ES → In $\mathbb{R}^3$ dotato del prodotto scalare usuale si consideri il sottospazio

$U: 2x + y - z = 0$

a) determinare una base ortonormale di U

b) Sia $\pi: \mathbb{R}^3 \mapsto \mathbb{R}^3$ la proiezione ortogonale su U; determinare $\pi\begin{pmatrix}3\\1\\1\end{pmatrix}$

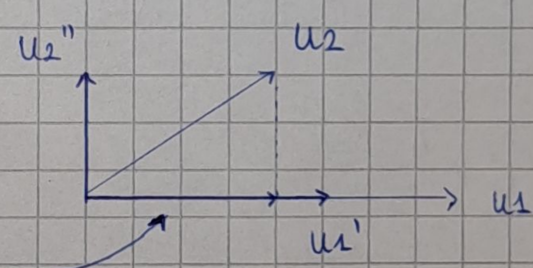
a) Base ortonormale: i) i vettori sono versori $\|v_i\| = 1 \quad \forall i$

ii) i vettori sono a due a due ortogonali $v_i \cdot v_j = 0 \quad \forall i \neq j$

Risolvendo l'equazione di U si ottiene: $U = \left\langle \begin{pmatrix}1\\-2\\0\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$

ortonormalizziamo la base di $U = \left\{ \underset{u_1}{\begin{pmatrix}0\\1\\1\end{pmatrix}} \underset{u_2}{\begin{pmatrix}1\\-2\\0\end{pmatrix}} \right\}$

$$u_1' = \frac{u_1}{\|u_1\|} = \frac{\begin{pmatrix}0\\1\\1\end{pmatrix}}{\sqrt{1^2 + 1^2 + 0^2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$



$$u_2'' = u_2 - \left(\begin{array}{l}\text{proiezione ortogonale}\\ \text{di } u_2 \text{ su } \langle u_1' \rangle\end{array}\right) = u_2 - \left(\frac{u_2 \cdot u_1'}{u_1' \cdot u_1'}\right) u_1'$$

ma $u_1' \cdot u_1' = 1$ u_1' è un versore

$$= u_2 - (u_2 \cdot u_1')\, u_1'$$

$$= \begin{pmatrix}1\\-2\\0\end{pmatrix} - \left[\begin{pmatrix}1\\-2\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} \frac{1}{\sqrt{2}}\right] \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\0\end{pmatrix} - \frac{1}{2}(-2)\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\-2\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$u_2' = \frac{u_2''}{\|u_2\|} = \frac{\begin{pmatrix}1\\-1\\1\end{pmatrix}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Una base ortonormale di U è: $\left\{ \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}}_{u_1'}, \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}}_{u_2'} \right\}$

b) $p_{\perp U}(v) = (v \cdot t_1)t_1 + (v \cdot t_2)t_2 + \dots + (v \cdot t_n)t_n$ con $T \subset \mathbb{R}^n$, $\{t_1, \dots, t_n\}$ base ortonormale di T

$$v = p_\perp U^\perp + p_\perp U$$

$p_\perp U = v - p_\perp U^\perp$ mi serve $U^\perp$!

$$\Rightarrow U^\perp = \{u^\perp \in \mathbb{R}^3 \mid u^\perp \cdot u = 0\}$$

$$\begin{cases} (x, y, 1) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = y + 1 = 0 \\ (x, y, 1) \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = x - 2y = 0 \end{cases} \qquad \begin{cases} x - 2y = 0 \\ y + 1 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\begin{cases} y = -1 \\ x = +2y = -2 \end{cases} \qquad \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow U^\perp = \left\langle \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

base ortonormale di $U^\perp \Rightarrow \Big\{ \underbrace{\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}_{u_1^{\perp\prime}} \Big\}$

$$p_\perp U = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{v \cdot u_1^{\perp\prime}}{\underbrace{u_1^{\perp\prime} \cdot u_1^{\perp\prime}}_{=1}} \right) u_1^{\perp\prime}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6} \left(3 \cdot 2 + \cancel{1 \cdot 1} + \cancel{1(-1)} \right) \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

