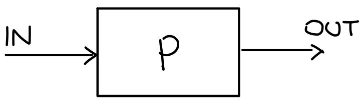
COMPUTABILITY (09/12/2025)

RICE - SHAPIRO'S THEOREM

properties of the behaviour of programs



two possible views :

(1) properties of the function computed by a program P

$$\mathcal{A} \subseteq \mathcal{C}$$

$\mathcal{T} = \{ f \in \mathcal{C} \mid f \text{ is total} \} = \{ f \in \mathcal{C} \mid dom(f) = \mathbb{N} \}$

$ONE = \{ \mathbb{1} \}$

$\mathcal{B}_n = \{ f \in \mathcal{C} \mid n \in cod(f) \}$ $\quad n \in \mathbb{N}$ fixed

$\vdots$

(2) extensional / saturated property of programs $A \subseteq \mathbb{N}$

$T = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{T} \}$

$P_{ONE} = \{ x \in \mathbb{N} \mid \varphi_x = \mathbb{1} \}$

$B_n = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{B}_n \}$

Rice's Theorem :

no extensional property, apart from the trivial ones (TRUE / FALSE), is decidable

Rice - Shapiro's Theorem :

an extensional property can be semi-decidable only if it is finitary

it depends only on a finite amount of inputs

Rice - Shapiro's Theorem

Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions.

and let $A \subseteq \mathbb{N}$ be $A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

If $\boxed{A \text{ is r.e.}}$ $(*)$ then

$$\boxed{\forall f \quad (f \in \mathcal{A} \Leftrightarrow \exists \vartheta \subseteq f, \vartheta \text{ finite}, \vartheta \in \mathcal{A})} \quad (**)$$

proof

We want to show that $(*) \Rightarrow (**)$

we prove $\neg (**) \Rightarrow \neg (*)$

This amounts to

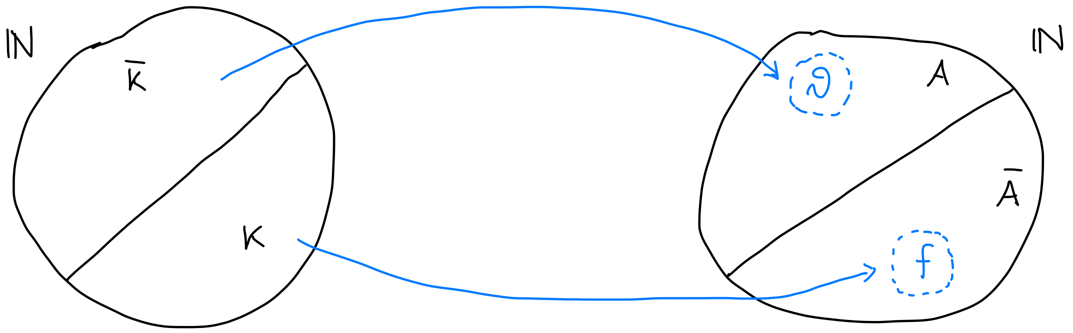
① $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f, \vartheta$ finite, $\vartheta \in \mathcal{A}$

② $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f, \vartheta$ finite, $\vartheta \notin \mathcal{A}$

① $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f, \vartheta$ finite, $\vartheta \in \mathcal{A} \Rightarrow A$ not r.e.

let $f \notin \mathcal{A}$ and let $\vartheta \subseteq f$, ϑ finite, $\vartheta \in \mathcal{A}$

We show $\underbrace{\bar{K} = \{ x \in \mathbb{N} \mid \varphi_x(x) \uparrow \}}_{\text{not r.e.}} \leq_m A$ $\rightsquigarrow$ A not r.e.



Define

$$g(x,y) = \begin{cases} \overset{\vartheta \subseteq f}{\vartheta(y)} & \text{if } x \in \bar{K} \\ f(y) & \text{if } x \in K \end{cases}$$

$$= \begin{cases} \vartheta(y) = f(y) & \text{if } x\in\overline{K} \text{ and } y\in dom(\vartheta) \\ \uparrow & \text{if } x\in \overline{K} \text{ and } y\notin dom(\vartheta)\\ f(y) & \text{if } x\in K\end{cases}$$

$$= \begin{cases} f(y) & \text{if } x\in K \text{ or } y \in dom(\vartheta)\\ \uparrow & \underbrace{\text{if } x\in\overline{K} \text{ and } y\notin dom(\vartheta)}_{\text{otherwise}}\end{cases}$$

