Posizione reciproca di 2 piani in $\mathbb{R}^4$

$\pi_1 : \begin{cases} a_1x+b_1y+c_1z+d_1t=e_1 \\ a_2x+b_2y+c_2z+d_2t=e_2 \end{cases}$ $\quad \dim \pi_1 = 2 = 4 - rg$

$rg\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix} = 2$

$\pi_2 : \begin{cases} a_3x+b_3y+c_3z+d_3t=e_3 \\ a_4x+b_4y+c_4z+d_4t=e_4 \end{cases}$ $\quad rg\begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} = 2$

$$\left(\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{array}\right) = (A|b)$$

$A \in M_{4,4}(\mathbb{R})$

$2 \le rg(A) \le 4$

$A|b \in M_{4,5}(\mathbb{R})$

$2 \le rg(A|b) \le 4$

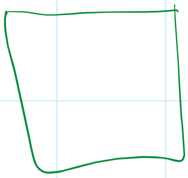
	$rg(A)$	$rg(A\|b)$	
$\pi_1 = \pi_2$	2	2	Piani uguali
$\pi_1 // \pi_2$ $\quad \pi_1 \cap \pi_2 = \emptyset$ $\quad V_{\pi_1} \cap V_{\pi_2} = V_{\pi_1} = V_{\pi_2}$	2	3	Piani disgiunti Piani paralleli
$\pi_1 \cap \pi_2 = r$ retta	3	3	Piani incidenti in una retta
$\pi_1 \cap \pi_2 = \emptyset$ $\quad \dim V_{\pi_1} \cap V_{\pi_2} = 4-3=1$	3	4	Piani disgiunti non sono paralleli (A)
$\pi_1 \cap \pi_2 = \{P\}$	4	4	Piani incidenti in un punto (B)

A

$$\pi_1: \begin{cases} x=0 \\ t=0 \end{cases} \qquad \pi_2: \begin{cases} y=0 \\ t=1 \end{cases} \qquad \begin{array}{c} \pi_1 \\ \\ \pi_2 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{pmatrix} 0 \\ y \\ z \\ 0 \end{pmatrix} \qquad \begin{array}{c} \\ \\ \\ 4^{\circ}-2^{\circ} \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$t=0$



$t=1$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ t \end{pmatrix}$$

B

$$\boxed{\pi_1: \begin{cases} x=0 \\ y=0 \end{cases}} \qquad \boxed{\pi_2: \begin{cases} z=0 \\ t=0 \end{cases}} \qquad \pi_1 \cap \pi_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

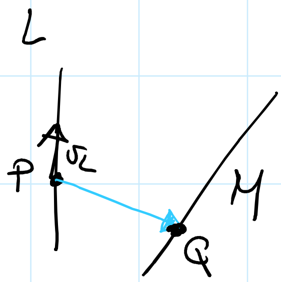
$$\begin{pmatrix} 0 \\ 0 \\ z \\ t \end{pmatrix}$$

Sottovarietà lineare generata.

$\mathbb{A} = \mathbb{R}^n$

$L: P + V_L \qquad V_L, V_M \leq \mathbb{R}^n$

$M: Q + V_M$



$$\boxed{L \vee M = P + \langle V_L, Q-P, V_M \rangle}$$

Formule di Grassmann affine:

Se L e M sono sottovarietà lineari incidenti oppure sghembe allora $\dim(L \vee M) = \dim L + \dim M - \dim(L \cap M)$

Esempio: due rette sghembe r, s

$$\dim(r \vee s) = \dim r + \dim s - \dim(\overset{\emptyset}{r \cap s}) = 1+1+1=3$$

Esempio:

Stabilire la posizione reciproca delle rette

$$r: \underset{P}{\begin{pmatrix}1\\0\\0\end{pmatrix}} + \underset{V_r}{<\begin{pmatrix}1\\1\\1\end{pmatrix}>} \qquad s: \underset{Q}{\begin{pmatrix}2\\0\\0\end{pmatrix}} + \underset{V_s}{<\begin{pmatrix}0\\1\\1\end{pmatrix}>} \qquad t: \begin{pmatrix}2\\1\\1\end{pmatrix} + <\begin{pmatrix}0\\1\\1\end{pmatrix}>$$

r e s:

$$<\begin{pmatrix}1\\1\\1\end{pmatrix}> \neq <\begin{pmatrix}0\\1\\1\end{pmatrix}>$$

$$V_r \neq V_s$$

$$\dim(r \vee s) = \dim <\begin{pmatrix}1\\0\\0\end{pmatrix}> + <\overset{V_r}{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \overset{Q-P}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \overset{V_s}{\begin{pmatrix}0\\1\\1\end{pmatrix}}> = 2$$

