In $BV(U)$ $\quad \|f\|_{BV} = \|f\|_{L^1} + \underbrace{V(f,U)}_{\text{variation of } f \text{ in } U}$.

$$(W^{1,1}(U), \|\cdot\|_{1,1}) \xrightarrow[\text{CONTINUOUSLY}]{} (BV(U), \|\cdot\|_{BV}) \qquad V(f,U) = \| |Df| \|_{L^1}$$

$$\|f\|_{BV} = \|f\|_{L^1} + \| |Df| \|_{L^1} \leq C \|f\|_{W^{1,1}} \leq \tilde{C} \|f\|_{BV}$$

NOTION OF CONVERGENCE in BV.

① $f_k \to f$ STRONGLY in BV if

$\|f_k - f\|_{L^1} \to 0$ and $V(f_k - f, U) \to 0$

② $f_k \to f$ STRICTLY in BV if

$\|f_k - f\|_{L^1} \to 0$ and $V(f_k, U) \to V(f,U)$

(ASSOCIATED TO A DISTANCE $d(f,g) = \|f-g\|_{L^1} + |V(f,U) - V(g,U)|$

Note that $V(f_k - f, U) \geq |V(f_k, U) - V(f, U)|$

so $f_k \to f$ STRONGLY in BV $\Longrightarrow$ $f_k \to f$ STRICTLY in BV

the viceverse is NOT TRUE

(observe that if $f_k \to f$ STRONGLY in BV and $f_k \in W^{1,1}(U) \Rightarrow$ then $f \in W^{1,1}(U)$ since f_k is Cauchy in $W^{1,1}(U)$.)

Ex
$$f_k = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ k(x - \frac{1}{2}) & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{k} \\ 1 & \frac{1}{2} + \frac{1}{k} \leq x \leq 1 \end{cases} \qquad k \in \mathbb{N} \quad k > 1$$

$$T'_{\chi_{(\frac{1}{2},1)}} = \delta_{\frac{1}{2}}$$

$f_k \in W^{1,1}(0,1)$ $\quad f_k \longrightarrow \chi_{(\frac{1}{2},1)}$ in $L^1(0,1)$

$$V(f_k, (0,1)) = \| f_k' \|_{L^1} = \int_{1/2}^{1/2 + 1/k} k \, dx = 1 \longrightarrow V(\chi_{(\frac{1}{2},1)}, (0,1))$$

$V(f_k,(0,1)) = 1 \longrightarrow 1 = V(\chi_{(\frac{1}{2},1)},(0,1))$

$f_k \longrightarrow \chi_{(\frac{1}{2},1)}$ in STRICT SENSE BUT NOT IN STRONG SENSE.

③ $f_k \xrightarrow{*} f$ WEAKLY STAR in $BV(U)$

If $f_k \to f$ in $L^1(U)$, $f \in BV(U)$ and

$\frac{\partial}{\partial x_i} T_{f_k} \longrightarrow \frac{\partial}{\partial x_i} T_f$ weakly as RADON MEASURES

$\forall \phi \in C_c(U) \quad \int_U \phi \, d\mu_{k,i}^+ - d\mu_{k,i}^- \longrightarrow \int_U \phi \, d\mu_i^+ - d\mu_i^- \quad \forall i$

(where $\frac{\partial}{\partial x_i} T_{f_k} = \mu_{k_i}^+ - \mu_{k_i}^- \qquad \frac{\partial}{\partial x_i} T_f = \mu_i^+ - \mu_i^-$

(STRICT CONVERGENCE $\Longrightarrow$ WEAK* CONVERGENCE
(vi: converse not true

Ex $\quad f_k = \dfrac{\sin(kx)}{k} \qquad x \in (-\pi,\pi)$

$f_k \to 0$ in $L^1(-\pi,\pi)$

$$V(f_k, \quad ,\pi)) = \|f_k'\|_{L^1} = \int_{-\pi}^{\pi} |\cos kx|\, dx = \frac{2}{k}\int_0^{k\pi} |\cos y|\, dy =$$

$$= \frac{2}{k}\, k \int_0^{\pi} |\cos y|\, dy = 4 \quad > V(0,(0,\pi)) \qquad f_k \not\to 0 \text{ STRICTLY.}$$

$$\forall \phi \in C_c(-\pi,\pi) \qquad \int_{-\pi}^{\pi} \phi \cos(kx)\, dx \longrightarrow 0 \qquad \text{(Riemann Lebesgue)}$$

