Ex (CALCULUS of VARIATIONS)

(DIRECT METHOD in the CALCULUS of VARIATIONS – by Tonelli)

Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1.

$g \in L^p(U)$ $\underline{p > 1}$

I want to find the "best approximation of g" in $W^{1,p}(U)$.

$$\inf_{f \in W^{1,p}(U)} \int_U |Df|^p + |f-g|^p \, dx = \inf_{f \in W^{1,p}(U)} E(f)$$

$E(f) \geq 0$ $\inf_{f \in W^{1,p}(U)} E(f) = \bar{c} \geq 0$

f_k be a MINIMIZING SEQUENCE

$\forall k \in \mathbb{N}$ $\exists f_k \in W^{1,p}(U)$ $\bar{c} \leq E(f_k) \leq \bar{c} + \frac{1}{k}$ (by definition of infimum)

$$\| |Df_k| \|_p^p + \underbrace{\|f_k - g\|_p^p}_{\substack{\vee\!\shortmid \\ (\|f_k\|_p - \|g\|_p)^p}} = E(f_k) \le \bar{c} + 1 \quad \forall k \ge 1$$

$$|a-b+b|^p \le (|a-b|+|b|)^p \le (|a-b|^p + |b|^p)\, 2^{p-1}$$

$$\|f_k - g\|_p^p \ge \|f_k\|_p^p \, 2^{1-p} - \|g\|_p^p$$

$$\| |Df_k| \|_p^p + C \|f_k\|_p^p \le C_0 \implies \|f_k\|_{W^{1,p}} \le \bar{C}$$

$\implies$ (RK) UP TO SUBSEQUENCE $\quad f_k \to f$ in $L^p(U)$

$f \in W^{1,p}(U)$ $\qquad \dfrac{\partial f_k}{\partial x_i} \to \dfrac{\partial f}{\partial x_i}$ in $L^p(U)$

$Df_k \to Df$ in $L^p(U; \mathbb{R}^n)$

$$\int_U |f_k - g|^p dx \to \int_U |f-g|^p dx$$ by strong convergence in $L^p(U)$

$$\int_U |Df_k|^p\,dx \geq \int_U |Df|^p\,dx + \int_U p\,\underbrace{|Df|^{p-2}Df}\cdot(Df_k - Df)\,dx$$

CONVEXITY of $\phi : \mathbb{R}^n \to \mathbb{R}$

$x \mapsto |x|^p$

$$|x|^p \geq |x_0|^p + p\,|x_0|^{p-2}\,x_0\cdot(x-x_0)$$

$$|Df|^{p-2}Df \in L^{p'}(U;\mathbb{R}^n)$$

by weak convergence

$$\liminf_{k\to+\infty} \int_U |Df_k|^p\,dx \geq \int_U |Df|^p\,dx + \underbrace{\liminf_{k} \int_U p\,|Df|^{p-2}Df\cdot(Df_k - Df)}$$

$$\liminf_{k}\ \bar{c} + \frac{1}{k} \geq \liminf_{k} E(f_k) \geq \int_U |Df|^p + |f-g|^p\,dx \overset{0}{=} E(f) \geq \bar{c}$$

f is A MINIMIZER

Moreover it is UNIQUE

$f_1 \neq f_2$ 2 minimizers.

$$\bar{C} \ni E(\lambda f_1 + (1-\lambda) f_2) = \int_U |\lambda Df_1 + (1-\lambda) Df_2|^p + |\lambda(f_1-g) + (1-\lambda)(f_2-g)|^p dx$$

$$< \lambda E(f_1) + (1-\lambda) E(f_2) = \min E(f)$$

by STRICT CONVEXITY

Contradiction.

(the same pb $\inf\limits_{f \in W^{1,1}(U)} \int_U |Df| + |f-g| \, dx$ in general

does not have a solution (infimum is NOT

a minimum)

f the minimizer $\quad \forall \varepsilon \in \mathbb{R} \quad \forall \phi \in C_c^\infty(U)$

$$E(f+\varepsilon\phi) \geq E(f) \quad \forall \varepsilon \in \mathbb{R}$$

$$\lim_{\varepsilon \to 0} \frac{E(f+\varepsilon\phi) - E(f)}{\varepsilon} = 0 = \lim_{\varepsilon \to 0} \int_U \frac{|Df+\varepsilon D\phi|^p - |Df|^p}{\varepsilon} + \frac{|f+\varepsilon\phi-g|^p - |f-g|^p}{\varepsilon}$$

GATEAUX DERIVATIVE

$$|a+\varepsilon b|^p - |a|^p = \varepsilon |a|^{p-2} a \cdot b + o(\varepsilon) \ldots$$

$$= \int_U \underbrace{|Df|^{p-2} Df}_{\in L^{p'}(U)} \cdot D\phi + |f-g|^{p-2}(f-g) \cdot \phi \, dx = 0 \qquad \forall \phi \in C_c^\infty(U)$$

$$\forall \phi \in C_c^\infty(U) \qquad \underbrace{\sum_i \int_U |Df|^{p-2} \frac{\partial f}{\partial x_i} \cdot \frac{\partial \phi}{\partial x_i}} + |f-g|^{p-2}(f-g)\phi \, dx = 0$$