$\Rightarrow$ complanari

$$r: \overset{P + a v_r}{\begin{pmatrix}1\\0\\0\end{pmatrix} + <\begin{pmatrix}1\\1\\1\end{pmatrix}>} \qquad s: \overset{Q + b v_s}{\begin{pmatrix}2\\0\\0\end{pmatrix} + <\begin{pmatrix}0\\1\\1\end{pmatrix}>}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+a\\a\\a\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\b\\b\end{pmatrix} \qquad \begin{cases}1+a=2\\a=b\\a=b\end{cases} \qquad \begin{cases}a=2-1=1\\b=1\end{cases}$$

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \qquad \begin{pmatrix}2\\1\\1\end{pmatrix} \qquad R = r \cap s \qquad R = \begin{pmatrix}2\\1\\1\end{pmatrix}.$$

$$s: \overset{Q}{\begin{pmatrix}2\\0\\0\end{pmatrix}} + <\overset{v_s}{\begin{pmatrix}0\\1\\1\end{pmatrix}}> \qquad t: \overset{T}{\begin{pmatrix}2\\1\\1\end{pmatrix}} + <\overset{v_t}{\begin{pmatrix}0\\1\\1\end{pmatrix}}>$$

Se $s = t^{= s \cap t}$ $\dim(s \vee t) = 1+1-1=1$ $\quad$ / $s = t = s \cap t = s \vee t$

$$\dim(s \vee t) = \dim \begin{pmatrix}2\\0\\0\end{pmatrix} + \overset{T-Q}{\left\langle \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle} = 1 \Rightarrow s = t$$

$r: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \qquad t: \begin{pmatrix}2\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle = s \qquad r \cap s = R$, $r \cap t = R$ perché $\underline{s = t}$

Es: Pos. reciproca $r: \overset{P}{\begin{pmatrix}1\\0\\0\end{pmatrix}} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$ e $\ell: \overset{L}{\begin{pmatrix}0\\0\\0\end{pmatrix}} + \left\langle \overset{v_\ell}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \right\rangle$

$$\dim r \vee \ell = \dim \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \overset{v_r}{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \overset{L-P}{\begin{pmatrix}-1\\0\\0\end{pmatrix}}, \overset{v_\ell}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \right\rangle = 3 \Rightarrow \boxed{r \text{ e } \ell \text{ sono sghembe}}$$

Esercizio: $\mathbb{R}$ r e S_a dipendente da un parametro $a \in \mathbb{R}$

$$r: \begin{cases} x - y = 1 \\ x - z = 3 \end{cases} \qquad S_a: \begin{cases} x + (a-1)z = a+4 \\ y - z = a+1 \end{cases}$$

1) Determinare la posizione reciproca fra r e S_a al variare di $a \in \mathbb{R}$

Svolgimento: $r \cap S_a$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 3 \\ 1 & 0 & a-1 & a+4 \\ 0 & 1 & -1 & a+1 \end{array}\right)$$

Riduciamo

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & a-1 & a+3 \\ 0 & 1 & -1 & a+1 \end{array}\right) \qquad \begin{array}{c} \\ \\ 3^\circ - 2^\circ \\ 4^\circ - 2^\circ \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & a & a+1 \\ 0 & 0 & 0 & a-1 \end{array}\right)$$

Per $a \in \mathbb{R} \setminus \{0,1\}$ $\quad rg(A) = 3 \neq rg(A|b) = 4 \Rightarrow$ rette sghembe

Se $a = 0$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{rg(A) = 2} \neq rg(A|b) = 3$ disgiunte $\quad \dim(V_r \cap V_{s_0}) = 3-2 = 1$

rette parallele disgiunte

Se $a = 1$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad rg A = rg(A|b) = 3$$