$f_k \overset{*}{\longrightarrow} 0$ in BV sense

FUNCTIONS of BOUNDED VARIATION in dim 1 -

Obs
Let $f: \mathbb{R} \to \mathbb{R}$ monotone non decreasing (or non inc)

$\Rightarrow$ then $f \in C^1_{loc}(\mathbb{R})$ and $(T_f)'$ is a positive distribution
(so associated to $\mu \in \mathcal{M}(\mathbb{R})$)

f monotone in $\mathbb{R}$ $\Rightarrow$ $f \in BV_{loc}(\mathbb{R})$ $\Rightarrow$ $f \in BV(a,b) \cap L^\infty(a,b)$
$\forall$ interval

f monotone $(T_f)' = \mu \in \mathcal{M}(\mathbb{R})$

$$V(f,(a,b)) = \mu(a,b) < +\infty \qquad \forall (a,b) \subseteq \mathbb{R} \text{ bounded}$$
$$\stackrel{!}{=} f(b^-) - f(a^+)$$

$$\|f\|_\infty = f(b^-) \leq \underbrace{\frac{1}{b-a}\int_a^b [f(t)]\,dt}_{f(a^+) \leq \int_a^b f(t)\,dt\,(b-a)} + f(b^-) - f(a^+)$$

Theorem (Characterization of BV function).

Let $(a,b) \subseteq \mathbb{R}$ bounded interval

$f \in BV(a,b) \iff \exists\, g_1, g_2$ monotone non decreasing functions in $\mathbb{R}$ such that

$$f(t) = g_1(t) - g_2(t) \quad \text{for a.e. } t \in (a,b)$$

Moreover $f \in L^\infty(a,b)$ $\quad \|f\|_\infty \le \frac{1}{b-a}\|f\|_{L^1} + V(f,(a,b)) \le$

$$\le C \|f\|_{BV}$$

f has RIGHT and LEFT LIMITS EVERYWHERE, has a COUNTABLE NUMBER OF JUMPS $\left(J_f = \{x \in (a,b) \ f(x^+) \neq f(x^-)\}\right)$

f has a RIGHT CONTINUOUS and a LEFT CONTINUOUS representative, and f is differentiable a.e.

PRECISELY

Let $\mu = (T_f)'$ $\quad (\mu = \mu^+ - \mu^-,\ \mu^+, \mu^- \in \mathcal{M}(a,b))$

Then $f(t) = f(a^+) + \mu^+(a,t] - \mu^-(a,t]$ $\underset{t}{a.e}$ (Right cont represent.)

$f(t) = f(a^+) + \mu^+(a,t) - \mu^-(a,t)$ $\underset{t}{a.e}$ (left cont. representative!)

$$V(f,(a,b)) = \mu^+(a,b) + \mu^-(a,b) = g_1(b^-) - g_1(a^+) + g_2(b^-) - g_1(a^+)$$

$$= \sup_{a < x_1 < x_2 < \dots x_m < b} \sum_{1}^{m} |f(x_i) - f(x_{i+1})|$$

proof $\Leftarrow$ if $f = g_1 - g_2 \Longrightarrow f \in BV(a,b)$ by the initial remark

$\rightarrow$ Let $f \in BV(a,b)$ $\quad (T_f)' = \mu^+ - \mu^-$ $\quad \mu^+, \mu^- \in \mathcal{M}(a,b)$ (FINITE RADON MEAS.)

$$g_1(t) = \begin{cases} 0 & t \leq a \\ \mu^+(a,t) & t \in (a,b) \\ \mu^+(a,b) & t \geq b \end{cases} \qquad g_2(t) = \begin{cases} 0 & t \leq a \\ \mu^-(a,t) & t \in (a,b) \\ \mu^-(a,b) & t \geq b \end{cases}$$

g_1, g_2 are monotone non decreasing functions.

