^2.\end{aligned}$$

If $w \geqslant 1$, noticed that $|X - Y| = w$ iff $X = Y \pm w$ we have

$$\begin{aligned}\mathbb{P}(Z = z,\ W = w) &= \mathbb{P}\left(\min(X, Y) = z,\ X = Y \pm w\right) \\ &= \mathbb{P}(X = Y - w,\ X = z) + \mathbb{P}(X = Y + w,\ Y = z) \\ &= \mathbb{P}(X = z,\ Y = z + w) + \mathbb{P}(Y = z,\ X = z + w) \\ &\overset{indep}{=} \mathbb{P}(X = z)\mathbb{P}(Y = z + w) + \mathbb{P}(Y = z)\mathbb{P}(X = z + w) \\ &= 2(1-p)^{z-1}p(1-p)^{z+w-1}p \\ &= 2(1-p)^{2z+w-2}p^2.\end{aligned}$$

ii) Z and W are independent iff

$$\mathbb{P}(Z = z,\ W = w) = \mathbb{P}(Z = z)\mathbb{P}(W = w).$$

From previous calculation we have

$$\begin{aligned}
\mathbb{P}(Z=z) &= \mathbb{P}\left(\bigsqcup_{w} Z=z, W=w\right) = \sum_{w=0}^{\infty} \mathbb{P}(Z=z, W=w) \\
&= (1-p)^{2z-2}p^2 + \sum_{w=1}^{\infty} 2(1-p)^{2z+w-2}p^2 = p^2(1-p)^{2z-2}\left(1+2\sum_{w=1}^{\infty}(1-p)^w\right) \\
&= p^2(1-p)^{2z-2}\left(1+2\left(\frac{1}{1-(1-p)}-1\right)\right) \\
&= p^2(1-p)^{2z-2}\frac{p+2(1-p)}{p} = p(1-p)^{2z-2}(2-p).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}(W=w) &= \mathbb{P}\left(\bigsqcup_{z\geqslant 1} Z=z, W=w\right) = \sum_{w=1}^{\infty} \mathbb{P}(Z=z, W=w) \\
&= \begin{cases} w=0, & \sum_{z=1}^{\infty}(1-p)^{2z-2}p^2 = p^2\frac{1}{1-(1-p)^2} = \frac{p}{2-p}, \\ w\geqslant 1 & 2p^2(1-p)^w\sum_{z=1}^{\infty}(1-p)^{2z-2} = 2(1-p)^w\frac{p}{2-p} \end{cases}
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{P}(Z=z)\mathbb{P}(W=w) &= \begin{cases} w=0, & p(1-p)^{2z-2}(2-p)\cdot\frac{p}{2-p} = p^2(1-p)^{2z-2}, \\ w\geqslant 1, & p(1-p)^{2z-2}(2-p)\cdot 2(1-p)^w\frac{p}{2-p} = 2p^2(1-p)^{2z+w-2} \end{cases} \\
&= \mathbb{P}(Z=z, W=w).
\end{aligned}$$

We conclude that Z and W are independent. □

Exercise 11. See notes for Borel–Cantelli's Lemma.
i) We prove that

$$\mathbb{P}\left(\bigcap_N \bigcup_n \{Y_n \geqslant 1+\varepsilon\}\right) = 0.$$

To this aim we apply first Borel–Cantelli's Lemma proving that

$$\sum_n \mathbb{P}\left(Y_n \geqslant 1+\varepsilon\right) < +\infty.$$

We have

$$\mathbb{P}\left(Y_n \geqslant 1+\varepsilon\right) = \mathbb{P}\left(X_n \geqslant \log n + (1+\varepsilon)\log(\log n) = \log\left(n(\log n)^{1+\varepsilon}\right)\right).$$

Being $X_n \sim \exp(1)$, we have

$$\mathbb{P}(X_n \geqslant \alpha) = e^{-\alpha},$$

so

$$\mathbb{P}(Y_n \geqslant 1+\varepsilon) = e^{-\log\left(n(\log n)^{1+\varepsilon}\right)} = \frac{1}{n(\log n)^{1+\varepsilon}},$$

from which

$$\sum_n \mathbb{P}(Y_n \geqslant 1+\varepsilon) = \sum_n \frac{1}{n(\log n)^{1+\varepsilon}} < +\infty.$$

ii) According to second Borel–Cantelli's Lemma, if the events E_n are independent and

$$\sum_n \mathbb{P}(E_n) = +\infty, \implies \mathbb{P}\left(\bigcap_N \bigcup_n E_n\right) = 1.$$

So, since the Y_n are independent, to prove that

$$\mathbb{P}\left(\bigcap_N \bigcup_n \{Y_n \geqslant 1-\varepsilon\}\right) = 1$$

we just need to verify that

$$\sum_n \mathbb{P}(\{Y_n \geqslant 1-\varepsilon\}) = +\infty.$$

By a calculation similar to that of i),

$$\mathbb{P}(Y_n \geqslant 1-\varepsilon) = \frac{1}{n(\log n)^{1-\varepsilon}}, \implies \sum_n \mathbb{P}(Y_n \geqslant 1-\varepsilon) = \sum_n \frac{1}{n(\log n)^{1-\varepsilon}} = +\infty,$$

and the conclusion follows. □

Exercise 13. i) We have

$$\lambda_2(A) = \int_{|x-y|<a,\ |x+y|<b} dxdy \overset{u=x-y, v=x+y}{=} \frac{1}{2}\int_{|u|<a,|v|<b} dudv = 2ab,$$

so

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f_{X,Y}(x,y)\,dx = \frac{1}{2ab}\int_{\mathbb{R}} 1_{y-a\leqslant x\leqslant y+a,\ -y-b\leqslant x\leqslant -y+b}(x)\,dx \\ &= \frac{1}{2ab}\int_{\mathbb{R}} 1_{\max(y-a,-y-b)\leqslant x\leqslant \min(y+a,-y+b)}\,dx = \frac{M(y)-m(y)}{2ab} 1_{m(y)\leqslant M(y)} \end{aligned}$$

where

$$m(y) = \max(y-a,-y-b) = \begin{cases} -y-b, & y < -\frac{b-a}{2}, \\ y-a, & y > -\frac{b-a}{2}. \end{cases} \quad M(y) = \min(y+a,-y+b) = \begin{cases} y+a, & y < \frac{b-a}{2}, \\ -y+b & y > \frac{b-a}{2}. \end{cases}$$