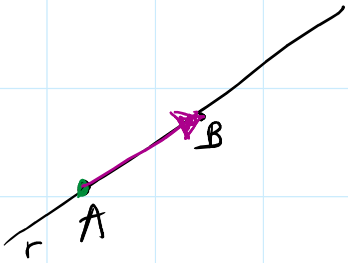
$$r \cap s = P = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

rette incidenti in un punto

$$\begin{cases} x - y = 1 \\ y - z = 2 \\ z = 2 \end{cases} \qquad \begin{cases} x = 1 + y = 1 + 4 = 5 \\ y = z + 2 = 4 \\ z = 2 \end{cases}$$

Retta per due punti distinti: $\quad A = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$

$$r: A \vee B = \overset{A}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} + \left\langle \overset{B-A}{\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}} \right\rangle$$



$A + \langle B - A \rangle$

$B + \langle B - A \rangle$

$A + \langle A - B \rangle$

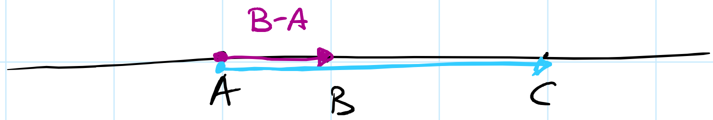
$B + \langle A - B \rangle$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

$$\begin{cases} x = 3 - 3t \\ y = 1 \\ z = 1 + 4t \end{cases} \qquad 3t = 3 - x \qquad t = \frac{3-x}{3} \qquad \begin{cases} y = 1 \\ z = 1 + \frac{4}{3}(3 - x) \end{cases}$$

Punti distinti A, B, C si dicono allineati se

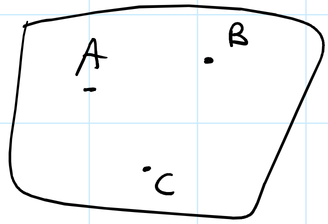
$\dim \langle A \vee B \vee C \rangle = 1$



$$A \vee B \vee C = A + \langle B-A,\ C-A \rangle \qquad A \neq B \quad B \neq C \quad A \neq C$$

Sono allineati $\Leftrightarrow$ $\dim \langle B-A, C-A \rangle = 1$

Piano per 3 punti non allineati



$$\pi = A \vee B \vee C = A + \langle B-A, C-A \rangle$$

$$\dim \langle B-A, C-A \rangle = 2$$

Esempio: $A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

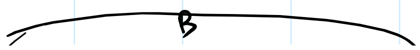
$$A \vee B \vee C = \overset{A}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + \underbrace{\left\langle \overset{B-A}{\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}}, \overset{C-A}{\begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}} \right\rangle}$$ è un piano

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \underbrace{a \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} - A$$

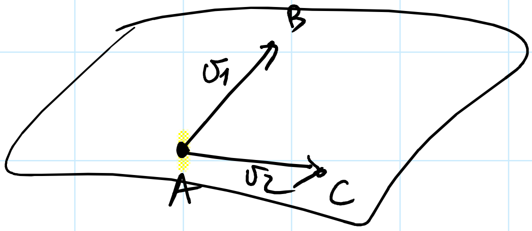
$$\begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix} \in \left\langle \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} \right\rangle \qquad \det \begin{pmatrix} x-1 & 0 & 0 \\ y-1 & -1 & -1 \\ z & 3 & 5 \end{pmatrix} = 0$$

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad A + \langle v_1, v_2 \rangle \qquad \det(P-A \ \ v_1 \ \ v_2) = 0$$

$$\det \begin{pmatrix} x-1 & 0 & 0 \\ y-1 & -1 & -1 \\ z & 3 & 5 \end{pmatrix} = 0 \qquad (x-1)(-5+3) = 0 \qquad \pi:\ x-1=0$$



$A + \langle B-A,\ C-A \rangle$

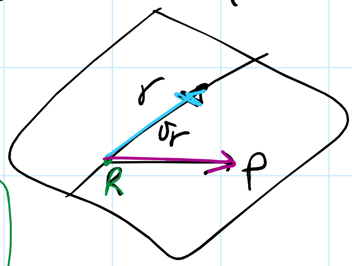


$$A + \langle \underbrace{B-A}_{v_1}, \underbrace{C-A}_{v_2} \rangle$$

Esercizio:

$$r: \begin{array}{cc} R & v_r \\ \begin{pmatrix}1\\0\\0\end{pmatrix} + & \left\langle \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle \end{array} \qquad \begin{pmatrix}1\\a\\a\end{pmatrix}$$

