$(\Omega, \mathcal{F}, \mathbb{P})$ $\qquad M^2 = \{X : \Omega \to \mathbb{R} \quad \mathbb{E}(|X|^2) < +\infty\}$

$\mathcal{P}(\mathbb{R}) =$ "probability measures on $\mathbb{R}$" $=$

$= \{\mu : \mathcal{B}(\mathbb{R}) \to [0, +\infty]$, finite $\quad \mu(\mathbb{R}) = 1\}$

all Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$

Theorem If $\mu \in \mathcal{P}(\mathbb{R})$, there exists $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space (NOT UNIQUE) and a random variable $X : \Omega \to \mathbb{R}$ such that

$$\mu = \mathcal{L}_X$$

$\mathcal{P}(\mathbb{R}) = \{$laws of all random variables taking values in $\mathbb{R}\}$

$\forall\, p \geq 1$

$$\mathcal{P}_p(\mathbb{R}) = \{ \mu \in \mathcal{P}(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} |x|^p \, d\mu(x) < +\infty \}$$

$\mathcal{P}_p(\mathbb{R})$ = { laws of random variables X with $E|X|^p < +\infty$ }

$$\mathbb{E}(|X|^p) = \int_{\mathbb{R}} |x|^p \, d\mathcal{L}_X(x)$$

By Jensen inequality

$$\mathcal{P}(\mathbb{R}) \supsetneq \mathcal{P}_1(\mathbb{R}) \supseteq \mathcal{P}_2(\mathbb{R}) \supseteq \dots$$

if $E|X|^p < +\infty \implies \mathbb{E}|X|^q < +\infty \quad \forall q \leq p$

$$\mathcal{P}_p(\mathbb{R}) = \bigcup_{(\Omega,\mathcal{F},\mathbb{P})} \left\{ \text{laws of } X \in M^p_{(\Omega,\mathcal{F},\mathbb{P})} \right\}$$

(every $\mu \in \mathcal{P}_p(\mathbb{R})$ is the law of some $X : (\Omega,\mathcal{F},\mathbb{P}) \to \mathbb{R}$
with $\mathbb{E}(|X|^p) < +\infty$, $X \in M^p$)

$X, Y \in M^p_{(\Omega,\mathcal{F},\mathbb{P})}$. $d_p(X,Y) = \left(\mathbb{E}|X-Y|^p\right)^{1/p}$

$p=2$ $$d_2(X,Y) = \sqrt{\mathbb{E}(|X-Y|^2)} = \sqrt{\int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\mathcal{L}_{(X,Y)}(x,y)}$$

$$(X,Y) : \Omega \longrightarrow \mathbb{R}\times\mathbb{R} \qquad \mathcal{L}_{(X,Y)} = (X,Y)\#\mathbb{P}$$
$$\omega \longmapsto (X(\omega), Y(\omega))$$

Definition (COUPLING BETWEEN MEASURES)

$\mu, \nu \in \mathcal{P}(\mathbb{R})$ $\left[\begin{array}{l}\mu, \nu : \mathcal{B}(\mathbb{R}) \to [0,+\infty] \\ \mu(\mathbb{R}) = 1 = \nu(\mathbb{R})\end{array}\right.$

A coupling between μ, ν is a BOREL MEASURE on $\mathbb{R} \times \mathbb{R}$

$$\pi : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \longrightarrow [0,+\infty]$$
$$A, B \longrightarrow \pi(A \times B)$$

such that the first marginal of π is μ and the second marginal is ν.

$\forall A \in \mathcal{B}(\mathbb{R})$
$$\pi(A \times \mathbb{R}) = \mu(A)$$
$$\pi(\mathbb{R} \times A) = \nu(A)$$

$\boxed{\text{The simplest possible coupling is } \quad \pi = \mu \otimes \nu \qquad \pi(A \times B) = \mu(A)\,\nu(B)}$

$\sim \cdot \sim$

$\mu, \nu \in \mathcal{P}(\mathbb{R})$ π is a coupling between μ, ν

$\rightarrow$ It is possible to prove that there exists at least one (more than one) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X, Y random variables

$X : \Omega \rightarrow \mathbb{R}$, $Y : \Omega \rightarrow \mathbb{R}$ such that

$\mu = \mathcal{L}_X$, $\nu = \mathcal{L}_Y$, $\pi = \mathcal{L}_{(X,Y)}$

$\mu, \nu \in \mathcal{P}(\mathbb{R})$

$\Pi(\mu,\nu) =$ {all couplings between μ, ν} that is

{all $\pi : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \longrightarrow [0,+\infty]$ such that

