2) COMPACTNESS THEOREM by KOLMOGOROV in L^p (see BREZIS)

Let f_k be a sequence in $L^p(\mathbb{R}^n)$ $p \in [1, +\infty)$ such that

L^p VERSION OF ASCOLI-ARZELA'

(1) $\forall \varepsilon > 0 \ \exists U_\varepsilon \subset\subset \mathbb{R}^n$ such that $\|f_k\|_{\mathbb{R}^n \setminus U_\varepsilon} \leq \varepsilon$ $\Rightarrow$ IF f_k are all SUPPORTED in $\overline{V} \subset\subset \mathbb{R}^n$ always TRUE

(f_k ARE "SUPPORTED UP TO ERROR" ON A COMPACT SET)

(2) $\|f_k\|_{L^p(\mathbb{R}^n)} \leq C$

(UNIFORM BOUNDEDNESS)

(3) $\forall \varepsilon \ \exists \delta$ such that if $h \in \mathbb{R}^n$ $|h| \leq \delta$

$$\|\tau_h f_k - f_k\|_{L^p}^p = \int_{\mathbb{R}^n} |f_k(x+h) - f_k(x)|^p \, dx \leq \varepsilon^p.$$

(equivalently $\|\tau_h f_k - f_k\|_p \to 0$ as $|h| \to 0$ UNIFORMLY in k

(EQUICONTINUITY)

Then (f_k) is precompact ($\exists f_{k_n}$ subsequence, $f \in L^p(\mathbb{R}^n)$ such that $f_{k_n} \to f$ in $L^p(\mathbb{R}^n)$, $\|f_{k_n} - f\|_{L^p} \to 0$

(proof is based on approximation with $C_c^\infty(\mathbb{R}^n)$)

" $\forall \varepsilon>0\ \exists N_\varepsilon \quad \|\rho_N * f_k - f_k\|_p \le \varepsilon \ \forall N > N_\varepsilon \quad \forall k \qquad \rho_N = N^n \rho(Nx)$

then

$$\|f_h - f_k\|_{L^p} \le \underbrace{\|f_h - f_h * \rho_N\|_p}_{\text{small}} + \|f_h * \rho_N - f_k * \rho_N\|_p + \underbrace{\|f_k * \rho_N - f_k\|_{L^p}}_{\text{small}}$$

by A.A. theorem

$f_k * \rho_N \xrightarrow[k\to+\infty]{} g_N$ converge uniformly up to a subsequence

$f_k * \rho_N$ is Cauchy in $\|\cdot\|_\infty$ "$\Rightarrow$" in $\|\cdot\|_p$

SEE BREZIS

—— . ——

RECALL also the WEAK COMPACTNESS THEOREM in $L^p(U)$

U bdd Let $f_k \in L^p(U)$ with $\|f_k\|_{L^p} \le C$

(EQUIBOUNDEDNESS)

Then, up to passing to a subsequence

1) for $1 < p < +\infty$ $\quad f_k \rightharpoonup f$ in $L^p(U)$ $\quad f \in L^p(U)$ $\quad \int_U f_k g\,dx \to \int_U fg\,dx \quad \forall g \in L^{p'}(U)$

2) for $p = +\infty$ $\quad f_k \overset{*}{\rightharpoonup} f$ in $L^\infty(U)$ $\quad \int_U f_k g\,dx \to \int_U fg\,dx \quad \forall g \in L^1(U)$

3) for $p = 1$ $\quad \exists\, \mu^+, \mu^- \in \mathcal{M}(U)$ $\quad \int_U f_k \phi\,dx \to \int_U \phi\,d\mu^+ - \int_U \phi\,d\mu \quad \forall \phi \in C_c^0(U)$

proof of Rellich - Kondrachov (on Evans a different proof - same ingredients)

Let f_k be a sequence bounded in $W^{1,p}(U)$

Fix $V \subset\subset \mathbb{R}^n$ such that $U \subseteq V$ and consider the extension $\quad E_V : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$

$$E_V(f_k) = \tilde{f}_k \in W^{1,p}(\mathbb{R}^n) \quad \circledast \quad \|\tilde{f}_k\|_{W^{1,p}(\mathbb{R}^n)} \leq C \quad \forall k$$

and $\operatorname{supp}(\tilde{f}_k) \subseteq V \subset\subset \mathbb{R}^n$.

by Property 1, $\forall h \in \mathbb{R}^n$ $\quad \odot \quad \|\tau_h \tilde{f}_k - \tilde{f}_k\| \leq |h|\,\|D\tilde{f}_k\|_p \leq C|h|$

we want to use Kolmogorov theorem.