if we define $Q(x,y) \equiv$ "$\underbrace{\underbrace{x\in K}_{\text{semidec.}} \text{ or } \underbrace{y\in dom(\vartheta)}_{\text{decidable } (dom(\vartheta) \text{ finite})}}_{\text{semidecidable}}$"

$$= \underbrace{f(y) \cdot \underbrace{sc_Q(x,y)}_{\begin{cases}1 & \text{if } Q(x,y)\\ \uparrow & \text{otherwise}\end{cases}}}_{\begin{cases}f(y) & \text{if } Q(x,y)\\ \uparrow & \text{otherwise}\end{cases}}$$

sc_Q computable

computable by composition. Hence by smn theorem there is $s:\mathbb{N}\to\mathbb{N}$ total and computable s.t. $\forall x,y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases}\vartheta(y) & \text{if } x\in\overline{K}\\ f(y) & \text{if } x\in K\end{cases}$$

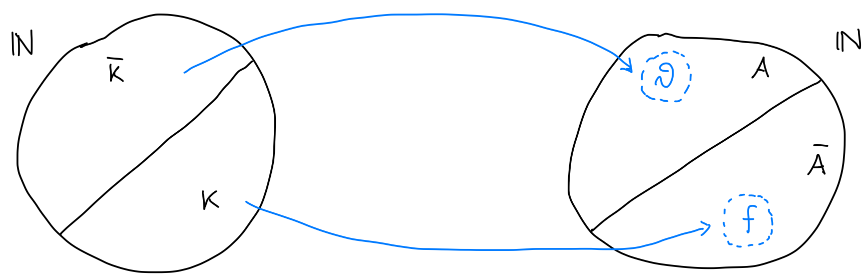
We show that s is the reduction function for $\overline{K} \leq_m A$

* if $x\in\overline{K}$ then $s(x)\in A$

let $x\in\overline{K}$. Then $\varphi_{s(x)}(y) = \vartheta(y)$ $\forall y$; Thus $\varphi_{s(x)} = \vartheta$

and hence

$s(x)\in A$



$*$ if $x \notin \overline{K}$ $(x \in K)$ then $s(x) \notin A$

let $x \in K$. Then $\forall y$ $\varphi_{s(x)}(y) = f(y)$. Hence $\varphi_{s(x)} = f$

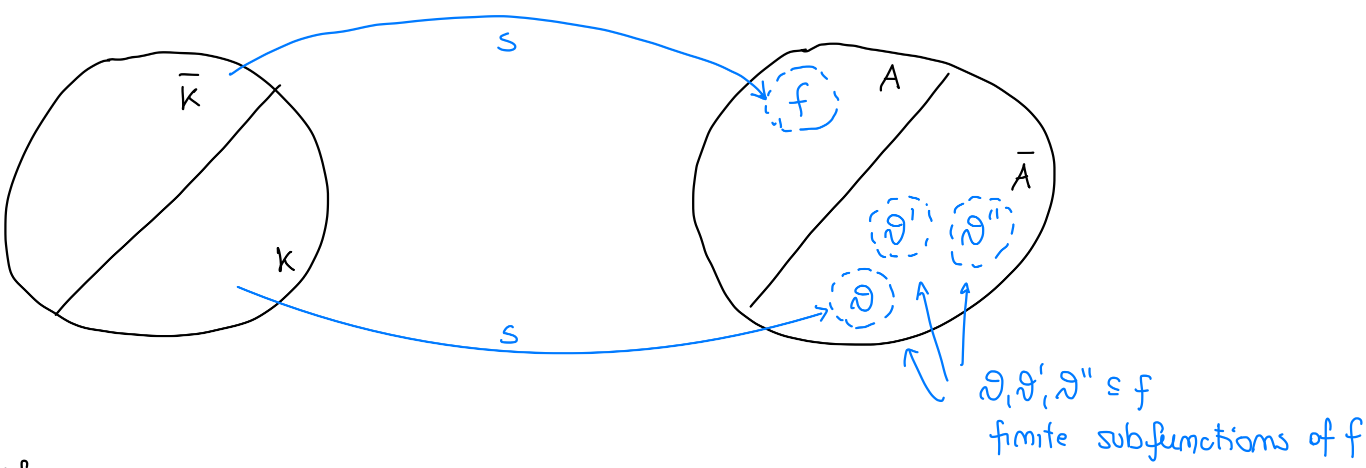
and thus $s(x) \in \overline{A}$

Thus $\overline{K} \leq_m A$ and, since $\overline{K}$ is not r.e., then A not r.e. $\bullet$

(2) $\exists f$ $f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite, $\vartheta \notin \mathcal{A}$ $\Rightarrow$ A not r.e.