Returning to $f_Y(y)$ we have

$$f_Y(y) = \begin{cases} y < -\frac{b-a}{2}, & = \frac{1}{2ab}\int_{\mathbb{R}} 1_{[-y-b,y+a]}(x)\, dx, \\ -\frac{b-a}{2} \leqslant y < \frac{b-a}{2}, & = \frac{1}{2ab}\int_{\mathbb{R}} 1_{[y-a,y+a]}(x)\, dx = \frac{1}{b}, \\ y \geqslant \frac{b-a}{2}, & = \frac{1}{2ab}\int_{\mathbb{R}} 1_{[y-a,-y+b]}(x)\, dx. \end{cases}$$

Now, $-y-b < y+a$ iff $y > -\frac{a+b}{2}$, and since $-\frac{a+b}{2} < -\frac{b-a}{2}$ we have

$$f_Y(y) = \begin{cases} 0, & y < -\frac{a+b}{2}, \\ \frac{1}{2ab}(2y+a+b) = \frac{1}{ab}\left(y + \frac{a+b}{2}\right), & -\frac{a+b}{2} \leqslant y < -\frac{b-a}{2}. \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} 0, & y > \frac{a+b}{2}, \\ \frac{1}{ab}\left(\frac{a+b}{2} - y\right), & \frac{b-a}{2} \leqslant y < \frac{b+a}{2}. \end{cases}$$

We conclude that

$$f_Y(y) = \begin{cases} 0, & y < -\frac{a+b}{2}, \\ \frac{1}{2ab}(2y+a+b) = \frac{1}{ab}\left(y + \frac{a+b}{2}\right), & -\frac{a+b}{2} \leqslant y < -\frac{b-a}{2}, \\ \frac{1}{b}, & -\frac{b-a}{2} \leqslant y < \frac{b-a}{2}, \\ \frac{1}{ab}\left(\frac{a+b}{2} - y\right), & \frac{b-a}{2} \leqslant y < \frac{b+a}{2}, \\ 0, & y > \frac{a+b}{2}. \end{cases}$$

ii) We have

$$\mathbb{E}[X \mid Y = y] = 0, \ |y| > \frac{a+b}{2},$$

while, for $|y| \leqslant \frac{a+b}{2}$,

$$\mathbb{E}[X \mid Y = y] = \int_{\mathbb{R}} x f_{X|Y}(x \mid y)\, dx = \frac{1}{f_Y(y)}\frac{1}{2ab}\int_{\mathbb{R}} x 1_{\max(y-a,-y-b) \leqslant x \leqslant \min(y+a,-y+b)}\, dx.$$

Denoting by $m(y) := \max(y-a, -y-b)$ and $M(y) := \min(y+a, -y+b)$, we have

$$\frac{1}{M(y)-m(y)}\int_{\mathbb{R}} x 1_{\max(y-a,-y-b) \leqslant x \leqslant \min(y+a,-y+b)}\, dx = \frac{m(y)+M(y)}{2},$$

and since $f_Y(y) = \frac{1}{2ab}(M(y) - m(y))$, we have

$$\mathbb{E}[X \mid Y = y] = \frac{m(y)+M(y)}{2}.$$

Here we have the following cases:

- case $a = b$. Then $m(y) = \max(y - a, -y - a) = |y| - a$, $M(y) = \min(y + a, -y + a) = -|y| + a$ so
$$\mathbb{E}[X \mid Y = y] = 0,\ |y| \leqslant a, \implies \mathbb{E}[X \mid Y = y] \equiv 0.$$
- case $0 \leqslant a < b$: if we have
$$\mathbb{E}[X \mid Y = y] = \begin{cases} \frac{-y-b+y+a}{2} = -\frac{b-a}{2}, & -\frac{a+b}{2} \leqslant y \leqslant -\frac{b-a}{2}, \\ \frac{y-a+y+a}{2} = y & -\frac{b-a}{2} \leqslant y \leqslant \frac{b-a}{2}, \\ \frac{y-a+(-y+b)}{2} = \frac{b-a}{2}, & \frac{b-a}{2} \leqslant y \leqslant \frac{b+a}{2} \end{cases}$$
- case $0 \leqslant b < a$: if we have
$$\mathbb{E}[X \mid Y = y] = \begin{cases} \frac{y-a+(-y+b)}{2} = \frac{b-a}{2}, & -\frac{a+b}{2} \leqslant y \leqslant -\frac{b-a}{2}, \\ \frac{-y-b+(-y+b)}{2} = -y & -\frac{b-a}{2} \leqslant y \leqslant \frac{b-a}{2}, \\ \frac{-y-b+(y+a)}{2} = -\frac{b-a}{2}, & \frac{b-a}{2} \leqslant y \leqslant \frac{b+a}{2} \end{cases}$$

iii) Since in all cases take constant values with positive probability, $\mathbb{E}[X \mid Y]$ is not absolutely continuous, so there is no density for it. □

Exercise 15. i) Let $X \geqslant 0$ natural valued. By monotone convergence,
$$\sum_{k=0}^{\infty} \mathbb{P}(X > k) = \sum_{k=0}^{\infty} \mathbb{E}[1_{X>k}] = \mathbb{E}\left[\sum_{k=0}^{\infty} 1_{X>k}\right].$$
Now, assume $X(\Omega) = n$. Then
$$\sum_{k=0}^{\infty} 1_{X(\omega)>k} = \sum_{k=0}^{n-1} 1 = n = X(\omega),$$
From this the conclusion follows.

ii) We have
$$\int_0^{+\infty} \mathbb{P}(X > t)\, dt = \int_0^{+\infty} \mathbb{E}[1_{X>t}]\, dt = \mathbb{E}\left[\int_0^{+\infty} \underbrace{1_{X>t}}_{=1_{[0,X[}(t)}\, dt\right] = \mathbb{E}\left[\int_0^{X} dt\right] = \mathbb{E}[X]. \quad \square$$

Exercise 16. i) To be a true covariance matrix C must be symmetric and positive definite. Clearly, C is symmetric. To be positive definite we need that
$$C\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} > 0,\ \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0.$$
A quick condition for positivity, if you remind, is that all the $k \times k$ sub-determinants on the diagonal are positive. These are 1 and $1 - \rho^2$, so the condition is $1 - \rho^2 > 0$, that is $-1 < \rho < 1$.