1) $\tilde{f}_k$ are all supported on $V \subset\subset \mathbb{R}^n$.

2) $\|\tilde{f}_k\|_{L^p} \le C$ (since by extension
$\|\tilde{f}_k\|_{L^p} \le \|\tilde{f}_k\|_{W^{1,p}(\mathbb{R}^n)} \le C \|f_k\|_{W^{1,p}(U)}$

3) $\|\tau_h \tilde{f}_k - \tilde{f}_k\|_p \le C|h|$ ok

$\Rightarrow \exists\ \tilde{f}_{k_n}$ and $\tilde{f}$ in $L^p(\mathbb{R}^n)$ such that

$\|\tilde{f}_{k_n} - \tilde{f}\|_{L^p(\mathbb{R}^n)} \to 0$ $\qquad f = \tilde{f}|_U$

$\Rightarrow \|f_{k_n} - f\|_{L^p(U)} \to 0$

$\Rightarrow \|f_{k_n} - f\|_{L^1(U)} \le \|f_{k_n} - f\|_{L^p(U)} |U|^{1-\frac{1}{p}} \to 0$

$\Rightarrow f_{k_n} \to f$ in $L^1(U) \Rightarrow f_{k_n}$ is Cauchy in $L^1(U)$

$\forall q \in (1, p^*)$ by interpolation $\quad \frac{1}{q} = \frac{\theta}{1} + \frac{(1-\theta)}{p^*}$

$$\|f_{k_n} - f_{k_m}\|_{L^q(U)} \leq \underbrace{\|f_{k_n} - f_{k_m}\|}_{}{}^{\theta}_{L^1(U)} \underbrace{\|f_{k_n} - f_{k_m}\|}_{\leq (GNS)}{}^{1-\theta}_{L^{p^*}(U)}$$

$$\underset{(GNS)}{\leq} C(n,p,U)\left(\|f_{k_n}\|_{W^{1,p}} + \|f_{k_m}\|_{W^{1,p}}\right)^{1-\theta} \|f_{k_n} - f_{k_m}\|^{\theta}_{L^1(U)}$$

$$\leq \bar{C}(n,p,U)\, \|f_{k_n} - f_{k_m}\|^{\theta}_{L^1(U)}$$

$\Rightarrow f_{k_n}$ is Cauchy in $L^q(U)$ $\quad \forall q \in [1, p^*)$

$\Rightarrow f_{k_n} \to f$ in $L^q(U)$ $\forall q$.

$\|f_{k_n}\|_{p^*} \overset{GNS}{\leq} C \Rightarrow$ up to a subsequence it converges <u>weakly</u> in $L^{p^*}(U)$.

IMPORTANT COROLLARY of R-K.

U open bdd of class C^1. $\forall p \geq 1$

$W^{1,p}(U) \hookrightarrow L^p(U)$ COMPACTLY.