(SINCE μ^+, μ^- are POSITIVE MEASURES)

Let us compute the derivative in the sense of distributions of $g_1 - g_2$ in (a,b).

Let $\phi \in C_c^\infty(a,b)$ $\quad T'_{g_1-g_2}(\phi) = -\int_a^b \phi'(t)\left[g_1(t) - g_2(t)\right] dt =$

$$= -\int_a^b \phi'(t)\left[\int_a^t d\mu^+(s) - d\mu^-(s)\right] dt \underset{\text{change order}}{=} -\int_a^b \left[\int_s^b \phi'(t)\,dt\right] d\mu^+(s) - d\mu^-(s) =$$

$$= -\int_a^b -\phi(s)\, d\mu^+(s) - d\mu^-(s) = +\int_a^b \phi(s)\, d\mu^+(s) - \int_a^b \phi(s)\, d\mu^-(s) =$$

$$= -\int_a^b \phi'(t) f(t)\, dt = T'_f(\phi)$$

$\forall \phi \in C_c^\infty(a,b)$ $\quad \int_a^b \phi'(t)\left[f(t) - g_1(t) + g_2(t)\right] dt = 0 \Rightarrow$ by the corollary of the fundamental lemma Calculus of var.

$\exists\, c \in \mathbb{R}$ $\quad f(t) = g_1(t) - g_2(t) + c \qquad (c = f(a^+))$

$\Rightarrow f$ has the same structure of monotone functions

f has a RIGHT CONTINUOUS REPRESENTATIVE

$$f(t) = f(a^+) + \mu^+(a,t] - \mu^-(a,t]$$

and a LEFT CONTINUOUS REPRESENTATIVE

$$f(t) = f(a^+) + \mu^+(a,t) - \mu^-(a,t) .$$

f is differentiable a.e (in general it has NOT the weak derivative!)

$$f(t) - f(s) = \mu^+(s,t] - \mu^-(s,t] \qquad \forall \; a<s<t<b$$

$$f(t) - f(s) = -\mu^+(t,s] + \mu^-(t,s] \qquad \forall \; a<t<s<b$$

$$|f(t)| \le |f(s)| + \mu^+(a,b) + \mu^-(a,b) \qquad \forall \; t,s \in (a,b)$$

integrate in ds

$$|f(t)| \, b-a \le \int_a^b |f(s)| ds + (b-a) V(f,(a,b)) = \|f\|_{L^1} + (b-a) V(f,(a,b))$$

$$\Rightarrow \quad \|f\|_\infty \le \tfrac{1}{b-a} \|f\|_{L^1} + V(f,(a,b)) \le \max\left(1, \tfrac{1}{b-a}\right) \|f\|_{BV}.$$

$f \in BV(a,b) \rightarrow$

$$\sup_{a<x_1<x_2<\dots x_n<b} \sum_{1}^{n-1} |f(x_i)-f(x_{i+1})| \leq \sup_{a<x_1<\dots<x_n<b} \sum_{i=1}^{n-1} |g_1(x_{i+1})-g_1(x_i)-(g_2(x_{i+1})-g_2(x_i))|$$

$$\leq \sup_{a<x_1<\dots x_n<b} \sum_{1}^{n} |g_1(x_{i+1})-g_1(x_i)| + |g_2(x_{i+1})-g_2(x_i)| =$$

$$= \sup_{a<x_1<\dots<x_n<b} g_1(x_n)-g_1(x_1) + g_2(x_n)-g_2(x_1) =$$

$$= g_1(b^-)-g_1(a^+)+g_2(b^-)-g_2(a^+) = V(f,(a,b))$$

It is also true that if $\sup_{a<x_1<x_2\dots x_n<b} \sum_{1}^{n-1} |f(x_i)-f(x_{i+1})| < +\infty \implies$

$f \in BV(a,b)$ and $V(f,(a,b)) = \sup_{a<x_1\dots<x_n<b} \sum_i |f(x_i)-f(x_{i+1})|$