Equivalently,
$$C\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \rho y \\ \rho x + y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x + \rho y)x + (\rho x + y)y = x^2 + y^2 + 2\rho xy$$

If $x = \pm y$ we have $x^2 + x^2 \pm 2\rho x^2 = 2x^2(1 \pm \rho) > 0$ iff $1 \pm \rho > 0$, that is $-1 < \rho < 1$. For such ρ,

$$x^2 + y^2 + 2\rho xy > x^2 + y^2 - 2|x||y| = (|x| + |y|)^2 > 0,\ \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0.$$

ii) We know that $(X, Y) = (R\cos\Theta, R\sin\Theta) = \Phi(R, \Theta)$, so

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(\Phi(r, \theta))|\det \Phi'(r, \theta)|$$

where

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{1}{\sqrt{(2\pi)^2 \det C}} e^{-\frac{1}{2}C^{-1}\binom{x}{y}\binom{x}{y}} = \frac{1}{\sqrt{(2\pi)^2(1-\rho^2)}} e^{-\frac{1}{2}\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\binom{x}{y}\cdot\binom{x}{y}} \\ &= \frac{1}{\sqrt{(2\pi)^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(x^2+y^2-2\rho xy\right)}. \end{aligned}$$

Thus,

$$f_{R,\Theta}(r, \theta) = \frac{1}{\sqrt{(2\pi)^2(1-\rho^2)}} e^{-\frac{r^2}{2(1-\rho^2)}(1-2\rho\cos\theta\sin\theta)} \cdot r 1_{[0,+\infty[}(r) 1_{[0,2\pi]}(\theta).$$

For the distribution of Θ we have

$$\begin{aligned} f_\Theta(\theta) &= \int_{\mathbb{R}} f_{R,\Theta}(r, \theta)\, dr = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{+\infty} r e^{-\frac{1-\rho\sin(2\theta)}{2(1-\rho^2)} r^2}\, dr \cdot 1_{[0,2\pi]}(\theta) \\ &= -\frac{1}{2\pi\sqrt{1-\rho^2}} \frac{2(1-\rho^2)}{1-\rho\sin(2\theta)} \left[e^{-\frac{1-\rho\sin(2\theta)}{2(1-\rho^2)} r^2} \right]_{r=0}^{r=+\infty} 1_{[0,2\pi]}(\theta) \\ &= \frac{1}{2\pi} \frac{\sqrt{1-\rho^2}}{1-\rho\sin(2\theta)} 1_{[0,2\pi]}(\theta) \end{aligned}$$

iii) R and Θ are independent iff $f_{R,\Theta} = f_R f_\Theta$. It is evident that this can happen iff $\rho = 0$, that is iff X and Y are independent. In this case we have

$$f_\Theta(\theta) = \frac{1}{2\pi} 1_{[0,2\pi]}(\theta),$$

and

$$f_R(r) = \frac{f_{R,\Theta}(r, \theta)}{f_\Theta(\theta)} = r e^{-\frac{r^2}{2}} 1_{[0,+\infty[}(r).$$

iv) We have

$$\begin{aligned} \mathbb{E}[R \mid \Theta = \theta] &= \int_{\mathbb{R}} r f_{R|\Theta}(r|\theta)\, dr = \frac{2\pi(1-\rho\sin(2\theta))}{\sqrt{1-\rho^2}} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{+\infty} r^2 e^{-\frac{1-\rho\sin(2\theta)}{2(1-\rho^2)} r^2}\, dr \\ &= \sqrt{\frac{1-\rho^2}{1-\rho\sin(2\theta)}} \int_0^{+\infty} u^2 e^{-\frac{u^2}{2}}\, du = \sqrt{\frac{\pi(1-\rho^2)}{2(1-\rho\sin(2\theta))}}. \end{aligned}$$

Similarly,

$$\mathbb{E}[R^2 \mid \Theta = \theta] \quad = \int_{\mathbb{R}} r^2 f_{R|\Theta}(r|\theta)\, dr = \frac{1-\rho\sin(2\theta)}{1-\rho^2} \int_0^{+\infty} r^3 e^{-\frac{1-\rho\sin(2\theta)}{2(1-\rho^2)} r^2}\, dr$$

$$= \frac{1-\rho^2}{1-\rho\sin(2\theta)} \int_0^{+\infty} u^3 e^{-\frac{u^2}{2}}\, du.$$

Noticed that

$$\int_0^{+\infty} u^3 e^{-\frac{u^2}{2}}\, du = \left[u^2(-e^{-\frac{u^2}{2}})\right]_{r=0}^{r=+\infty} + \int_0^{+\infty} 2u e^{-\frac{u^2}{2}}\, du = 2$$

we deduce

$$\mathbb{R}[R^2 \mid \Theta = \theta] = 2\frac{1-\rho^2}{1-\rho\sin(2\theta)}. \quad \square$$

Exercise 17. i) We have to show that

$$\mathbb{P}\left(\frac{|X_n|}{\log n} \leqslant 1+\varepsilon \text{ for all but finitely many } n\right) = \mathbb{P}\left(\bigcup_N \bigcap_{n \geqslant N} \frac{|X_n|}{\log n} \leqslant 1+\varepsilon\right) = 1,$$

that is, taking the complementaries,

$$\mathbb{P}\left(\bigcap_N \bigcup_{n \geqslant N} \left\{\frac{|X_n|}{\log n} \geqslant 1+\varepsilon\right\}\right) = 0.$$

We apply first Borel-Cantelli's Lemma. Notice first that

$$\mathbb{P}\left(\frac{|X_n|}{\log n} \geqslant 1+\varepsilon\right) = \mathbb{P}\left(|X_n| \geqslant (1+\varepsilon)\log n\right).$$

Now, for $\alpha > 0$,

$$\mathbb{P}(|X_n| \geqslant \alpha) = \int_{]-\infty,-\alpha]\cup[\alpha,+\infty[} \frac{1}{2} e^{-|x|}\, dx = \int_\alpha^{+\infty} e^{-x}\, dx = e^{-\alpha},$$

so

$$\mathbb{P}(|X_n| \geqslant (1+\varepsilon)\log n) = e^{-(1+\varepsilon)\log n} = \frac{1}{n^{1+\varepsilon}}.$$

Therefore

$$\sum_n \mathbb{P}\left(\frac{|X_n|}{\log n} \geqslant 1+\varepsilon\right) = \sum_n \frac{1}{n^{1+\varepsilon}} < +\infty,\ \forall \varepsilon > 0,$$

so the conclusion follows.

ii) We now have to prove that

$$\mathbb{P}\left(\frac{|X_n|}{\log n} \geqslant 1-\varepsilon \text{ for infinitely many } n\right) = \mathbb{P}\left(\bigcap_N \bigcup_{n \geqslant N} \left\{\frac{|X_n|}{\log n} \geqslant 1-\varepsilon\right\}\right) = 1.$$

Here we can apply the second Borel-Cantelli Lemma: since the X_n are independent, the conclusion will follow once we prove

$$\sum_n \mathbb{P}\left(\left\{\frac{|X_n|}{\log n} \geqslant 1-\varepsilon\right\}\right) = +\infty.$$