More precisely: if $f_k \in W^{1,p}(U)$ is a bdd sequence then $\exists f_{k_i}$ such that $f_{k_i} \to f$ in $L^p(U)$

AND

(1) $p = 1$ $\quad f \in L^{\frac{n}{n-1}}(U)$ (but in general $f \notin W^{1,1}(U)$)

(2) $p > 1$ $\quad f \in W^{1,p}(U)$ and $\boxed{\frac{\partial}{\partial x_j} f_{k_i} \xrightarrow[\text{WEAKLY}]{} \frac{\partial}{\partial x_j} f \text{ in } L^p(U) \; \forall j}$

(3) Moreover $p < n$ $\quad f \in L^{p^*}(U)$, $p = n$ $\quad f \in L^q(U)$ $q < +\infty$

$p > n$ $\quad f \in C^{0,1-n/p}(U)$ and $f_k \to f$ in $C^{0,\alpha}(U)$ $\forall \alpha < 1 - \frac{n}{p}$

PROOF $\forall\, p<n \Rightarrow p^* > p$ by R-K $W^{1,p}(U) \xhookrightarrow{\text{COMP.}} L^p(U)$

$p \geq n \Rightarrow f \in W^{1,p}(U) \Rightarrow f \in W^{1,q}(U) \quad \forall\, q < p \Rightarrow$

take $q<n$ such that $q^* > p$ (this exists since as $p \nearrow n$ $p^* \nearrow +\infty$) and apply R-K:

$$W^{1,p}(U) \underset{\substack{\text{CONTINUOUS} \\ \text{since } U \text{ bdd}}}{\hookrightarrow} W^{1,q}(U) \underset{\text{COMPACT}}{\hookrightarrow} L^p(U) \quad \text{since } p<q^*$$

We INDICATE f_k the SUBSEQUENCE

1. $\forall\, p<n \quad f_k \to f$ in $L^{p^*}(U)$ so $f \in L^{p^*}(U)$.

(2) for $p>1$ since $\frac{\partial}{\partial x_j}(f_k)$ is bounded in $L^p(U) \Rightarrow$ it admits a subsequence converging weakly in $L^p(U)$ to $g_j \in L^p(U)$

$\forall\, \phi \in C_c^\infty(\mathbb{R}^n)$

$$\int_U \left(\frac{\partial}{\partial x_j}\phi\right) \cdot f_k\,dx = -\int_U \phi \frac{\partial}{\partial x_j} f_k\,dx \xrightarrow[\text{CONV}]{\text{by WEAK}} -\int_U \phi g_j\,dx$$

by L^p conv. $\to \int \frac{\partial}{\partial x_j}\phi \cdot f$

$\Rightarrow g_j = \frac{\partial}{\partial x_j} f \in L^p(U)$!

Therefore for $p > 1$ $\quad f_k \to f$ strongly in $L^p(U)$

$\frac{\partial}{\partial x_j} f_k \rightharpoonup \frac{\partial}{\partial x_j} f$ weakly in $L^p(U)$

$f \in W^{1,p}(U) \cap L^{p^*}(U) \quad 1 < p < n$

$f \in W^{1,p}(U) \cap L^q(U) \quad \forall q < +\infty \quad p = n$

$f \in C^{0,1-\frac{n}{p}}(U) \cap W^{1,p}(U) \quad p > n$

For $p = 1$ NOT TRUE

for $f_k \in W^{1,1}(U)$ $\cdot \|f_k\|_{W^{1,1}} \le C$

up to subsequence

$$f_k \to f \in L^{1^*}(U) = L^{\frac{n}{n-1}}(U) \quad \text{STRONGLY in } L^1$$

$\forall i = 1 .. n \quad \exists \mu_i^+, \mu_i^- \in \mathcal{M}(U)$ such that

$$\forall \phi \in C_c^0(U) \int_U \frac{\partial f_k}{\partial x_i} \phi \, dx \to \int_U \phi \, d\mu_i^+ - \int_U \phi \, d\mu_i^-$$

$$\frac{\partial f_n}{\partial x_i} \longrightarrow \mu_i^+ - \mu_i^-$$

WEAKLY in the sense of RADON MEASURE

If $\mu_i^+ \ll \mathcal{L}$ and has density g_i^+

$\mu_i^- \ll \mathcal{L}$ and has density g_i^-

$$\Rightarrow \underbrace{g_i^+ - g_i^-}_{\text{this coincides with the weak derivative}} = \frac{\partial f}{\partial x_i}$$

But in general we cannot expect μ_i^+, μ_i^- to be absolutely continuous w.r. to Lebesgue.