Let $g_i = |Df|^{p-2} \frac{\partial f}{\partial x_i}$

Note that

$$\sum_{1}^{n} i \int_U \overbrace{|Df|^{p-2} \frac{\partial f}{\partial x_i}}^{g_i} \cdot \frac{\partial \phi}{\partial x_i} = \sum_i T_{g_i}\left(\frac{\partial \phi}{\partial x_i}\right)$$

$$= \sum_i^n T_{g_i}\left(\frac{\partial \phi}{\partial x_i}\right) =$$ by def of derivative in the sense of distributions

$$= \sum_i \left(-\frac{\partial}{\partial x_i} T_{g_i}\right)(\phi) = -\underbrace{\sum_1^n i \frac{\partial}{\partial x_i} T_{|Df|^{p-2}\frac{\partial f}{\partial x_i}}}(\phi) =$$

$$= -(\operatorname{div} T_g)(\phi)$$ where $T_g(\phi) = (T_{g_1}(\phi), \dots, T_{g_n}(\phi))$

$g = |D\bar{f}|^{p-2} D\bar{f}$

So $\bar{f}$ solves in the sense of distributions

$$-\operatorname{div}\left(|D\bar{f}|^{p-2} D\bar{f}\right) + |\bar{f} - g|^{p-2}(\bar{f} - g) = 0 \text{ in } U$$

For $p=2$ $$-\operatorname{div}(D\bar{f}) + \bar{f} - g = 0$$

$$-\Delta \bar{f} + \bar{f} - g = 0$$

POINCARÉ INEQUALITY

$U \subseteq \mathbb{R}^n$ open, bold, of class C^1 and CONNECTED

$\exists\, C = C(n,p,U)$ such that $\forall f \in W^{1,p}(U)$

$$\left\| f - \frac{1}{|U|} \int_U f(y)\,dy \right\|_{L^p(U)} \leq C \, \| |Df| \|_{L^p(U)} .$$

Obs $f \in W^{1,p}(U) \rightarrow f \in L^1(U)$ $\quad g = f - \frac{1}{|U|} \int_U f(y)dy \in W^{1,p}(U)$

with $\int_U g(y)dy = 0$

$C = \{ g \in W^{1,p}(U) \;\; \int_U g(y)dy = 0 \}$ is CLOSED in $W^{1,p}(U)$

In C $\|g\|_{1,p}$ is equivalent to $\| |Dg| \|_{L^p}$.

<u>proof</u> by contradiction.

· Assume not true: $\forall k>0\ \exists\ f_k \in W^{1,p}(U)$ such that

$$\| f_k - \frac{1}{|U|}\int_U f_k(y)\,dy \|_{L^p} > k\, \| |Df_k| \|_{L^p}.$$

$$v_k(y) = \frac{f_k(y) - \frac{1}{|U|}\int_U f_k(y)\,dy}{\| f_k - \frac{1}{|U|}\int_U f_k(y)\,dy\|_p} \in W^{1,p}(U) \qquad \int_U v_k(y)\,dy = 0$$

$\|v_k\|_{L^p} = 1 \qquad \||Dv_k|\|_{L^p} \leq \frac{1}{k} < 1 \qquad v_k$ are uniformly bold in $W^{1,p}(U)$

$v_{k_j} \to v$ in $L^q(U)$ $\forall q < p^*$ $\quad (\forall q < +\infty$ if $p \geq n)$

Moreover $\forall \phi \in C_c^\infty(U)$

$$\int_U \frac{\partial \phi}{\partial x_i} v_{k_j} = -\int_U \phi \frac{\partial}{\partial x_i} v_{k_j} \leq \|\phi\|_{p'} \|Dv_{k_j}\|_p \longrightarrow 0$$

Therefore $\frac{\partial}{\partial x_i} T v_{k_j} \to 0 = \frac{\partial}{\partial x_i} T v$ in the sense of distributions

$\Rightarrow$ U is connected $\frac{\partial}{\partial x_i} T v = 0 \ \forall i \Rightarrow Tv$ is constant

$\Rightarrow v$ is constant but $\begin{cases} \|v\|_{L^p} = 1 \\ \int_U v(y)\,dy = 0 \end{cases} \longrightarrow$ CONTRADICTION

Obs 1 If $U = B(x,r)$ $\quad C(n,p,B(x,r)) = C(n,p,B(0,1))\,r$

$f \in W^{1,p}(B(x,r))$ $\quad y \mapsto g(y) := f(x+ry) \in W^{1,p}(B(0,1))$

$\|f\|_p = r^{\frac{n}{p}} \|g\|_{L^p(B(0,1))}$ $\quad \||Df|\|_{L^p} = r^{\frac{n}{p}-1} \||Dg|\|_{L^p(B(0,1))}$

$\|f\|_p = r^{n/p} \|g\|_{L^p} \leq C(n,p,B(x,r))\, r^{n/p-1} \||Dg|\|_{L^p}$

$\frac{C(n,p,B(x,r))}{r} = C(n,p,B(0,1))$