The space $W^{1,n}(\mathbb{R}^n)$ (NOT EMBEDDED in $L^\infty(\mathbb{R}^n)$)

We have the following result

Theorem $W^{1,n}(\mathbb{R}^n) \xrightarrow[\text{CONTINUOUSLY}]{} BMO(\mathbb{R}^n)$

In particular: $\exists\, C(n)$ s.t. $\forall f \in W^{1,n}(\mathbb{R}^n)$

$$\|f\|_{BMO(\mathbb{R}^n)} \leq C(n) \, \| |Df| \|_{L^n}$$

Def: $f \in BMO(\mathbb{R}^n)$ (function of Bounded Mean Oscillation)

if

$$\boxed{\sup_{B_r \subseteq \mathbb{R}^n} \frac{1}{|B_r|} \int_{B_r} \left| f(x) - \frac{1}{|B_r|} \int_{B_r} f(y)\,dy \right| dx} < +\infty$$

B_r ball of radius r

$= \|f\|_{BMO}$

$f, g \in BMO(\mathbb{R}^n)$ $\quad f \sim g$ iff $f = g + c$

$\left(BMO(\mathbb{R}^n)/\sim \, , \|\cdot\|_{BMO}\right)$ is a Banach space

Obs $f \in L^\infty(\mathbb{R}^n) \rightarrow f \in BMO(\mathbb{R}^n)$ $\left| \fint_{B_r} f \right| \leq \|f\|_\infty$ $\|f\|_{BMO} \leq 2\|f\|_\infty$

$L^\infty(\mathbb{R}^n) \underset{\neq}{\subset} BMO(\mathbb{R}^n)$

$f(x) = \lg|x| \in BMO(\mathbb{R}^n)$ indeed $\frac{1}{r^n \omega_n} \int_{B_r} \lg|x| \, dx = \frac{1}{r^n \cancel{\omega_n}} \int_0^r \lg s \; n s^{n-1} \cancel{\omega_n} \, ds$

(by parts) $= \frac{1}{r^n} \left[\lg s \; s^n \right]_0^r - \frac{1}{r^n} \int_0^r \frac{1}{s} s^n \, ds = \lg r - \frac{1}{n}$

$\| \lg|x| \|_{BMO} = \sup_{B_r} \frac{1}{\omega_n r^n} \int_{B_r} \left| \lg|x| - \lg r + \frac{1}{n} \right| dx = \sup_{B_r} \frac{1}{\cancel{\omega_n} r^n} \int_0^r n \cancel{\omega_n} s^{n-1} \left| \lg \frac{s}{r} + \frac{1}{n} \right| ds =$

$= \sup_{B_r} \int_0^r n \left(\frac{s}{r} \right)^{n-1} \left| \lg\left(\frac{s}{r}\right) + \frac{1}{n} \right| \frac{ds}{r} = \int_0^1 n t^{n-1} \left| \lg t + \frac{1}{n} \right| dt < +\infty$

Theorem $\exists \, C(n) > 0$ $\forall f \in W^{1,n}(\mathbb{R}^n)$ it holds $f \in BMO(\mathbb{R}^n)$

$\|f\|_{BMO} \leq C(n) \| |Df| \|_{L^n(\mathbb{R}^n)}$

proof $\quad B_r \subseteq \mathbb{R}^n \quad f \in W^{1,n}(\mathbb{R}^n) \Rightarrow f \in W^{1,n}(B_r) \Rightarrow f \in W^{1,p}(B_r)$
$\forall\, p < n.$

$$\int_{B_r} \left| f(x) - \frac{1}{|B_r|}\int_{B_r} f(y)\,dy \right| dx = \left\| f - \frac{1}{|B_r|}\int_{B_r} f(y)\,dy \right\|_{L^1(B_r)} \leq$$ POINCARÉ for $W^{1,1}(B_r)$

$$\leq C(n,1,B_r)\, \| |Df| \|_{L^1(B_r)} = r\, C(n,1,B(0,1)) \int_{B_r} |Df(y)|\,dy \leq$$

$\leq$ Holder
$|Df| \in L^n$
$\chi_{B_r} \in L^{n/n-1}$

$$\leq r\, C(n,1,B(0,1))\, \|Df\|_{L^n(B_r)}\, |B_r|^{1-\frac{1}{n}} =$$ $|B_r| = \omega_n r^n$

$$\leq r\, C(n,1,B(0,1))\, \| |Df| \|_{L^n(\mathbb{R}^n)}\, \omega_n^{1-\frac{1}{n}}\, r^{n-1}$$