As above,

$$\mathbb{P}\left(\left\{\frac{|X_n|}{\log n} \geqslant 1-\varepsilon\right\}\right) = \mathbb{P}\left(|X_n| \geqslant (1-\varepsilon)\log n\right) = e^{-(1-\varepsilon)\log n} = \frac{1}{n^{1-\varepsilon}},$$

so

$$\sum_n \mathbb{P}\left(\left\{\frac{|X_n|}{\log n} \geqslant 1-\varepsilon\right\}\right) = \sum_n \frac{1}{n^{1-\varepsilon}} = +\infty. \quad \square$$

Exercise 19. In L^2, conditional expectation is the orthogonal projection ΠX. In particular $X - \Pi X \perp \Pi X$ so, by Pythagorean's theorem,

$$\|X\|_2^2 = \|\Pi X\|_2^2 + \|X-\Pi X\|_2^2 \geqslant \|\Pi X\|_2^2,$$

that is

$$\|\mathbb{E}[X \mid \mathscr{G}]\|_2^2 \leqslant \|X\|_2^2,$$

from which the conclusion follows.

i) By the Pythagorean theorem,

$$\|X\|_2^2 = \|\mathbb{E}[X \mid Y]\|_2^2 + \|X - \mathbb{E}[X \mid Y]\|_2^2 = \|Y\|_2^2 + \|X-Y\|_2^2, \implies \|X-Y\|_2^2 = \|X\|_2^2 - \|Y\|_2^2.$$

For the same argument, from $\mathbb{E}[Y \mid X] = X$ we have

$$\|Y-X\|_2^2 = \|Y\|_2^2 - \|X\|_2^2,$$

and summing these identities we get

$$2\|X-Y\|_2^2 = 0, \implies X = Y \text{ a.s.}$$

ii) Arguing as in i) we get

$$\begin{cases} \|X-Y\|_2^2 = \|X\|_2^2 - \|Y\|_2^2, \\ \|Y-Z\|_2^2 = \|Y\|_2^2 - \|Z\|_2^2, \\ \|Z-X\|_2^2 = \|Z\|_2^2 - \|X\|_2^2, \end{cases} \implies \|X-Y\|_2^2 + \|Y-Z\|_2^2 + \|Z-X\|_2^2 = 0, \implies X = Y = Z \text{ a.s.} \quad \square$$

Exercise 20. i) Clearly F is well defined, increasing, $F(-\infty) = e^{-(+\infty)} = 0$ and $F(+\infty) = e^{-0} = 1$. Moreover $F \in \mathscr{C}(\mathbb{R})$, so F is a cdf.

ii) We have

$$\begin{aligned} F_{Y_n}(y) &= \mathbb{P}(Y_n \leqslant y) = \mathbb{P}\left(\max\{X_1, \dots, X_n\} \leqslant y\right) \\ &= \mathbb{P}\left(X_1 \leqslant y, \dots, X_n \leqslant y\right) = \prod_{k=1}^{n} \mathbb{P}(X_k \leqslant y) \\ &= \begin{cases} 0, & y < 0, \\ \left(1 - e^{-y}\right)^n, & y > 0. \end{cases} \end{aligned}$$

iii) Let $Z_n := Y_n - \log n$. Then

$$F_{Z_n}(z) = \mathbb{P}(Z_n \leqslant z) = \mathbb{P}(Y_n \leqslant z + \log n) = F_{Y_n}(z + \log n).$$

For $z \in \mathbb{R}$ fixed and n large enough, $z + \log n > 0$ so

$$F_{Z_n}(z) = \left(1 - e^{-(z + \log n)}\right)^n = \left(1 - \frac{e^{-z}}{n}\right)^n \longrightarrow e^{-e^{-z}} = F(z), \ \forall z \in \mathbb{R}.$$

Since F is a continuous cdf, we conclude that $Z_n \xrightarrow{d} Z$ where $F_Z = F$. □

Exercise 21. We notice that

$$\begin{aligned} \mathbb{P}(X + Y \in E) &= \int_{\mathbb{R}^2} 1_E(x+y) f_{X,Y}(x,y)\, dxdy = \int_{\mathbb{R}^2} 1_E(x+y) f_X(x) f_Y(y)\, dxdy \\ &\overset{u=x+y,\ v=x}{=} \int_{\mathbb{R}^2} 1_E(u) f_X(v) f_Y(u-v)\, dudv \\ &= \int_E \left(\int_{\mathbb{R}} f_X(v) f_Y(u-v)\, dv \right) du \\ &= \int_E f_X * f_Y(u)\, du. \end{aligned}$$

This says that $f_{X+Y}(u) = f_X * f_Y(u)$.

i) By induction,

$$f_{X_1 + \cdots + X_n} = f_{X_1} * \cdots * f_{X_n}.$$

Applying the FT,

$$\widehat{f_{X_1 + \cdots + X_n}}(\xi) = \widehat{f_{X_1}}(\xi) \cdots \widehat{f_{X_n}}(\xi).$$

Notice that

$$\begin{aligned} \widehat{f_{X_k}}(\xi) &= \int_{\mathbb{R}} f_{X_k}(x) e^{-i\xi x}\, dx = \int_0^{+\infty} \lambda e^{-\lambda x} e^{-i\xi x}\, dx = \lambda \int_0^{+\infty} e^{-(\lambda + i\xi)x}\, dx \\ &= \lambda \left[\frac{e^{-(\lambda+i\xi)x}}{-(\lambda + i\xi)} \right]_{x=0}^{x=+\infty} = \lambda \left(0 - \frac{1}{-(\lambda + i\xi)} \right) = \frac{\lambda}{\lambda + i\xi}. \end{aligned}$$

So,

$$\widehat{f_{X_1+\cdots+X_n}}(\xi) = \left(\frac{\lambda}{\lambda+i\xi}\right)^n = \frac{\lambda^n}{(-i)^{n-1}(n-1)!}\partial_\xi^{n-1}(\lambda+i\xi)^{-1}.$$

Applying the FT,

$$\begin{aligned} 2\pi f_{X_1+\cdots+X_n}(-x) &= \widehat{\widehat{f_{X_1+\cdots+X_n}}}(x) = \frac{\lambda^n}{(-i)^{n-1}(n-1)!}\widehat{\partial_\xi^{n-1}(\lambda+i\sharp)^{-1}}(x) \\ &= (-1)^{n-1}\frac{\lambda^n}{(n-1)!}x^{n-1}\widehat{(\lambda+i\sharp)^{-1}}(x). \end{aligned}$$

