First corollary of the extension theorem

Density

Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1, then

$$\overline{C^\infty(\bar{U})}^{\|\cdot\|_{1,p}} = W^{1,p}(U) \qquad p < +\infty .$$

($C^\infty(\bar{U})$ = functions which are $C^\infty(U)$ and such that it can be extended continuously (together with all the derivatives) up to the bdry).

$$C^\infty(\bar{U}) \supseteq C_c^\infty(\mathbb{R}^n)|_U \qquad \phi \in C_c^\infty(\mathbb{R}^n) \quad \phi|_U \in C^\infty(\bar{U})$$

Same result holds for $W^{k,p}(U)$

U bdd of class C^1, then $\overline{C^\infty(\bar{U})}^{\|\cdot\|_{k,p}} = W^{k,p}(U)$.

with a direct proof (NO EXTENSION)

COROLLARY of the EXTENSION THEOREM (LOCAL VERSION) of GNS & MORREY

Let $U \subseteq \mathbb{R}^n$ open bounded of class C^1.

(1) $p < n$ local version of (GNS)

$\forall q \in [1, p^*]$ $\exists$ $C(n, p, q, U) \geq 0$ such that $\forall f \in W^{1,p}(U)$

$$\|f\|_{L^q(U)} \leq C(n, p, q, U) \|f\|_{W^{1,p}(U)} \quad \forall q \leq p^*$$

(so $f \in L^q(U)$ $\forall q \leq p^*$).

proof: $f \in W^{1,p}(U) \rightarrow$ Fix $V \supset\supset U$ and consider the extension $E_V(f)$ $\Rightarrow$ by GNS applied to the extension

$$\|E_V(f)\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p) \| |DE_V(f)| \|_{L^p(\mathbb{R}^n)}$$

$\|E_V(f)\|_{L^{p^*}(U)} = \|f\|_{L^{p^*}(U)}$

$\| |D E_V(f)| \|_{L^p(\mathbb{R}^n)} \leq C \|E_V(f)\|_{W^{1,p}(\mathbb{R}^n)} \overset{\text{EXTENSION}}{\leq} \overline{C} \|f\|_{W^{1,p}(U)}$

$$\Rightarrow \|f\|_{L^{p^*}(U)} \leq C(U,V)\, C(n,p)\, \|f\|_{W^{1,p}(U)}.$$

$$\|f\|_{L^q(U)} \leq \|f\|_{L^{p^*}(U)} \cdot |U|^{\frac{p^*-q}{p^* q}}. \qquad \square$$

② For $p = n$ $\quad W^{1,p}(U) \hookrightarrow L^q(U) \quad \forall\, q < +\infty$

③ For $p > n$ local version of Morrey.

$\exists\, C(n,p,U)$ s. that $\forall f \in W^{1,p}(U) \Rightarrow f \in C^{0,1-\frac{n}{p}}(U)$

$$\|f\|_{C^{0,1-\frac{n}{p}}(U)} = \|f\|_{L^\infty(U)} + \sup_{x \neq y \in U} \frac{|f(x)-f(y)|}{|x-y|^{1-n/p}} \leq C(n,p,U)\, \|f\|_{W^{1,p}(U)}$$

$W^{1,p}(U) \hookrightarrow C^{0,1-n/p}(U)$ continuously

take $V \supset\supset U$ $\quad E_V(f) \in W^{1,p}(\mathbb{R}^n)$ and apply Morrey.

$$\|E_V(f)\|_{L^\infty(U)} = \|f\|_{L^\infty(U)} \leq C(n,p,U)\, \|E_V(f)\|_{W^{1,p}(\mathbb{R}^n)} \leq \tilde{C}\, \|f\|_{W^{1,p}(U)}$$

$$|E_V(f(x)) - E_V f(y)| = |f(x)-f(y)| \leq C\, |x-y|^{1-\frac{n}{p}}\, \|DE_V f\|_{L^p(\mathbb{R}^n)} \leq C\, |x-y|^{1-\frac{n}{p}}\, \|f\|_{W^{1,p}(U)}$$
$x, y \in U$

REMARK Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1

$\forall p > n \quad W^{1,p}(U) \hookrightarrow C^{0,\alpha}(U)$ COMPACT $\forall \alpha \in [0, 1-\frac{n}{p})$

(that is $f_k \in W^{1,p}(U)$ with $\|f_k\|_{W^{1,p}(U)} \leq C \Rightarrow \exists f_{k_j}$ and $f \in C^{0,1-\frac{n}{p}}(U)$ such that $f_{k_j} \to f$ UNIFORMLY and in $C^{0,\alpha}$ sense $\forall \alpha < 1-\frac{n}{p}$.

THEOREM of RELLICH - KONDRACHOV.

Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1. $p \leq n$

1) Let $p < n$ then $\forall q < p^*$ $W^{1,p}(U) \hookrightarrow L^q(U)$ is COMPACT (that is if f_k is a bdd sequence in $W^{1,p}(U)$, then $\exists f_{k_j}$ such that $f_{k_j} \to f$ in $L^q(U)$ $\forall q < p^*$) where $f \in L^{p^*}(U)$ (and also $f_{k_j} \xrightarrow{\text{WEAKLY}} f$ in $L^{p^*}(U)$)

2) Let $p = n$. Then $W^{1,p}(U) \hookrightarrow L^q(U)$ compact $\forall q < +\infty$
$\forall f_k$ bdd $\exists f_{k_j}$, $f \in L^q(U)$ $\forall q < +\infty$ $f_{k_j} \to f$ in $L^q(U)$.