let $f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite, $\vartheta \notin \mathcal{A}$

and we show $\overline{K} \leq_m A$



We define

$$g(x,y) = \text{"}\begin{cases} f(y) & \text{if } x \in \overline{K} \\ \vartheta(y) & \text{if } x \in K \end{cases}\text{"}$$

$\varphi_x(x)\uparrow$ $P_x(x)$ never halts

$\varphi_x(x)\downarrow$ $P_x(x)$ halts after a finite number of steps

$\vartheta(y)$: some finite $\vartheta \subseteq f$

$$= \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

$$= f(y) + \underbrace{\mu w. \underbrace{\chi_H(x,x,y)}_{\substack{0 \text{ if } \neg H(x,x,y) \\ 1 \text{ if } H(x,x,y)}}}_{\substack{0 \text{ if } \neg H(x,x,y) \\ \uparrow \text{ otherwise}}}$$

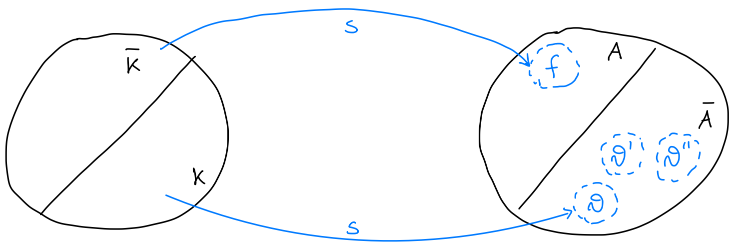
computable

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable such that

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

We show that s is the reduction function for $\bar{K} \leq_m A$

* if $x \in \bar{K}$ then $s(x) \in A$



let $x \in \bar{K}$ i.e. $\varphi_x(x)\uparrow$

i.e. $\forall y \quad \neg H(x,x,y)$

hence
$$\varphi_{s(x)}(y) = f(y) \quad \forall y$$

therefore $\varphi_{s(x)} = f$. Thus $s(x) \in A$.

* if $x \in K$ then $s(x) \notin A$

let $x \in K$ i.e. $\varphi_x(x)\downarrow$. There is $y_0 \in \mathbb{N}$, number of steps such that $P_x(x)\downarrow$, i.e.

$$\forall y < y_0 \quad \neg H(x,x,y)$$
$$\forall y \geq y_0 \quad H(x,x,y)$$

Hence
$$\varphi_{s(x)}(y) = \begin{cases} f(y) & \text{if } y < y_0 \\ \uparrow & \text{if } y \geq y_0 \end{cases}$$

Observe that $\varphi_{s(x)} \subseteq f$ and $dom(\varphi_{s(x)}) \subseteq [0, y_0)$ finite

hence $s(x) \in \bar{A}$.

Thus $\bar{K} \leq_m A$ and, since $\bar{K}$ is not r.e., neither A is.

$\square$

Typical use of Rice-Shapiro's theorem

Prove that $A \subseteq \mathbb{N}$ is not r.e.

(a) A is saturated / extensional $\qquad A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$

$\mathcal{A} \subseteq \mathcal{C}$

(b) Show that $\mathcal{A}$ is not finitary (showing (1) or (2))

(1) $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$, ϑ finite, $\vartheta \in \mathcal{A}$

(2) $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite, $\vartheta \notin \mathcal{A}$

Example: Totality $\qquad \mathcal{T} = \{f \in \mathcal{C} \mid f \text{ is total}\}$

$T = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{T}\}$

* T is not r.e.

$id \in \mathcal{T} \qquad dom(id) = \mathbb{N}$

$\forall \vartheta \subseteq id \quad \vartheta$ finite $\quad dom(\vartheta)$ finite $\neq \mathbb{N} \quad \Rightarrow \quad \vartheta \notin \mathcal{T}$ (2)

$\Rightarrow$ by Rice-Shapiro T is not r.e.

* $\overline{T}$ is not r.e.

$id \notin \overline{\mathcal{T}}$ and $\vartheta = \emptyset \subseteq id$, $\vartheta \in \overline{\mathcal{T}}$ (1)

$\emptyset(x) \uparrow$
$\forall x$

$\Rightarrow$ $\overline{T}$ is not r.e. by Rice-Shapiro

Example: $\qquad ONE = \{x \mid \varphi_x \in \{\mathbb{1}\}\}$

- ONE is not r.e.

$\mathbb{1} \in \{\mathbb{1}\}$ and $\forall \vartheta \subseteq \mathbb{1}$, ϑ finite $\quad \vartheta \notin \{\mathbb{1}\}$ (2)

hence by Rice-Shapiro ONE is not r.e.