$\pi(A \times \mathbb{R}) = \mu(A)$

$\pi(\mathbb{R} \times A) = \nu(A)$ }

$=$ { all possible joint laws of random variables X, Y with $\mathcal{L}_X = \mu$, $\mathcal{L}_Y = \nu$ }

$$= \bigcup_{(\Omega, \mathcal{F}, \mathbb{P})} \bigcup_{\substack{X : \Omega \to \mathbb{R} \\ Y : \Omega \to \mathbb{R} \\ \mathcal{L}_X = \mu,\ \mathcal{L}_Y = \nu}} \{ \mathcal{L}_{(X,Y)} \}$$

Among all couplings there are

(1) the coupling associated with the joint law of X, Y <u>independent</u>

$$\pi = \mu \otimes \nu \qquad \pi(A \times B) = \mu(A)\,\nu(B)$$

(2) DETERMINISTIC COUPLINGS (NOT ALWAYS PRESENT)

$\varphi : \mathbb{R} \to \mathbb{R}$ MEASURABLE with respect to μ

(for example φ <u>CONTINUOUS</u>, or <u>φ MONOTONE</u>)

such that

$$\boxed{\varphi_{\#}\mu = \nu} \qquad \mathbb{R}, \mu \xrightarrow{\varphi} \mathbb{R}, \nu$$

$\forall A \in \mathcal{B}(\mathbb{R})$ $\quad \nu(A) = \mu\{x \in \mathbb{R},\ \varphi(x) \in A\}$

This means that if $\mu = \mathcal{L}_X$ $\nu = \mathcal{L}_Y$

$$X : \Omega \to \mathbb{R}$$
$$Y : \Omega \to \mathbb{R}$$

then actually $Y = \psi(X)$

$$Y : \Omega \longrightarrow \mathbb{R}$$

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend right, "\psi(X) = Y"'] & \mathbb{R} \arrow[r, "\psi"] & \mathbb{R} \end{tikzcd}$$

$$\pi = \underline{(\mathrm{Id}, \psi)} \# \mu$$

μ

$$(\mathrm{Id}, \psi) : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$$
$$x \longmapsto (x, \psi(x))$$

π is a deterministic coupling if $\exists \psi : \mathbb{R} \to \mathbb{R}$

$\pi = (Id, \psi) \# \mu$

$$\pi(A \times B) = \mu \{ x \in \mathbb{R} \quad (Id, \psi)(x) = (x, \psi(x)) \in A \times B \}$$
$$\| $$
$$= \mu \{ x \in \mathbb{R}, \quad x \in A, \ \psi(x) \in B \}$$
$$\|$$
$$= \mu \{ x \in A, \psi(x) \in B \}$$

$$\mathcal{L}_{(X, \psi(X))}(A \times B) = \mathbb{P} \underbrace{\{ \omega \in \Omega \quad X(\omega) \in A,}\ \psi(X(\omega)) \in B \}$$
$$\|$$
$$= \mu \{ x \in \mathbb{R} \quad x \in A, \quad \psi(x) \in B \}$$

OBSERVATION

Deterministic couplings between μ and ν does not exist always.

ex $\Rightarrow$ $\mu = \delta_0$ $\qquad \mu(A) = \begin{cases} 0 & 0 \notin A \\ 1 & 0 \in A \end{cases}$

$\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_3$ $\qquad \nu(A) = \begin{cases} 0 & 1,3 \notin A \\ \frac{1}{2} & 1 \in A,\ 3 \notin A \\ \frac{1}{2} & 1 \notin A,\ 3 \in A \\ 1 & 1 \in A,\ 3 \in A \end{cases}$

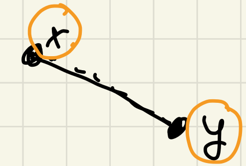
$\psi : \mathbb{R} \to \mathbb{R}$

$0 \to 1$, $0 \to 3$

NOT A FUNCTION

it DOES NOT EXIST A DETERMINISTIC COUPLING BETWEEN μ and ν.

Problem of optimal transport



$\mu, \nu \in \mathcal{P}(\mathbb{R})$

$c(x,y) : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$

"COST"