$$P = \begin{pmatrix}1\\5\\3\end{pmatrix}$$



$$\boxed{P \vee r = \begin{array}{ccc} R & v_r & P-R \\ \begin{pmatrix}1\\0\\0\end{pmatrix} + & \left\langle \begin{pmatrix}0\\1\\1\end{pmatrix}, \right. & \left. \begin{pmatrix}0\\5\\3\end{pmatrix} \right\rangle \end{array}}$$

$$P + V_\pi$$

Esempio: $\pi: \begin{pmatrix}2\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\5\\1\end{pmatrix}, \begin{pmatrix}8\\8\\7\end{pmatrix} \right\rangle$ determinare il piano // a π e passante per $Q = \begin{pmatrix}11\\0\\0\end{pmatrix}$

$$\underset{\text{tau}}{\tau} : Q + V_\pi = \begin{pmatrix}11\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\5\\1\end{pmatrix}, \begin{pmatrix}8\\0\\7\end{pmatrix} \right\rangle$$

$r: \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$ $\qquad s // r$ passante per $P = \begin{pmatrix}5\\1\\-2\end{pmatrix}$

$s: \begin{pmatrix}5\\1\\-2\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

Fasci di piani di sostegno una retta (proprio)

$r \subseteq \pi \qquad rg(A) = rg(A|b) = 2$

$$r: \begin{cases} 3x+y-z=1 \\ 2x+5z=0 \end{cases}$$

$$\begin{array}{c} \alpha \\ \beta \\ \\ \end{array} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 2 & 0 & 5 & 0 \\ a & b & c & d \end{array}\right)$$

Determiniamo tutti i piani $\pi: ax+by+cz=d$ $\quad rg(a\ b\ c)=1$

che contengono r $\qquad r = \begin{cases} 3x+y-z-1=0 \\ 2x+5z=0 \end{cases}$

$$\pi_{\alpha,\beta} = \alpha(3x+y-z-1)+\beta(2x+5z)=0$$

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \in \mathbb{R}^2 \setminus \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

Data $\quad r: \begin{cases} 3x+y-z=1 \\ 2x+5z=0 \end{cases}$ retta in $\mathbb{R}^3$ determinare il piano

1) Contenente r e passante per $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$$(*) \quad \boxed{\pi_{\alpha,\beta} = \alpha(3x+y-z-1)+\beta(2x+5z)=0}$$

$P=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$x=0$
$y=1$
$z=0$

$$\alpha(0+\cancel{1}-0-\cancel{1})+\beta(0+0)=0$$

$\alpha(0)+\beta(0)=0 \qquad 0=0 \Rightarrow$ Per tutti i piani del fascio vanno bene

2) determinare il piano contenente r e passante per $Q=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad x=1,\ y=0,\ z=0$ in $(*)$

$$\alpha(3+0-0-1)+\beta(2+50)=0$$

$$2\alpha+2\beta=0 \qquad \boxed{\alpha=-\beta}$$

$\beta=1$, $\alpha=-1$ sostituiamo in $(*)$

$$\boxed{\pi_{-1,1}: \ -(3x+y-z-1)+2x+5z=0}$$

$$\pi: \ -x-y+6z+1=0$$

3) Determinare un piano contenente r e parallelo alla retta $s: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle$.

$$r \begin{cases} 3x+y-z=1 \\ 2x+5z=0 \end{cases}$$

$$(*) \quad \boxed{\pi_{\alpha,\beta} = \alpha(3x+y-z-1)+\beta(2x+5z)=0}$$

$V_s \subseteq V_\pi$ condizione $s // \pi$

$$\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle \subseteq V_\pi : \alpha(3x+y-z)+\beta(2x+5z)=0$$

Basta richiedere che $\boxed{\begin{pmatrix}1\\1\\0\end{pmatrix} \in V_\pi} \Rightarrow \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle \subseteq V_\pi$

$x=1$
$y=1$
$z=0$

$$\alpha(3+1-0)+\beta(2)=0$$

$$\cancel{4}^{2}\alpha+\cancel{2}\beta=0 \qquad \boxed{\beta=-2\alpha} \qquad \boxed{\begin{matrix}\alpha=1\\ \beta=-2\end{matrix}} \; (*)$$

$$\boxed{\pi: 3x+y-z-1-2(2x+5z)=0}$$