FUNCTIONS OF BOUNDED VARIATION

We observed that if $f \in W^{1,p}(\mathbb{R}^n)$ $p \geq 1$ then

$$\underbrace{\|\tau_h f - f\|_p \leq \| |Df| \|_{L^p} \cdot |h|}_{} \qquad \forall\, h \in \mathbb{R}^n$$

Also the viceversa is true in $W^{1,p}(\mathbb{R}^n)$ for $p > 1$

Let $f \in L^p(\mathbb{R}^n)$ such that $\exists\, C$ $\|\tau_h f - f\|_p \leq C|h|$ then $f \in W^{1,p}(\mathbb{R}^n)$, for $p > 1$.

proof Let $\phi \in C_c^\infty(\mathbb{R}^n)$, $h \in \mathbb{R}^n$
$$\int_{\mathbb{R}^n} \left(\frac{\phi(x-h) - \phi(x)}{|h|}\right) \cdot f(x) = \int_{\mathbb{R}^n} \phi(x) \left[\frac{f(x+h) - f(x)}{|h|}\right] dx$$

$$\overset{\text{Holder}}{\leq} \frac{1}{|h|} \|\phi\|_{L^{p'}} \|\tau_h f - f\|_{L^p} \leq \|\phi\|_{L^{p'}} \frac{1}{\cancel{|h|}} C \cancel{|h|} = C\|\phi\|_{L^{p'}}$$

$h = t e_i$ $\quad t \to 0^+$ $\quad \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{\phi(x - t e_i) - \phi(x)}{t} \cdot f(x) = -\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i}(x) f(x)$

$$\Longrightarrow \quad \frac{\partial}{\partial x_i} T_f(\phi) \leq C\|\phi\|_{L^{p'}} \qquad \forall\, \phi \in C_c^\infty(\mathbb{R}^n) \qquad p' \in [1, +\infty) \text{ since } p > 1.$$

Since $\overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{p'}} = L^{p'}(\mathbb{R}^n) \quad \forall p' \in [1,+\infty)$ (+∞ EXCLUDED)

$\frac{\partial}{\partial x_i} T_f : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ with $\left|\frac{\partial}{\partial x_i} T_f(\phi)\right| \leq C\|\phi\|_{L^{p'}(\mathbb{R}^n)}$ can

be extended by density to a linear continuous functional

on $L^{p'}(\mathbb{R}^n)$

$\frac{\partial}{\partial x_i} T_f : L^{p'}(\mathbb{R}^n) \to \mathbb{R} \implies \left(\frac{\partial}{\partial x_i} T_f\right)$ is in the DUAL of

$L^{p'}(\mathbb{R}^n) \implies \left(L^{p'}(\mathbb{R}^n)\right)^* = L^p(\mathbb{R}^n)$ for $p' \in [1,+\infty)$

$\left(\frac{1}{p'} + \frac{1}{p} = 1\right)$

$\implies \exists\, g_i \in L^p(\mathbb{R}^n)$ such that

$\frac{\partial}{\partial x_i} T_f(h) = \int_U h \cdot g_i \, dx \quad \forall h \in L^{p'}(\mathbb{R}^n).$

$\implies$ also $\forall h \in C_c^\infty(\mathbb{R}^n) \subseteq L^{p'}(\mathbb{R}^n) \implies$

$g_i = \frac{\partial}{\partial x_i} f$ is the weak derivative $\implies f \in W^{1,p}(\mathbb{R}^n)$.

$\square$

For $p=1$ Assume $f \in L^1(\mathbb{R}^n)$ and $\exists\, C>0$ such that

$$\|\tau_h f - f\|_{L^1} \leq C|h| \quad \forall\, h \in \mathbb{R}^n$$

$$\Downarrow$$

$$\boxed{\left|\frac{\partial}{\partial x_i} T_f(\phi)\right| \leq C\|\phi\|_\infty} \quad \forall\, \phi \in C_c^\infty(\mathbb{R}^n)$$

[$\frac{\partial}{\partial x_i} T_f$ is a DISTRIB. OF ORDER ZERO]

$\Rightarrow$ by density $\left(\overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_\infty} = C_0(\mathbb{R}^n)\right)$ (functions which are continuous and $\lim_{|x|\to+\infty} f(x)=0$)