- $\overline{ONE}$ is not r.e.

$\mathbb{1} \notin \overline{\{\mathbb{1}\}}$ and $\vartheta = \emptyset \subseteq \mathbb{1}$, $\vartheta \in \overline{\{\mathbb{1}\}}$ (1)

then by Rice Shapiro $\overline{ONE}$ is not r.e.

Example

$\mathcal{B}_m = \{ f \in \mathcal{C} \mid m \in cod(f) \}$ $\qquad m \in \mathbb{N}$ fixed

$B_m = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{B}_m \}$

$\boxed{\mathcal{B}_m \text{ is finitary}}$

$\forall f \quad f \in \mathcal{B}_m$ iff $m \in cod(f)$

iff $\exists y$ s.t. $f(y) = m$ $\Longrightarrow$ define $\vartheta(z) = \begin{cases} m & \text{if } z = y \\ \uparrow & \text{otherwise} \end{cases}$

$\vartheta \subseteq f$, ϑ finite, $\vartheta \in \mathcal{B}_m$

$\Longleftarrow$ conversely if $\vartheta \subseteq f$, ϑ finite

$m \in cod(\vartheta)$ then $m \in cod(f)$

$\cancel{\Longrightarrow}$ B_m r.e.

NO !!! (it is r.e., but this does not follow by Rice-Shapiro)

OBSERVATION: The converse implication of Rice-Shapiro is false

$\mathcal{A} \subseteq \mathcal{C}$ $\qquad A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

$\forall f \ (f \in \mathcal{A} \Leftrightarrow \exists \vartheta, \vartheta \subseteq f, \vartheta \in \mathcal{A})$ $\quad \not\Longrightarrow \quad A$ r.e.

counter example

$\mathcal{A} = \{ f \mid dom(f) \cap \bar{K} \neq \emptyset \}$

(a) $\forall f \ (f \in \mathcal{A} \Leftrightarrow \exists \vartheta, \vartheta \subseteq f, \vartheta \in \mathcal{A})$

(b) A not r.e.

(a) let f be a function

* $f \in \mathcal{A}$ then $\mathrm{dom}(f) \cap \overline{K} \neq \emptyset$. let $x_0 \in \mathrm{dom}(f) \cap \overline{K}$

and let

$$\vartheta(x) = \begin{cases} f(x) & \text{if } x = x_0 \\ \uparrow & \text{otherwise} \end{cases}$$

Then $\vartheta \subseteq f$, ϑ finite , $\mathrm{dom}(\vartheta) = \{x_0\}$

thus $\mathrm{dom}(\vartheta) \cap \overline{K} = \{x_0\} \neq \emptyset$ i.e. $\vartheta \in \mathcal{A}$

* let $\vartheta \subseteq f$, ϑ finite , $\vartheta \in \mathcal{A}$. Then

$$\underset{\mathrm{dom}(f)}{\underset{\cap\!\shortmid}{\mathrm{dom}(\vartheta)}} \cap \overline{K} \neq \emptyset$$

hence $\mathrm{dom}(f) \cap \overline{K} \supseteq \mathrm{dom}(\vartheta) \cap \overline{K} \neq \emptyset$

and therefore $f \in \mathcal{A}$.

(b) $A = \{x \mid \varphi_x \in \mathcal{A}\} = \{x \mid W_x \cap \overline{K} \neq \emptyset\}$ not r.e.

assume A is r.e. (semidecidable)

then you can semi-decide $\overline{K}$: given $x \in \mathbb{N}$

create a program

```
def P(y):
    if y=x
        return 0
    else loop (↑)
```

then $\overbrace{\underset{\{x\}}{\underset{\shortparallel}{\mathrm{dom}(P)}} \cap \overline{K} \neq \emptyset}^{P \in A}$

$$\Updownarrow$$

$$x \in \overline{K}$$

formally $\overline{K} \leq_m A$

define

$$g(x,y) = \begin{cases} 0 & \text{if } x = y \\ \uparrow & \text{otherwise} \end{cases} = \mu z.\, |x-y| \quad \text{computable}$$

$$= \varphi_{s(x)}(y)$$

by smn theorem $s : \mathbb{N} \to \mathbb{N}$ total computable

s is the reduction function $\overline{K} \leq_m A$

$W_{s(x)} = \mathrm{dom}(\varphi_{s(x)}) = \{x\}$

$x \in \overline{K} \iff \underset{\{x\}}{\underset{\shortparallel}{W_{s(x)}}} \cap \overline{K} = \{x\} \neq \emptyset \iff s(x) \in A$

hence $\overline{K} \leq_m A$, thus A not r.e.

$\square$