oppure: $$p_\perp U(v) = \underbrace{\left(\frac{v \cdot u_1'}{\underbrace{u_1' \cdot u_1'}_{=1}} \right) u_1'}_{\text{proiez. ortog. di } v \text{ su } u_1'} + \underbrace{\left(\frac{v \cdot u_2'}{\underbrace{u_2' \cdot u_2'}_{=1}} \right) u_2'}_{\text{proiez. ort. di } v \text{ su } u_2'}$$

$$= \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{3}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \checkmark$$

ESERCIZIO

In $\mathbb{R}^3$ dotato del prodotto scalare usuale si considerino i sottospazi

$$U = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix}\right\rangle \quad e \quad W = \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\rangle = \langle w_1, w_2 \rangle$$

siano $P_U : \mathbb{R}^3 \to \mathbb{R}^3$ e $P_W : \mathbb{R}^3 \to \mathbb{R}^3$ le proiezioni ortogonali su U e W rispettivamente

a) Determinare una base ortonormale di $U \cap W$ e completare a una base ortonormale di $U + W$

→ Ricavo eq. cartesiana di U: $\begin{cases} x = a + 2b \\ y = a + b \\ z = -2b \end{cases}$ $\begin{cases} x = a - z \\ y = a - \frac{z}{2} \\ b = -z/2 \end{cases}$ $\begin{cases} x = y + \frac{z}{2} - z \\ a = y + \frac{z}{2} \\ b = -z/2 \end{cases}$

$$x = y - \frac{z}{2} \to x - y + \frac{z}{2} = 0 \Rightarrow U: 2x - 2y + z = 0$$

intersecando con W si ottiene:

$$U \cap W = \left\{ \alpha \begin{pmatrix}0\\1\\-1\end{pmatrix} + \beta \begin{pmatrix}1\\2\\-1\end{pmatrix} \mid 2x - 2y + z = 0 \right\}$$

$$\begin{pmatrix}\beta\\ \alpha + 2\beta \\ -\alpha - \beta\end{pmatrix} \in U \Rightarrow 2(\beta) - 2(\alpha + 2\beta) + 1(-\alpha - \beta) = 0$$

$$-3\beta - 3\alpha = 0 \Rightarrow \alpha = -\beta$$

$$U \cap W = \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}-1\\-1\\0\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle$$

una base ortonormale di $U \cap W$ è $\frac{v}{\|v\|}$ con $v = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$$\frac{\begin{pmatrix}1\\1\\0\end{pmatrix}}{\sqrt{1^2+1^2+0}} \Rightarrow B = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$$

per completare B ad una base ortonormale di $U + W = \mathbb{R}^3$:

$$(U \cap W)^\perp = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle^\perp = \{ v^\perp \in \mathbb{R}^3 \mid v^\perp \cdot v = 0 \}$$

$$(x \; y \; z) \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = x + y = 0 \to \begin{cases} x = -y \\ z = z \end{cases} \quad \begin{pmatrix}-y\\x\\z\end{pmatrix} \in (U \cap W)^\perp$$

$(U \cap W)^\perp = \langle \begin{pmatrix}1\\-1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle$ $\rightarrow$ normalizziamo i vettori $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$,

dunque un completamento alla base B di $U \cap W$ ad una base ortonormale di $\mathbb{R}^3 = U + W$ è : $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} \right\}$

b) determinare $p_W(u)$, con $u = \begin{pmatrix}2\\1\\-2\end{pmatrix}$

$u = p_\perp W + p_\perp W^\perp \rightarrow p_\perp W = u - p_\perp W^\perp$

$W^\perp : \begin{cases} y - z = 0 \\ x + 2y - z = 0 \end{cases}$ $\begin{cases} y = z \\ x = -z \\ z = z \end{cases}$ $\Rightarrow W^\perp = \langle \begin{pmatrix}-1\\1\\1\end{pmatrix} \rangle$ una base ortonormale di $W^\perp$ è : $B = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix} \right\}$