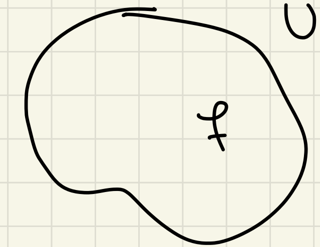
$$\frac{1}{\omega_n r^n} \int_{B_r} \left| f(x) - \frac{1}{|B_r|}\int_{B_r} f(y)\,dy \right| dx \leq \frac{1}{\cancel{\omega_n} \cancel{r^n}}\, \cancel{r}\, C(n,1,B(0,1))\, \|Df\|_{L^n}\, \omega_n^{\cancel{1}-\frac{1}{n}}\, \cancel{r^{n-1}}$$

$$\|f\|_{BMO} = \sup_{B_r} \frac{1}{\omega_n r^n} \int_{B_r} \left| f(x) - \frac{1}{|B_r|}\int_{B_r} f(y)\,dy \right| dx \leq \frac{C(n,1,B(0,1))}{\omega_n^{1/n}}\, \| |Df| \|_{L^n(\mathbb{R}^n)}$$

$\square$.

HARMONIC EXTENSION

$\boxed{\text{Let } U \text{ be a bdd open set of class } C^1 \text{ and } f \in W^{1,2}(U) \text{ be a fixed function}}$



The harmonic extension of f in U is the function $g \in W^{1,2}(U)$ such that

① $-\Delta g = 0$ in U (in the sense of distributions
that is $\int_U g \Delta \phi \, dx = 0 \ \forall \ \phi \in C_c^\infty(U)$)

② $Tr\, g = Tr\, f$ ("$g = f$ on ∂U")

$\Updownarrow$

$g = f + h$ for some $h \in W_0^{1,2}(U)$

$(g \in f + W_0^{1,2}(U))$

Obs 1 : If f is harmonic $g = f$

Obs 2 : the harmonic extension is unique

Let g_1, g_2 2 harmonic extension

$g_1 - g_2 \in W_0^{1,2}(U)$ $\qquad \int_U \Delta\phi (g_1 - g_2) dx = -\int_U D\phi \cdot D(g_1 - g_2) = 0$

$\forall \phi \in C_c^\infty(U)$

by density $\exists \phi_k \in C_c^\infty(U)$ $\quad \phi_k \to g_1 - g_2$ in $W^{1,2}$

$$0 = \int_U D\phi_k \cdot D(g_1 - g_2)\, dx \longrightarrow \int_U D(g_1 - g_2) \cdot D(g_1 - g_2)\, dx = \int_U |D(g_1 - g_2)|^2 dx$$

$\Rightarrow D(g_1 - g_2) = 0$ a.e in U $\Rightarrow g_1 - g_2 = c$ on every connected component of U but $tr(g_1 - g_2) = c = 0$

$\Rightarrow g_1 = g_2$ a.e $x \in U$.

To construct the harmonic extension we use a variational argument.

$$E(g) = \text{Dirichlet energy} = \int_U |Dg|^2 \, dx$$

$\forall\ g \in \{f + W_0^{1,2}(U)\} = \{ g \in W^{1,2}(U) \ \ tr(g) = tr(f) \text{ on } \partial U,$
or equivalently $g - f \in W_0^{1,2}(U) \}$

$$0 \leq E(g) < +\infty \quad \forall\, g \in W^{1,2}(U)$$

$$0 \leq \underbrace{\inf_{g \in f + W_0^{1,2}(U)} E(g)} \leq E(f) \qquad (\text{since } f \in f + W_0^{1,2}(U)!)$$

We want to prove the infimum is a minimum.

Let $\bar{c} = \inf\limits_{g \in f + W_0^{1,2}(U)} \int_U |Dg|^2 dx = \inf\limits_{g \dots} E(g)$

$\forall k \in \mathbb{N}$ $\exists g_k \in f + W_0^{1,2}(U)$ such that

$$\bar{c} \leq \int_U |Dg_k|^2 dx \leq \bar{c} + \frac{1}{k} \leq \bar{c} + 1$$

(by definition of infimum!)