Now, since

$$\lambda(\lambda+i\xi)^{-1} = \lambda\widehat{e^{-\lambda\sharp}1_{[0,+\infty[}}(\xi),$$

we have

$$\lambda\widehat{(\lambda+i\sharp)^{-1}}(x) = \lambda\widehat{\widehat{e^{-\lambda\sharp}1_{[0,+\infty[}}}(x) = 2\pi\lambda e^{\lambda x}1_{[0,+\infty[}(-x),$$

from which, finally,

$$f_{X_1+\cdots+X_n}(x) = \lambda\frac{(-\lambda x)^{n-1}}{(n-1)!}e^{-\lambda x}1_{[0,+\infty[}(x).$$

ii) Let $S_n := X_1+\cdots+X_n$. We compute the cdf of S_N: for $x \geqslant 0$,

$$\begin{aligned} \mathbb{P}(S_N \leqslant x) &= \sum_{n=1}^{\infty}\mathbb{P}(S_n \leqslant x, N=n) = \sum_{n=1}^{\infty}\mathbb{P}(S_n \leqslant x)(1-p)^{n-1}p \\ &= \lambda p\sum_{n=1}^{\infty}\int_0^x \frac{(-\lambda y)^{n-1}}{(n-1)!}e^{-\lambda y}(1-p)^{n-1}\,dy \\ &\overset{mon.conv,}{=} \lambda p\int_0^x\sum_{n=1}^{\infty}\frac{(-(1-p)\lambda y)^{n-1}}{(n-1)!}e^{-\lambda y}\,dy = \lambda p\int_0^x e^{-(1-p)\lambda y}e^{-\lambda y}\,dy \end{aligned}$$

that is

$$F_{S_N}(x) = \lambda p\int_0^x e^{-\lambda p y}\,dy, \implies f_{S_N}(x) = \lambda p e^{-\lambda p x}1_{[0,+\infty[}(x). \quad \square$$

Exercise 22. i) See notes.

ii) Let $\phi_X(\xi) := \mathbb{E}[e^{i\xi X}]$. To show that ϕ_X is differentiable we apply the differentiability under integral theorem. Formally

$$\partial_\xi\phi_X(\xi) = \mathbb{E}[iXe^{i\xi X}].$$

Notice that $|iXe^{i\xi X}| = |X| \in L^1(\Omega)$ being, by Cauchy-Schwarz inequality, $\mathbb{E}[|X|] \leqslant \mathbb{E}[X^2]^{1/2} < +\infty$. Thus differentiability theorem applies. Then,

$$\partial_\xi^2\phi_X(\xi) = \mathbb{E}[(iX)^2e^{i\xi X}].$$

Again, being $|(iX)^2e^{i\xi X}| = X^2 \in L^1(\Omega)$, because $\mathbb{E}[X^2] < +\infty$. In particular,

$$\partial_\xi^2\phi_X(0) = \mathbb{E}[(iX)^2] = -\mathbb{E}[X^2].$$

iii) Notice that $\partial_\xi \phi_X(\xi) = -c4\xi^3 e^{-c\xi^4}$ and $\partial^2_\xi \phi_X(\xi) = e^{-c\xi^4}(16c^2\xi^6 - 12c\xi^2)$, so, in particular, $\partial^2_\xi \phi_X(0) = 0$. By ii), $\mathbb{E}[X^2] = 0$, thus $X = 0$ a.s., from which $\phi_X(\xi) = \mathbb{E}[1] = 1 = e^{-c\xi^4}$ iff $c = 0$. □

Exercise 23. i) See notes.

ii) Let $\mathscr{F}_t := \sigma(W_s \,:\, s \leqslant t)$. We have to verify

$$\mathbb{E}\left[W_t^3 - 3tW_t \mid \mathscr{F}_s\right] = W_s^3 - 3sW_s,\ \forall t \geqslant s \geqslant 0.$$

We have

$$W_t^3 = (W_t - W_s + W_s)^3 = (W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3,$$

so, recalling that $W_t - W_s$ independent of $\mathscr{F}_s$ and W_s is $\mathscr{F}_s$, by the properties of the conditional expectation we have

$$\begin{aligned}\mathbb{E}[W_t^3 \mid \mathscr{F}_s] &= \mathbb{E}[(W_t - W_s)^3] + 3W_s\mathbb{E}\left[(W_t - W_s)^2 \mid \mathscr{F}_s\right] + 3W_s^2\mathbb{E}[W_t - W_s \mid \mathscr{F}_s] + W_s^3 \\ &= 0 + 3W_s\mathbb{E}[(W_t - W_s)^2] + 3W_s^2\mathbb{E}[W_t - W_s] + W_s^3 \\ &= 3W_s(t - s) + W_s^3\end{aligned}$$

so

$$\begin{aligned}\mathbb{E}\left[W_t^3 - 3tW_t \mid \mathscr{F}_s\right] &= 3W_s(t - s) + W_s^3 - 3t\mathbb{E}[W_t \mid \mathscr{F}_s] = 3W_s(t - s) + W_s^3 - 3tW_s \\ &= W_s^3 - 3sW_s,\end{aligned}$$

which is the conclusion.

iii) We start noticing that

$$W_t^4 = (W_t - W_s + W_s)^4 = (W_t - W_s)^4 + 4(W_t - W_s)^3 W_s + 6(W_t - W_s)^2 W_s^2 + 4(W_t - W_s)W_s^3 + W_s^4,$$

so

$$\mathbb{E}[W_t^4 \mid \mathscr{F}_s] = 3(t - s)^2 + 6(t - s)W_s^2 + W_s^4$$

Now, notice also that

$$\mathbb{E}[W_t^2 \mid \mathscr{F}_s] = t - s + W_s^2, \quad \Longleftrightarrow \quad \mathbb{E}[W_t^2 - t \mid \mathscr{F}_s] = W_s^2 - s,$$

so

$$6tW_s^2 = 6t\left(\mathbb{E}[W_t^2 - t \mid \mathscr{F}_s] + s\right) = \mathbb{E}[6tW_t^2 \mid \mathscr{F}_s] - 6t(t - s),$$

from which

$$\begin{aligned}\mathbb{E}[W_t^4 - 6tW_t^2 \mid \mathscr{F}_s] &= W_s^4 - 6sW_s^2 + 3(t - s)^2 - 6t(t - s) = \\ &= W_s^4 - 6sW_s^2 - 3(t^2 - s^2),\end{aligned}$$

so, finally,

$$\mathbb{E}[W_t^4 - 6tW_t^2 + 3t^2 \mid \mathscr{F}_s] = W_s^4 - 6sW_s^2 + 3s^2,$$

that is, $W_t^4 - 6tW_t^2 + 3t^2$ is a martingale. □

Exercise 25 i) Assume (X,Y) are independent, so $f_{X,Y} = f_X f_Y$. Then

$$\begin{aligned}\phi_{X,Y}(\xi,\eta) &= \mathbb{E}[e^{i(\xi,\eta)\cdot(X,Y)}] = \int_{\mathbb{R}^2} e^{i(\xi,\eta)\cdot(x,y)} f_{X,Y}(x,y)\, dxdy = \int_{\mathbb{R}^2} e^{i(\xi,\eta)\cdot(x,y)} f_X(x) f_Y(y)\, dxdy \\ &= \int_{\mathbb{R}} e^{i\xi x} f_X(x)\, dx \int_{\mathbb{R}} e^{i\eta y} f_Y(y)\, dy = \mathbb{E}[e^{i\xi X}]\mathbb{E}[e^{i\eta Y}] = \phi_X(\xi)\phi_Y(\eta)\end{aligned}$$

this for every $(\xi,\eta) \in \mathbb{R}^2$.