$$p_W(u) = u - p_{W^\perp}(u)$$

$$= \begin{pmatrix}2\\1\\-2\end{pmatrix} - \left(\begin{pmatrix}2\\1\\-2\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix} \right) \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-2\end{pmatrix} - \left(\frac{1}{3} \cdot (-3) \right) \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\\-2\end{pmatrix} + \begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

oppure cerco una base ortonormale di W :

$$w_1' = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$w_2'' = w_2 - (w_2 \cdot w_1')\, w_1' = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \left(\begin{pmatrix}1\\2\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} \right) \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\-1\end{pmatrix} - \left(\frac{1}{2} \cdot 3 \right) \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\1/2\\1/2\end{pmatrix}$$

$$w_2' = \frac{(1 \ \ 1/2 \ \ 1/2)}{\frac{\sqrt{6}}{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$B = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix} \right\}$$

$$p_W(u) = (u \cdot w_1')\, w_1' + (u \cdot w_2')\, w_2' = \left[\begin{pmatrix}2\\1\\-2\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} \right] \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} +$$

$$+ \left[\begin{pmatrix}2\\1\\-2\end{pmatrix} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix} \right] \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= \frac{1}{2} \cdot 3 \begin{pmatrix}0\\1\\-1\end{pmatrix} + \frac{1}{6} \cdot 3 \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} \quad \checkmark$$

c) determinare un vettore $w \in W$ t.c. $p_U(w) = u$

$w = u + U^\perp$ infatti $w = p_U^{-1}(u) = u + \ker p_U = u + U^\perp$

$= \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \alpha \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $\quad U^\perp : \begin{cases} x + y = 0 \\ 2x + y - 2z = 0 \end{cases}$ $\quad \begin{pmatrix} -y \\ y \\ -y/2 \end{pmatrix} \in U^\perp$

inoltre $w \in W \Rightarrow$ poiché $W: -x + y + z = 0$ $\quad \left(W^\perp = \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \right)$

impongo : $\begin{pmatrix} 2+2\alpha \\ 1-2\alpha \\ -2+\alpha \end{pmatrix} \in W \rightarrow -(2+2\alpha) + (1-2\alpha) + (-2+\alpha) = 0$

$-3 - 3\alpha = 0 \rightarrow \alpha = -1$

$\Rightarrow w = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$

d) determinare tutti i vettori $v \in \mathbb{R}^3$ t.c. $\|p_U(v)\| = 0$ e $\|p_W(v)\| = \sqrt{6}$

$\|p_U(v)\| = 0$ significa $p_U(v) = 0_{\mathbb{R}^3} \Rightarrow v \in U^\perp \Rightarrow v = \alpha \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\|p_W(v)\| = \sqrt{6}$: determino $p_W(v) = p_W\left(\alpha \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right) = \alpha\, p_W \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\Rightarrow \|p_W(v)\| = |\alpha| \cdot \left\| p_W \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\|$

$\Rightarrow p_W \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \circ \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = v - p_{W^\perp}(v)$

$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \left(\frac{-3}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow \|p_W(v)\| = |\alpha| \cdot \left\| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\| = |\alpha| \cdot \sqrt{6} = \sqrt{6} \Rightarrow \alpha = \pm 1$

$\Rightarrow v = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ e $v = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$ soddisfano le richieste del punto d)