$\Rightarrow \int_U |Dg_k|^2 dx \leq \bar{c} + 1$

$g_k \in f + W_0^{1,2}(U) \Rightarrow \forall k \; \exists h_k \in W_0^{1,2}(U)$ such that

$f + h_k = g_k$ $\qquad |Dg_k| = |Df + Dh_k| \geq |Dh_k| - |Df|$

$|Dh_k| \leq |Dg_k| + |Df|$

$$\int_U |Dh_k|^2 dx \le \int_U (|Dg_k| + |Df|)^2 dx \le$$

$$\le 2\int_U |Dg_k|^2 dx + 2\int_U |Df|^2 dx \le$$

$$\le 2(\overline{C}+1) + 2\int_U |Df|^2 dx \le \overline{C}$$

by Poincaré inequality on $W_0^{1,2}(U)$

$$\|h_k\|_{W^{1,2}(U)} \le C\left(\|h_k\|_{L^2} + \||Dh_k|\|_{L^2}\right) \le$$

$$\le \overline{C}\, \||Dh_k|\|_{L^2} \le \overline{C}$$

h_k is equibounded in $W^{1,2}(U)$ $\rightarrow$ by Rellich-Kondrachov

$h_{k_n} \rightarrow h$ in $L^2(U)$ $\quad \exists\, h \in W^{1,2}(U)$

$Dh_{k_n} \rightarrow Dh$ weakly in $L^2(U)$

Note that since $W_0^{1,2}(U)$ is weakly closed $\Rightarrow$

$\Rightarrow h \in W_0^{1,2}(U)$

$g_{k_m} = f + h_{k_m} \longrightarrow f + h = \bar{g}$ strongly in $L^2(U)$ and weakly in $W^{1,2}(U)$

$$\underbrace{Dg_k \rightharpoonup D\bar{g} \text{ weakly in } L^2}$$

$$C + \frac{1}{k} \geq \int_U |Dg_{k_m}|^2 \, dx \underset{\text{convexity}}{\geq} \int_U |D\bar{g}|^2 \, dx + \underbrace{\int_U 2(Dg_k - D\bar{g}) \cdot D\bar{g} \, dx}_{\to 0}$$

$|Dg_k(x)|^2 \geq |Dg(x)|^2 + 2(Dg_k(x) - D\bar{g}(x)) \cdot D\bar{g}(x)$

$2 \int_U (Dg_k - D\bar{g}) \cdot \underbrace{D\bar{g}}_{\in L^2} \, dx \longrightarrow 0$ by weak convergence in $L^2(U)$

as $k \to +\infty$ $\quad C \geq \liminf_k \int_U |Dg_k|^2 dx \geq \int_U |D\bar{g}|^2 dx \geq C$

$\bar{g}$ is a MINIMIZER $\quad C = E(\bar{g}) = \int_U |D\bar{g}|\, dx = \inf_{g \in f + W_0^{1,2}} \int_U |Dg|^2$

we compute the Gateaux derivative of $E(g)$ at $\bar{g}$

$\varepsilon \in \mathbb{R} \quad \phi \in C_c^\infty(U) \qquad \bar{g} + \varepsilon\phi \in f + W_0^{1,2}(U)$ since $\varepsilon\phi \in W_0^{1,2}(U)$

$$\frac{E(\bar{g}+\varepsilon\phi) - E(\bar{g})}{\varepsilon} \quad \begin{array}{ll} \geq 0 & \varepsilon > 0 \\ \leq 0 & \varepsilon < 0 \end{array}$$

$$\frac{E(\bar{g}+\varepsilon\phi) - E(\bar{g})}{\varepsilon} = \int_U \frac{|D\bar{g}+\varepsilon D\phi|^2 - |D\bar{g}|^2}{\varepsilon}\, dx =$$

$$= \int_U \frac{2\varepsilon\, D\bar{g}\cdot D\phi + \varepsilon^2 |D\phi|^2}{\varepsilon}\, dx = 2\int_U D\bar{g}\cdot D\phi\, dx + \overbrace{\varepsilon \int_U |D\phi|^2 dx}^{\xrightarrow{\varepsilon \to 0}\ 0}$$

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{E(\bar{g}+\varepsilon\phi) - E(\bar{g})}{\varepsilon} = \int_U D\bar{g}\cdot D\phi\, dx = 0$$

$$\Rightarrow 0 = \int_U D\bar{g} \cdot D\phi \, dx = -\int_U \bar{g} \cdot \Delta\phi \, dx = -\Delta T_{\bar{g}}(\phi)$$

$\Rightarrow$ $\bar{g}$ is a solution of $-\Delta \bar{g} = 0$ in U in the sense of distributions

$\bar{g}$ is the harmonic extension of f

$$\begin{cases} -\Delta \bar{g} = 0 & U \\ \bar{g} = f & \partial U \end{cases}$$