Vice versa: assume $\phi_{X,Y} \equiv \phi_X \phi_Y$. The previous calculation shows that

$$\widehat{f_{X,Y}} = \widehat{f_X f_Y},$$

and since both $f_{X,Y}, f_X f_Y \in L^1(\mathbb{R}^2)$ because they are probability densities, by the injectivity of L^1 FT we conclude that $f_{X,Y} \overset{a.s.}{=} f_X f_Y$ as claimed.

ii) We can use the characteristic functions:

$$\begin{aligned}\phi_{X+Y,X-Y}(\xi,\eta) &= \mathbb{E}\left[e^{i(X+Y,X-Y)\cdot(\xi,\eta)}\right] = \mathbb{E}\left[e^{(\xi+\eta)X+(\xi-\eta)Y}\right] = \mathbb{E}\left[e^{i(\xi+\eta)X}\right]\mathbb{E}\left[e^{i(\xi-\eta)Y}\right] \\ &= e^{-\frac{1}{2}(\xi+\eta)^2} e^{-\frac{1}{2}(\xi-\eta)^2} = e^{-\frac{1}{2}(2\xi^2+2\eta^2)} = e^{-\xi^2} e^{-\eta^2}.\end{aligned}$$

On the other hand

$$\phi_{X\pm Y}(t) = \mathbb{E}\left[e^{i(X\pm Y)t}\right] = \mathbb{E}\left[e^{itX}\right]\mathbb{E}\left[e^{\pm itY}\right] = e^{-\frac{t^2}{2}} e^{\frac{-t^2}{2}} = e^{-t^2}.$$

We can conclude that $\phi_{X+Y,X-Y} = \phi_{X+Y}\phi_{X-Y}$, thus $X+Y, X-Y$ are independent and that they are both Gaussian $\mathcal{N}(0,2)$.

iii) As suggested, $(x+y)^2 - (x-y)^2 = 4xy$, so

$$\mathbb{E}[XY \mid X-Y] = \frac{1}{4}\mathbb{E}\left[(X+Y)^2 - (X-Y)^2 \mid X-Y\right]$$

By independence,

$$\mathbb{E}\left[(X+Y)^2 \mid X-Y\right] = \mathbb{E}\left[(X+Y)^2\right] = 2,$$

while

$$\mathbb{E}\left[(X-Y)^2 \mid X-Y\right] = (X-Y)^2.$$

Therefore

$$\mathbb{E}[XY \mid X-Y] = \frac{1}{4}\left(2 - (X-Y)^2\right). \quad \square$$

Exercise 26 i) We start noticing that

$$X_k = \alpha X_{k-1} + N_{k-1} = \alpha(\alpha X_{k-2} + N_{k-2}) + N_{k=1} = \alpha^2 X_{k-2} + N_{k-1} + \alpha N_{k-2}.$$

Iterating this we get

$$X_k = \alpha^k X_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j} = \alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j}.$$

Clearly,

$$\mathbb{E}[X_k] = \alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j \mathbb{E}[N_{k-1-j}] = \alpha^k x_0.$$

About the variance we have

$$\mathbb{V}[X_k] = \mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2 = \mathbb{E}\left[\left(\alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j}\right)^2\right] - \alpha^{2k} x_0^2.$$

Noticed that $\mathbb{E}[\alpha^k x_0 \alpha^j N_{k-1-j}] = 0$, and being the N_k independent, we have

$$\mathbb{V}[X_k] = \alpha^{2k} x_0^2 + \sum_{j=0}^{k-1} \alpha^{2j} \mathbb{E}[N_{k-1-j}^2] - \alpha^{2k} x_0 = \sigma^2 \sum_{j=0}^{k-1} (\alpha^2)^j = \sigma^2 \frac{1-\alpha^{2k}}{1-\alpha^2},$$

because of the formula $\sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}$.

ii) We have

$$\mathbb{E}[X_{k+1} \mid \mathscr{F}_k] = \mathbb{E}[\alpha X_k + N_k \mid \mathscr{F}_k] = \alpha X_k + \mathbb{E}[N_k \mid \mathscr{F}_k].$$

Since $\mathscr{F}_k = \sigma(X_1, \dots, X_k) = \sigma(N_0, \dots, N_{k-1})$ and N_k being idependent of N_j for $j < k$, we have

$$\mathbb{E}[N_k \mid \mathscr{F}_k] = \mathbb{E}[N_k] = 0.$$

Therefore, $\mathbb{E}[X_{k+1} \mid \mathscr{F}_k] = X_k$, so (X_k) is a martingale w.r.t. $\mathscr{F}_k$ iff $\alpha = 1$.

iii) We have $X_{k+1} - X_k = (1-\alpha)X_k + N_k$. If $\alpha = 1$,

$$\|X_{k+1} - X_k\|_2^2 = \mathbb{E}[(X_{k+1} - X_k)^2] = \mathbb{E}[N_k^2] = \sigma^2,$$

whereas, if $\alpha \neq 1$,

$$\begin{aligned} \|X_{k+1} - X_k\|_2^2 &= \mathbb{E}[(X_{k+1} - X_k)^2] = \mathbb{E}[(1-\alpha)^2 X_k^2 + N_k^2 + 2(1-\alpha) X_k N_k] \\ &= (1-\alpha)^2 \frac{1-\alpha^{2k}}{1-\alpha} \sigma^2 + \sigma^2 = \Big((1-\alpha)(1-\alpha^{2k}) + 1\Big) \sigma^2. \end{aligned}$$

In each case, $\|X_{k+1} - X_k\|_2 \not\longrightarrow 0$ when $k \to +\infty$, and this must happens to have L^2 convergence. Thus, X_k is not convergent in L^2.

iv) We have

$$\phi_{X_k}(\xi) = \mathbb{E}\left[e^{i\xi X_k}\right] = \mathbb{E}\left[e^{i\xi\left(\alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j N_{k-1-j}\right)}\right] = e^{i\xi\alpha^k x_0} \prod_{j=0}^{k-1} \mathbb{E}\left[e^{i\xi\alpha^j N_{k-1-j}}\right].$$