ESERCIZIO 3 In $\mathbb{R}^4$, dotato del prodotto scalare usuale, siano:

$$U = \left\langle \underset{u_1}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}} \underset{u_2}{\begin{pmatrix}2\\0\\0\\1\end{pmatrix}} \underset{u_3}{\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}} \right\rangle \quad e \quad W = \left\langle \underset{w}{\begin{pmatrix}t\\4\\-2\\1\end{pmatrix}} \right\rangle$$ con $t \in \mathbb{R}$, sottospazi di $\mathbb{R}^4$

a) per quali valori di t, U e W non sono in somma diretta

$U \cap W \neq \{0_{\mathbb{R}^4}\} \iff \delta w \in U \quad \delta \in \mathbb{R}$

ricavo l'equazione che definisce U:

$$\begin{cases} x_1 = a + 2b \\ x_2 = c \\ x_3 = a - c \\ x_4 = b + c \end{cases} \rightarrow \begin{cases} x_1 = a + 2b \\ c = x_2 \\ x_3 = a - x_2 \\ x_4 = b + x_2 \end{cases} \begin{matrix} \\ \\ \rightarrow \\ \rightarrow \end{matrix} \begin{cases} x_1 = x_3 + x_2 + 2(x_4 - x_2) \\ c = x_2 \\ a = x_3 + x_2 \\ b = x_4 - x_2 \end{cases}$$

$U: \quad x_1 + x_2 - x_3 - 2x_4 = 0$

Vediamo per quali t $\delta w \in U$

$$\delta \cdot t + \delta 4 - (-2\delta) - 2(\delta \cdot 1) = 0 \Rightarrow \delta(t + 4 + \cancel{2} - \cancel{2}) = 0$$

Se $\delta = 0 \vee t \neq -4$ somma diretta

Se $\delta \neq 0 \wedge t = -4$ U, W non sono in somma diretta

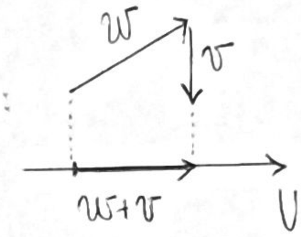
$\Rightarrow t = -4 \quad U \cap W \neq \{0_{\mathbb{R}^4}\}$

b) per $t = 3$, si determini una base di $U^\perp$ e si trovi un vettore v di norma minima tale che $w + v \in U^\perp$

$w = \begin{pmatrix}3\\4\\-2\\1\end{pmatrix}$, Base di $U^\perp$: $\left\{ \underbrace{\begin{pmatrix}1\\1\\-1\\-2\end{pmatrix}}_{u^\perp} \right\}$ infatti se $u \in U$

soddisfa l'equazione $x_1 + x_2 - x_3 - 2x_4 = 0 \Rightarrow \begin{pmatrix}1\\1\\-1\\-2\end{pmatrix} \cdot \underbrace{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}}_{u} = 0$

Ora $w + v \in U^\perp \Rightarrow w + v = p_{U^\perp}(w)$



La proiezione ortogonale definisce il vettore di minima norma:

$$P_{U^\perp}(w) = (w \cdot u^{\perp\prime})\, u^{\perp\prime} \qquad u^{\perp\prime} = \frac{u^\perp}{\|u^\perp\|} = \frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\-1\\-2\end{pmatrix}$$

$$= \left[\begin{pmatrix}3\\4\\-2\\1\end{pmatrix} \cdot \frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\-1\\-2\end{pmatrix}\right] \frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\-1\\-2\end{pmatrix}$$

$$= \frac{3+4+2-2}{7}\begin{pmatrix}1\\1\\-1\\-2\end{pmatrix} = \begin{pmatrix}1\\1\\-1\\-2\end{pmatrix}$$

$$\Rightarrow v = -w + p_{U^\perp}(w) = \begin{pmatrix}1\\1\\-1\\-2\end{pmatrix} - \begin{pmatrix}3\\4\\-2\\1\end{pmatrix} = \begin{pmatrix}-2\\-3\\+1\\-3\end{pmatrix}$$