$\frac{\partial}{\partial x_i} T_f$ can be extended to a linear continuous functional on $C_0(\mathbb{R}^n)$

$\frac{\partial}{\partial x_i} T_f$ is a DISTRIBUTION of ORDER 0 and ALSO CONTINUOUS on $C_0(\mathbb{R}^n)$

$\frac{\partial}{\partial x_i} T_f$ is A FINITE SIGNED RADON MEASURE

$$\frac{\partial}{\partial x_i} T_f = \mu_i^+ - \mu_i^- \qquad \mu_i^+, \mu_i^- \in \mathcal{M}(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} d\mu_i^+, \int_{\mathbb{R}^n} d\mu_i^- < +\infty$$

$\Longrightarrow W^{1,1}(\mathbb{R}^n) \subsetneq \{ f \in L^1(\mathbb{R}^n) \;\; \frac{\partial}{\partial x_i} T_f$ is a distribution of order 0, that can be extended to a continuous linear functional on $\overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_\infty}$ $\}$

$= \{ f \in L^1(\mathbb{R}^n) \;\; \forall i \;\; \exists \mu_i^+, \mu_i^- \in \mathcal{M}(\mathbb{R}^n)$ FINITE

$\frac{\partial}{\partial x_i} T_f = \mu_i^+ - \mu_i^- \} := BV(\mathbb{R}^n)$

$BV(\mathbb{R}^n)$ are functions in $L^1(\mathbb{R}^n)$ such that their derivatives (in the sense of distributions) are finite signed Radon measures).

$W^{1,1}(\mathbb{R}^n)$ are elements in $BV(\mathbb{R}^n)$ whose derivative in the sense of distributions are finite signed Radon measures ABSOLUTELY CONTINUOUS W.R.T. Lebesgue

$W^{1,1}(\mathbb{R}^n) \subsetneq BV(\mathbb{R}^n)$ $\quad n=1$ CANTOR FUNTION!

$U \subseteq \mathbb{R}^n$ open set (it can be also $\mathbb{R}^n$)

$BV(U) := \{ f \in L^1(U)$ $\frac{\partial}{\partial x_i} T_f$ " $=$ " $\mu_i^+ - \mu_i^-$ $\mu_i^+, \mu_i^- \in \mathcal{M}(U)$ finite

that is $\forall \phi \in \mathcal{C}_c^\infty(U)$ (ACTUALLY $\forall \phi \in \mathcal{C}_c^1(U)$)

$$\forall i \quad \int_U \frac{\partial \phi}{\partial x_i} f \, dx = -\left[\int_U \phi \, d\mu_i^+ - \int_U \phi \, d\mu_i^-\right] \leq \|\phi\|_\infty \left[\mu_i^+(U) + \mu_i^-(U)\right] \}$$

$V(f, U) =$ TOTAL VARIATION of f in U $=$

$$= \sup \left\{ \int_U f \operatorname{div} \Phi \, dx \, , \quad \Phi \in \mathcal{C}_c^1(U, \mathbb{R}^n) \quad \|\Phi\|_\infty = \sqrt{\textstyle\sum_i \|\varphi_i\|_\infty^2} \leq 1 \right\}$$

$\Phi = (\varphi_1, \varphi_2, \dots \varphi_n)$

Note that $\int_U f \operatorname{div} \Phi \, dx = \int_U f \left(\sum_1^n \frac{\partial}{\partial x_i} \varphi_i(x) \right) dx =$

$$= \sum_i \int_U f \frac{\partial}{\partial x_i} \phi_i \, dx = -\sum_i \int_U \phi_i \, d\mu_i^+ - \int_U \varphi_i \, d\mu_i^- \leq$$

$$\leq \sum_i \|\phi_i\|_\infty \left(\mu_i^+(U) + \mu_i^-(U) \right) \leq \left(\sum_i \|\varphi_i\|_\infty^2 \right)^{1/2} \left(\sum_i \left[\mu_i^+(U) + \mu_i^-(U) \right]^2 \right)^{1/2}$$

$$V(f,U) = \sup \left\{ \int_U f \,\mathrm{div}\,\Phi \,dx \quad \|\Phi\|_\infty = \sqrt{\textstyle\sum_i \|\varphi_i\|_\infty^2} \le 1 \right.$$
$$\left. \Phi \in C^1_c(U;\mathbb{R}^n) \right\}$$