$x, y \in \mathbb{R}$ $\quad c(x,y)$ measures the cost of "moving" x to y

$$c(x,y) = |x-y|^p \qquad p \geq 1$$

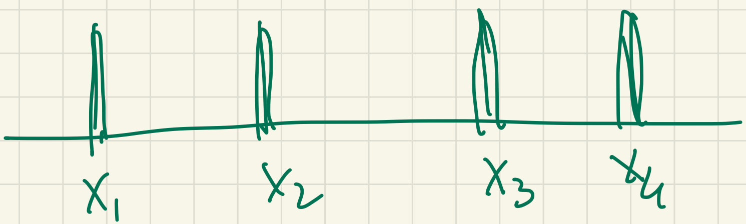
find

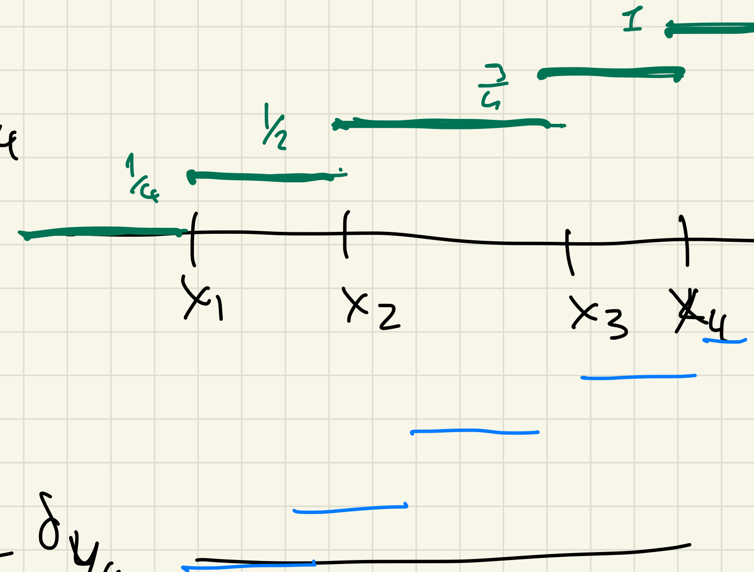
$$\inf_{\pi \in \Pi(\mu,\nu)} \left[\int_{\mathbb{R}\times\mathbb{R}} c(x,y) \, d\pi(x,y) \right]$$

$d\pi(x,y)$ "amount" of mass "$\mu(x)$" moving to "$\nu(y)$"

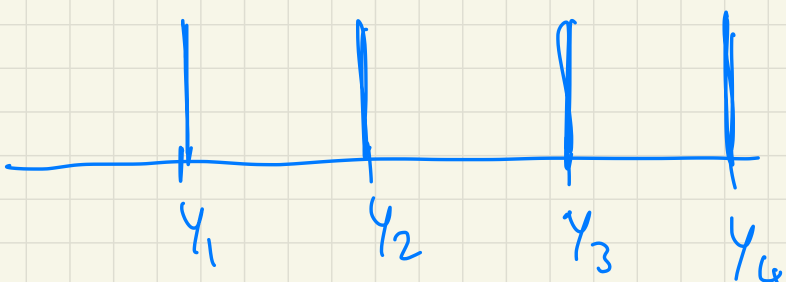
Example $\quad c(x,y) = |x-y| \quad$ or $\quad |x-y|^2 = c(x,y)$

$$\mu = \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_{x_2} + \frac{1}{4}\delta_{x_3} + \frac{1}{4}\delta_{x_4}$$





$$\nu = \frac{1}{4}\delta_{y_1} + \frac{1}{4}\delta_{y_2} + \frac{1}{4}\delta_{y_3} + \frac{1}{4}\delta_{y_4}$$



$$\psi : \begin{array}{l} x_1 \longrightarrow y_1 \\ x_2 \longrightarrow y_2 \\ x_3 \longrightarrow y_3 \\ x_4 \longrightarrow y_4 \end{array}$$

$$\nu = \psi \# \mu$$

(all possible ψ sending x_i to y_j one-to-one are all deterministic coupling)

$$\psi : x_i \longrightarrow y_j$$

$$\pi = (Id, \psi) \# \mu = \frac{1}{4} \sum_{i=1}^{4} \delta_{(x_i, \psi(x_i))}$$

$$0 \leq \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y| \, d\pi(x,y) \leq \inf_{\substack{\psi \\ \text{one-to-one}}} \frac{1}{4} \sum_{i=1}^{4} |x_i - \psi(x_i)|$$

How to interpret $\pi \in \Pi(\mu,\nu)$ coupling between μ and ν as a sort of "TRANSPORT PLAN" between μ and ν.

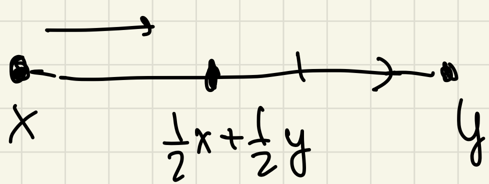
$$\pi : \mathcal{B}(\mathbb{R}\times\mathbb{R}) = \mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}) \longrightarrow [0,+\infty]$$

Fix $t\in[0,1]$

$$\phi_t : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,y) \mapsto (1-t)x + ty$$

ϕ_t "LEGGE ORARIA"

$$\phi_0 : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R} \qquad \phi_1 : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,y) \longrightarrow x \qquad (x,y) \longrightarrow y$$