Oppure equivalentemente $P_U(w) = -v$

c) Si determini una base ortonormale di U

$$u_1' = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$u_2'' = u_2 - (u_2 \cdot u_1')\, u_1' = \begin{pmatrix}2\\0\\0\\1\end{pmatrix} - \left[\begin{pmatrix}2\\0\\0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right] \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\0\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

$$u_2' = \frac{u_2''}{\|u_2''\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

$$u_3'' = u_3 - (u_3 \cdot u_1')\, u_1' - (u_3 \cdot u_2')\, u_2'$$

$$= \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} - \left[\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right] \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \left[\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right] \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} - \frac{-1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} = \begin{pmatrix}-1/6\\1\\1/6\\1/3\end{pmatrix}$$

$$= +\frac{1}{6}\begin{pmatrix}-1\\6\\1\\2\end{pmatrix}$$

$$u_3' = \frac{u_3''}{\|u_3''\|} = \frac{(-1, 6, 1, 2)}{\sqrt{1^2+6^2+1^2+2^2}} = \frac{1}{\sqrt{42}}\begin{pmatrix}-1\\6\\1\\2\end{pmatrix}$$

$\Rightarrow B = \{u_1', u_2', u_3'\}$ é ortonormale

ESERCIZIO

Si consideri $U = \left\langle \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}\begin{pmatrix}-1\\0\\1\\1\end{pmatrix} \right\rangle$ di $\mathbb{R}^4$

a) Determinare una base di U e delle equazioni che definiscono U.

Una base di U é : $\left\{\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\right\}$ infatti sono generatori l.i. di U

$$\begin{cases} x = a-b \\ y = -a \\ z = b \\ w = a+b \end{cases} \rightarrow \begin{cases} x = -y-z \\ a = -y \\ b = z \\ w = -y+z \end{cases} \Rightarrow U: \begin{cases} x+y+z = 0 \\ y-z+w = 0 \end{cases}$$

b) Sia $W = U^\perp$ il sottospazio ortogonale di U, determinare una base ortonormale di W.

$$W = U^\perp = \left\langle \begin{pmatrix}1\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \right\rangle \quad (\text{Vedi equazioni di } U!)$$

Ora applico Gram-Schmidt a W

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$$u_2' = \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} - \left[\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} - \frac{0}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\\1\end{pmatrix}$$

$$u_2 = \frac{u_2'}{\|u_2\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}$$

Una base ortonormale di W é : $\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}\right\}$

c) Sia $\pi_W : \mathbb{R}^4 \to \mathbb{R}^4$ la proiezione ortogonale su W, determinare $\pi_W \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\pi_W \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \left[\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right] \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \underbrace{\left[\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right]}_{=0} \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{3}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

d) Sia $V = \left\langle \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$, determinare la controimmagine $\pi_W^{-1}(V)$ del sottospazio V tramite la proiezione ortogonale π_W.

Poiché $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \notin W = \operatorname{Im} \pi_W$ si ha che

$\pi^{-1}_W \left(\alpha \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right)$ è non vuoto se e solo se $\alpha = 0$,

$$\Rightarrow \pi_W^{-1}(V) = \pi_W^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \ker \pi_W = U$$

Nota al punto b) Esercizio 3

possiamo scomporre v e w nelle loro proiezioni su U e $U^\perp$:

$$v = P_U(v) + P_{U^\perp}(v) \quad (*)$$

$$w = P_U(w) + P_{U^\perp}(w)$$

$$\Rightarrow \quad v + w = P_U(v) + P_{U^\perp}(v) + P_U(w) + P_{U^\perp}(w)$$

Se $v + w \in U^\perp \quad \Rightarrow \quad P_U(v) + P_U(w) = \vec{0}$

$$\Rightarrow \quad P_U(w) = -P_U(v) \quad (**)$$

quindi abbiamo $v + w = P_{U^\perp}(v) + P_{U^\perp}(w)$

poiché v deve avere norma minima, l'altra sua componente é nulla: $P_{U^\perp}(v) = 0$

$\Rightarrow$ si conclude che $v + w = P_{U^\perp}(w)$

e analogamente (da (**) e (*)) segue che: $P_U(w) = -v$