Let ϕ be the common characteristic function of the N_j. We have $\phi(\xi) = e^{-\frac{1}{2}\sigma^2\xi^2}$, so

$$\phi_{X_k}(\xi) = e^{i\xi\alpha^k x_0} \prod_{j=0}^{k-1} e^{-\frac{1}{2}\sigma^2(\alpha^j\xi)^2} = e^{i\xi\alpha^k x_0 - \frac{1}{2}\sigma^2\xi^2 \sum_{j=0}^{k-1} \alpha^{2j}} = e^{i\xi\alpha^k x_0 - \frac{1}{2}\sigma^2 \frac{1-\alpha^{2k}}{1-\alpha^2}\xi^2}.$$

Letting $k \to +\infty$, $\alpha^k \longrightarrow 0$ being $|\alpha| < 1$, so

$$\phi_{X_k}(\xi) \longrightarrow e^{-\frac{1}{2}\frac{\sigma^2}{1-\alpha^2}\xi^2},$$

that is

$$X_k \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{1-\alpha^2}\right). \quad \square$$

Exercise 27. i) See notes for the definitions.
ii) We start noticing that

$$\begin{aligned} X_t &= W_t^3 - 3\int_0^t W_r\, dr = (W_t - W_s + W_s)^3 - 3\int_0^s W_r\, dr - 3\int_s^t W_r\, dr \\ &= (W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + \underbrace{W_s^3 - 3\int_0^s W_r\, dr}_{=X_s} \\ &\quad -3\int_s^t (W_r - W_s)\, dr - 3W_s(t-s). \end{aligned}$$

We now apply the conditional expectation w.r.t $\mathscr{F}_s$. Notice first that $X_s \in \mathscr{F}_s$. Then

$$\begin{aligned} &\mathbb{E}[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 \mid \mathscr{F}_s] = \\ &= \underbrace{\mathbb{E}[(W_t - W_s)^3]}_{=0} + 3W_s \underbrace{\mathbb{E}[(W_t - W_s)^2]}_{t-s} + 3W_s^2 \underbrace{\mathbb{E}[W_t - W_s]}_{=0} \\ &= 3W_s(t-s). \end{aligned}$$

Finally,

$$\mathbb{E}\left[\int_s^t (W_r - W_s)\, dr \mid \mathscr{F}_s\right] = \int_s^t \underbrace{\mathbb{E}[W_r - W_s \mid \mathscr{F}_s]}_{=\mathbb{E}[W_r - W_s]=0}\, dr = 0.$$

Therefore, in conclusion,

$$\mathbb{E}[X_t \mid \mathscr{F}_s] = X_s + 3W_s(t-s) - 3W_s(t-s) = X_s,$$

that is, X_t is an $\mathscr{F}_t$ martingale. $\square$

Exercise 28. i) To be a well defined covariance matrix, C must be symmetric (evident) and strictly positive definite. This last follows from positivity of $k \times k$ ($k = 1, 2$) sub-determinants that are 1 and $1 - \frac{1}{4} = \frac{3}{4}$.
ii) We use the characteristic function:

$$\begin{aligned} \phi_{Y,2X-Y}(\xi,\eta) &= \mathbb{E}\left[e^{i(\xi,\eta)\cdot(Y,2X-Y)}\right] = \mathbb{E}\left[e^{i(\xi Y + \eta(2X-Y))}\right] = \mathbb{E}\left[e^{i(2\eta,\xi-\eta)\cdot(X,Y)}\right] \\ &= e^{-\frac{1}{2}C(2\eta,\xi-\eta)(2\eta,\xi-\eta)} = e^{-\frac{1}{2}\left(4\eta^2 + 2\frac{1}{2}2\eta(\xi-\eta) + (\xi-\eta)^2\right)} = e^{-\frac{1}{2}\left(3\eta^2+\xi^2\right)} \\ &= \phi_{\mathcal{N}(0,1)}(\xi)\phi_{\mathcal{N}(0,3)}(\eta) \end{aligned}$$

and since $Y \sim \mathcal{N}(0,1)$ and $2X - Y \sim \mathcal{N}(0,3)$ we have that

$$\phi_{Y,2X-Y}(\xi,\eta) \equiv \phi_Y(\xi)\phi_{2X-Y}(\eta),\ \forall(\xi,\eta) \in \mathbb{R}^2,$$

from which we deduce the independence.

iii) Since $X = \frac{1}{2}(2X - Y) + Y$, we have

$$X^2Y = \left(\frac{1}{2}(2X-Y) + \frac{1}{2}Y\right)^2 Y = \frac{1}{4}\left((2X-Y)^2Y + 2(2X-Y)Y^2 + Y^3\right)$$

and since Y and $2X - Y$ are independent

$$\begin{aligned}
\mathbb{E}[X^2Y \mid 2X-Y] &= \frac{1}{4}\left(\mathbb{E}\left[(2X-Y)^2Y \mid 2X-Y\right] + 2\mathbb{E}\left[(2X-Y)Y^2 \mid 2X-Y\right] + \mathbb{E}[Y^3 \mid 2X-Y]\right)\\
&= \frac{1}{4}\left((2X-Y)^2\mathbb{E}[Y \mid 2X-Y] + 2(2X-Y)\mathbb{E}[Y^2 \mid 2X-Y] + \mathbb{E}[Y^3]\right)\\
&= \frac{1}{4}\left((2X-Y)^2\mathbb{E}[Y] + 2(2X-Y)\mathbb{E}[Y^2] + \mathbb{E}[Y^3]\right)\\
&= \frac{2X-Y}{2}. \quad \square
\end{aligned}$$

Exercise 29. i) See LN.