actually =

$$\le \left[\sum_i \left[\mu_i^+(U) + \mu_i^-(U)\right]^2 \right]^{1/2}.$$

If $f \in W^{1,1}(U)$ $\quad \forall \Phi \in C^1_c(U;\mathbb{R}^n)$

$$\int_U f \cdot \mathrm{div}\,\Phi \,dx = -\int_U Df \cdot \Phi \,dx$$ (Df weak gradient)

$$V(f,U) = \sup \left\{ \int_U Df \cdot \Phi \,dx \;,\; \Phi \in C^1_c(U;\mathbb{R}^n) \;,\; \|\Phi\|_\infty \le 1 \right\}$$
$$= \int_U |Df| \,dx$$

($Df \cdot \Phi \le |Df| \cdot \|\Phi\|_\infty$ CAUCHY SCHWARTZ)

then I approximate $\frac{Df}{|Df|}$ with Φ_k

For $f \in W^{1,1}(U)$ $\quad V(f,U) = \| \, |Df| \, \|_1$

Proposition Let $f_k \in BV(U)$ such that $f_k \to f$ in $L^1(U)$

1) if $\exists C>0$ $V(f_k,U) \le C \implies f \in BV(U)$ and

and $\liminf_k V(f_k,U) \ge V(f,U)$

2) if $f \in BV(U)$ then $\liminf_k V(f_k,U) \ge V(f,U)$.

In particular $V(f,U)$ is LSC with respect to L^1 convergence.

proof $\boxed{(1)\&(2)}$ $V(f_k,U) = \sup\left[\int_U f_k \operatorname{div} \Phi \quad \Phi \in C^1_c(U,\mathbb{R}^n) \right.$ $\|\Phi\|_\infty \le 1 \}$

fix Φ

$$\ge \int_U f_k \operatorname{div} \Phi \, dx \xrightarrow[L^1 \text{ conv.}]{} \int_U f \operatorname{div} \Phi \, dx$$

$$\liminf_k V(f_k,U) \ge \int_U f \operatorname{div} \Phi \, dx \quad \forall \Phi \in C^1_c(U,\mathbb{R}^n) \ \|\Phi\|_\infty \le 1$$

take the supremum. $\square$

Observe that

$$0 \le V(f,U) = V(-f,U) \qquad \forall f \in BV(U)$$

$$V(f+g,U) = \sup \left\{ \int_U (f+g) \operatorname{div} \Phi \, dx \quad \Phi \in C_c^1(U;\mathbb{R}^n) \ \|\Phi\|_\infty \le 1 \right\}$$

$$\le \sup \left\{ \int_U f \operatorname{div} \Phi \, dx, \ \Phi \in C_c^1(U;\mathbb{R}^n) \ \|\Phi\|_\infty \le 1 \right\} +$$

$$+ \sup \left\{ \int_U g \operatorname{div} \Phi \, dx, \ \Phi \in C_c^1(U,\mathbb{R}^n) \ \|\Phi\|_\infty \le 1 \right\} =$$

$$\le V(f,U) + V(g,U)$$

<u>Thus</u> $BV(U)$, $\|f\|_{BV} = \|f\|_{L^1} + \underbrace{V(f,U)}_{}$ is a BANACH SPACE $=\mu$

$$W^{1,1}(U), \|\cdot\|_{1,1} \xhookrightarrow[\text{CONTINUOUSLY}]{} (BV(U), \|\cdot\|_{BV})$$

<u>proof</u> f_k Cauchy in $BV(U) \Rightarrow f_k$ Cauchy in $L^1 \Rightarrow f_k \to f$ in L^1

$$V(f_k,U) \le C \to f \in BV(U) \qquad V(f,U) \le \liminf_k V(f_k,U)$$

$V(f_k, U)$ is Cauchy $\qquad f_k - f_h \to f - f_h \quad k \to +\infty$

$0 \le V(f - f_h, U) \le \liminf_{k} V(f_k - f_h, U) \implies$ send $h \to +\infty$

$\lim_h V(f - f_h, U) = 0 \implies \|f_k - f\|_{BV} \to 0$

Obs if $f \in W^{1,1}(U)$ $\quad \|f\|_{BV} = \|f\|_{L^1} + \||Df|\|_{L^1} \le C\|f\|_{W^{1,1}}$