"t" time

time 0 I am in x

time 1 I am in y

$t \in [0,1]$ FIXED

$$\phi_t : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto (1-t)x + ty$$

π

$\phi_t \# \pi$

$A \in \mathcal{B}(\mathbb{R})$

$$\phi_t \# \pi \, (A) = \pi \{ (x,y) \in \mathbb{R}^2 \;\; \phi_t(x,y) \in A \}$$

$$= \pi \{ (x,y) \in \mathbb{R}^2 \quad \underline{(1-t)x + t}\, y \in A \}$$

$$\phi_0 \# \pi \, (A) = \pi \{ (x,y) \in \mathbb{R}^2 \quad x \in A \} =$$

$$= \pi (A \times \mathbb{R}) = \mu(A)$$

$$\phi_1 \# \pi (A) = \pi \{ (x,y) \in \mathbb{R}^2 \quad y \in A \} =$$

$$= \pi (\mathbb{R} \times A) = \nu(A)$$

Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, the problem is find (if it exists) an "optimal coupling" $\bar{\pi} \in \Pi(\mu,\nu)$ such that

$$\int_{\mathbb{R}\times\mathbb{R}} c(x,y)\, d\bar{\pi}(x,y) = \min_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} c(x,y)\, d\pi(x,y)$$

$$c(x,y) = |x-y|^p$$

Fix $\pi \in \Pi(\mu,\nu) \to \exists\ (\Omega, \mathcal{F}, \mathbb{P})$ and $X, Y : \Omega \to \mathbb{R}$ such that $\mathcal{L}_X = \mu \quad \mathcal{L}_Y = \nu \quad \mathcal{L}_{(X,Y)} = \pi$

$$\int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, d\pi(x,y) = \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, d\mathcal{L}_{(X,Y)}(x,y) =$$

$$= \mathbb{E}\left(|X-Y|^p\right)$$

$$\inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}\times\mathbb{R}} |x-y|^p d\pi(x,y)\right)^{1/p} = \inf_{(\Omega,\mathcal{F},\mathbb{P})} \ \inf_{\substack{X,Y \\ X:\Omega\to\mathbb{R}\ \mathcal{L}_X=\mu \\ Y:\Omega\to\mathbb{R}\ \mathcal{L}_Y=\nu}} \left(\mathbb{E}|X-Y|^p\right)^{1/p}$$

the optimal transport problem between two measures $\mu, \nu \in \mathcal{P}(\mathbb{R})$ with cost $|x-y|^p$ is actually finding among all possible random variables (defined on whatever probability space) X, Y with $\mathcal{L}_X = \mu$ $\mathcal{L}_Y = \nu$, the couple X, Y with minimal distance (in the M^p sense, that $\mathbb{E}(|X-Y|^p)$ has to be minimal)

$\boxed{p=2}$

$$\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y)$$

$\mu,\nu \in \mathcal{P}_2(\mathbb{R})$

$$= \inf_{\substack{(\Omega,\mathcal{F},\mathbb{P}) \\ X:\Omega\to\mathbb{R} \quad \mathcal{L}_X=\mu \\ Y:\Omega\to\mathbb{R} \quad \mathcal{L}_Y=\nu}} \mathbb{E}|X-Y|^2 = \underline{\underline{\mathbb{E}(|X|^2)}} + \underline{\underline{\mathbb{E}(|Y|^2)}} - 2\sup_{X,Y} \mathbb{E}(X\cdot Y)$$

$$\mathbb{E}|X-Y|^2 = \mathbb{E}\left(X^2+Y^2-2X\cdot Y\right) = \mathbb{E}|X|^2+\mathbb{E}|Y|^2-2\,\mathbb{E}(X\cdot Y)$$

$$= \inf_{\substack{(\Omega,\mathcal{F},\mathbb{P}) \\ X \quad \mathcal{L}_X=\mu \\ Y \quad \mathcal{L}_Y=\nu}} \underbrace{\mathbb{E}(|X|^2)}_{\substack{\text{fixed} \\ \int_{\mathbb{R}}|x|^2 d\mu(x)}} + \underbrace{\mathbb{E}(|Y|^2)}_{\substack{\text{fixed} \\ \int_{\mathbb{R}}|x|^2 d\nu(x)}} - 2\,\mathbb{E}(X\cdot Y)$$

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y) =$$

$$= \underbrace{\int_{\mathbb{R}} |x|^2 \, d\mu}_{\text{fixed}} + \underbrace{\int_{\mathbb{R}} |y|^2 \, d\nu}_{\text{fixed}} - \sup_{\substack{(\Omega,\mathcal{F},\mathbb{P}) \\ X:\Omega\to\mathbb{R} \ \mathcal{L}_X=\mu \\ Y:\Omega\to\mathbb{R} \ \mathcal{L}_Y=\nu}} E(X\cdot Y)$$

minimizing $\int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y)$ among all possible couplings between μ,ν is equivalent to maximizing the covariance between X,Y, for X random variable with law μ and Y random variable with law ν, (obviously X,Y defined on the same probability space, which is NOT FIXED A PRIORI).