ii) Let $E_k := \{X_k = X_{k+1} = X_{k+2} = 1\} \in \mathcal{F}$ (X_k are random variables). Then,

$$E = \bigcap_n \bigcup_{k \geqslant n} E_k = \limsup E_k \in \mathcal{F}.$$

Moreover, since

$$\mathbb{P}(E_k) \overset{indep}{=} \mathbb{P}(X_k = 1)\mathbb{P}(X_{k+1} = 1)\mathbb{P}(X_{k+2} = 1) = \frac{1}{\sqrt{k(k+1)(k+2)}},$$

and

$$\sum_k \mathbb{P}(E_k) = \sum_k \frac{1}{\sqrt{k(k+1)(k+2)}} \sim \sum_k \frac{1}{k^{3/2}} < \infty,$$

by the first Borel-Cantelli lemma we deduce that

$$\mathbb{P}(E) = \mathbb{P}(\limsup_k E_k) = 0.$$

iii) As in ii), let $F_k := \{X_k = X_{k+1} = 1\} \in \mathcal{F}$ and

$$F = \bigcap_n \bigcup_{k \geqslant n} F_k = \limsup F_k \in \mathcal{F}.$$

Notice that the events F_k are not independent, while F_{2k} are independent. Since,

$$F \supset \bigcap_n \bigcup_{k \geqslant n} F_{2k} = \limsup F_{2k},$$

we are led to assess $\mathbb{P}(\limsup_k F_{2k})$. As in ii),

$$\mathbb{P}(F_k) \overset{indep}{=} \mathbb{P}(X_k = 1)\mathbb{P}(X_{k+1} = 1) = \frac{1}{\sqrt{k(k+1)}}, \implies \mathbb{P}(E_{2k}) = \frac{1}{\sqrt{2k(2k+1)}} \geqslant \frac{1}{\sqrt{4k^2}} = \frac{1}{2k},$$

so

$$\sum_k \mathbb{P}(F_{2k}) \geqslant \sum_k \frac{1}{2k} = +\infty.$$

By the second Borel-Cantelli lemma we conclude that $\mathbb{P}(\limsup_k F_{2k}) = 1$, so $\mathbb{P}(F) \geqslant 1$ from which $\mathbb{P}(F) = 1$. □

Exercise 30. i) Since $0 \leqslant X_k \leqslant 1$ with $\mathbb{P} = 1$, we have that $S_{n+1} \geqslant S_n$, so $N > n$ iff $S_1 \leqslant 1, \dots, S_n \leqslant 1$, that is $S_n \leqslant 1$. In other words

$$\{N > n\} = \{S_n \leqslant 1\}.$$

Therefore,

$$\mathbb{P}(N > n) = \mathbb{P}(S_n \leqslant 1) = \mathbb{P}(X_1 + \cdots + X_n \leqslant 1) = \int_{x_1+\cdots+x_n \leqslant 1} \prod_{k=1}^n 1_{[0,1]}(x_k)\, dx_1 \cdots dx_n = \frac{1}{n!}.$$

From this we have

$$\mathbb{P}(N = n) = \mathbb{P}(\{N > n-1\} \backslash \{N > n\}) = \mathbb{P}(N\, n-1) - \mathbb{P}(N > n) = \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n-1}{n!}.$$

ii) We have

$$\mathbb{E}[N] = \sum_{n=1}^\infty n\mathbb{P}(N = n) = \sum_{n=1}^\infty n\frac{n-1}{n!} = \sum_{n=2}^\infty \frac{1}{(n-2)!} = \sum_{n=0}^\infty \frac{1}{n!}.$$

Recalling of the exponential series $\sum_{k=0}^\infty \frac{x^k}{k!} = e^x$ we have

$$\mathbb{E}[N] = e.$$

iii) We have

$$\mathbb{E}[S_N] = \sum_{n=2}^\infty \mathbb{E}[S_n 1_{N=n}].$$

We notice that

$$\{N = n\} = \{S_{n-1} \leqslant 1, S_n > 1\}$$

so

$$\begin{aligned}\mathbb{E}[S_n 1_{N=n}] &= \mathbb{E}[S_n 1_{S_{n-1}\leqslant 1} 1_{S_n > 1}] = \mathbb{E}\left[\mathbb{E}[S_n 1_{S_{n-1}\leqslant 1} 1_{S_n>1} \mid S_{n-1}]\right] \\ &= \mathbb{E}[1_{S_{n-1}\leqslant 1}\mathbb{E}[(S_{n-1} + X_n)1_{X_n > 1 - S_{n-1}} \mid S_{n-1}]]\end{aligned}$$

Now, if $s = S_{n-1} < 1$,

$$\mathbb{E}[(S_{n-1} + X_n)1_{X_n>1-S_{n-1}} \mid S_{n-1} = s] = s\mathbb{E}[1_{X_n>1-s} \mid S_{n-1} = s] + \mathbb{E}[X_n 1_{X_n>1-s} \mid S_{n-1}],$$

and by the independence of X_n from $S_{n-1} = X_1 + \cdots + X_{n-1}$, we have

$$\begin{aligned}\mathbb{E}[(S_{n-1} + X_n)1_{X_n > 1 - S_{n-1}} \mid S_{n-1} = s] &= s\mathbb{E}[1_{X_n > 1-s}] + \mathbb{E}[X_n 1_{X_n > 1-s}] \\ &= s(1 - (1-s)) + \textstyle\int_{1-s}^{1} x\, dx \\ &= s^2 + \left[\tfrac{x^2}{2}\right]_{x=1-s}^{1} = s^2 + \tfrac{1}{2}(1 - (1-s)^2) \\ &= \tfrac{s^2}{2} + s.\end{aligned}$$

Therefore

$$\mathbb{E}[S_n 1_{N=n}] = \mathbb{E}\left[\left(\frac{1}{2}S_{n-1}^2 + S_{n-1}\right) 1_{S_{n-1} \leqslant 1}\right] = \frac{1}{2} I_{n-1}^2 + I_{n-1}^1 = \frac{1}{2}\frac{1}{(n-2)!(n+1)} + \frac{1}{(n-2)!n}.$$

From this

$$\mathbb{E}[S_N] = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{(n-2)!(n+1)} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!n}$$

To compute the exact value of the sum we notice that

$$\sum_{n=2}^{\infty} \frac{1}{(n-2)!n} = \sum_{n=2}^{\infty} \frac{n-1}{n!} = \sum_{n=2}^{\infty} \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right) = e - 1 - (e-2) = 1,$$

while

$$\begin{aligned}\textstyle\sum_{n=2}^{\infty} \frac{1}{(n-2)!(n+1)} &= \textstyle\sum_{n=2}^{\infty} \frac{n(n-1)}{(n+1)!} = \sum_{n=2}^{\infty} \left(\frac{(n+1)(n-1)}{(n+1)!} - \frac{n+1}{(n+1)!} + \frac{2}{(n+1)!}\right) \\ &= \textstyle\sum_{n=2}^{\infty} \left(\frac{n-1}{n!} - \frac{1}{n!} + \frac{2}{(n+1)!}\right) = \sum_{n=2}^{\infty} \left(\frac{1}{(n-1)!} - \frac{2}{n!} + \frac{2}{(n+1)!}\right) \\ &= \textstyle e - 1 - 2\sum_{n=2}^{\infty} \frac{1}{n!} + 2\sum_{n=3}^{\infty} \frac{1}{n!} = e - 1 - 2\frac{1}{2} = e - 2.\end{aligned}$$

Thus,

$$\mathbb{E}[S_N] = \frac{1}{2}(e-2) + 1 = \frac{e}{2}. \quad \square$$