$\boxed{\text{Observation 1}}$ $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ IS NEVER COMPACT

Take $U = B(0,1)$ $\qquad \varrho(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|>1 \end{cases}$

$$f_k(x) = k^{n/p^*} \varrho(kx) \in C_c^\infty(B(0,1)) \quad \forall k > 1$$

$f_k \in C_c^\infty(\mathbb{R}^n)$ $\quad \operatorname{supp} f_k = B(0, \frac{1}{k})$: $f_k(x) \to 0$ for $k \to +\infty$ for all $x \neq 0$.

$$\|f_k\|_{L^{p^*}(B)} = \left(\int_{B(0,\frac{1}{k})} k^n (\varrho(kx))^{p^*} dx\right)^{1/p^*} = \left(\int_{B(0,1)} (\varrho(x))^{p^*} dx\right)^{\frac{1}{p^*}} = \|\varrho\|_{L^{p^*}(B)}$$

$$\|f_k\|_{L^p(B)} \leq \|f_k\|_{L^{p^*}} |B(0,1)|^{\frac{p^*-p}{p^*p}} = \|\varrho\|_{L^{p^*}} \omega_n^{\frac{p^*-p}{p^*p}}$$

UNIFORMLY BDD in k !

$$|Df_k|^p = k^{\frac{n}{p^*}p+p} |D\varrho(kx)|^p$$

$\frac{n}{p^*} + 1 - \frac{n}{p} = 0$!

$$\left(\int_{B(0,1)} |Df_k|^p\right)^{1/p} = k^{n/p^*+1} k^{-n/p} \|D\varrho\|_{L^p(B(0,1))} = \|D\varrho\|_{L^p(B(0,1))}$$

$\Rightarrow \quad \|f_k\|_{W^{1,p}(B(0,1))} \le C$ (it is uniformly bdd)

Nonetheless $\nexists\ f_{k_j} \to 0$ in $L^{p^*}(B(0,1))$ since

$$\|f_k\|_{L^{p^*}(B(0,1))} = \|e\|_{L^{p^*}(B(0,1))} \not\to 0$$

Whereas $\displaystyle \|f_k\|_{L^q}^q = \int_{B(0,1/k)} k^{\frac{n}{p^*}q} |e(kx)|^q \, dx =$

$q < p^*$

$$= k^{n/p^* q}\, k^{-n} \int_{B(0,1)} |e(y)|^q \, dy = \underbrace{k^{\frac{n}{p^*}q - n}}_{\downarrow\, 0} \|e\|_{L^q}^q$$

So $f_k \to 0$ in $L^q(B(0,1))$ $\forall q < p^*$ $\qquad \frac{n}{p^*}q - n < 0$! $q < p^*$!

$f_k \not\to 0$ in $L^{p^*}(B(0,1))$

RELLICH KONDRACHOV THEOREM

$U \subseteq \mathbb{R}^n$ open bdd of class C^1. $p < n$

then $W^{1,p}(U) \hookrightarrow L^q(U)$ $\forall q < p^*$ COMPACTLY.

(that is $\forall f_k \in W^{1,p}(U)$ bdd, $\exists f_{k_j}$ and $\underline{f \in L^{p^*}(U)}$

such that $\|f_{k_j} - f\|_{L^q(U)} \to 0$ as $k_j \to +\infty$ $\forall q < p^*$.

whereas $f_{k_j} \rightharpoonup f$ in $L^{p^*}(U)$ (weakly)

proof based on 2 important facts

1) Let $f \in W^{1,p}(\mathbb{R}^n)$ $\boxed{h \in \mathbb{R}^n}$ $\tau_h f(x) := f(x+h)$

$\|\tau_h f - f\|_{L^p} \leq |h| \|Df\|_{L^p}$ $\circledast$

prove (*) for $\phi \in C_c^\infty(\mathbb{R}^n)$ and then conclude by DENSITY.

$$\int_{\mathbb{R}^n} |\tau_h \phi - \phi|^p \, dx = \int_{\mathbb{R}} |\phi(x+h) - \phi(x)|^p \, dx = \int_{\mathbb{R}^n} \left| \int_0^1 \frac{d}{dt} \phi(x+th) \, dt \right|^p dx$$

$$= \int_{\mathbb{R}^n} \left| \int_0^1 D\phi(x+th) \cdot h \, dt \right|^p dx \underset{\text{Jensen}}{\leq} \int_{\mathbb{R}^n} \int_0^1 |D\phi(x+th) \cdot h|^p \, dt \, dx$$

F convex $\quad F\left(\int_0^1 h(t)\,dt\right) \leq \int_0^1 F(h(t))\,dt$

F convex. $\forall y_0 \ \exists C(y_0)$ s.that

$F(x_0) \geq F(y_0) + C(y_0)(x_0 - y_0)$ $\quad$ take $x_0 = h(t)$

$\int_0^1 F(h(t))\,dt \geq \int_0^1 F\left(\int_0^1 h(s)\,ds\right) dt + \int_0^1 C(y_0) \cancel{\left[h(t) - \int_0^1 h(s)\,ds\right] dt}$ $\quad y_0 = \int_0^1 h(s)\,ds$

CAUCHY-SCHWARTZ

$$\leq \int_{\mathbb{R}^n} \int_0^1 |D\phi(x+th)|^p \, |h|^p \, dt \, dx \overset{\text{Fubini/Tonelli}}{=} \int_0^1 \int_{\mathbb{R}^n} |D\phi(x+th)|^p \, |h|^p \, dx \, dt$$

$$= |h|^p \, \| |D\phi| \|_{L^p} \, . \ \square$$

Observation :

If $f \in L^p(\mathbb{R}^n)$ s. that $\exists\, C>0$ such that
$\forall\, h \in \mathbb{R}^n$ $\|f(x+h)-f(x)\|_{L^p} = \|\tau_h f - f\|_{L^p} \leq C\,|h|$

then if $\underline{\underline{p>1}}$ $f \in W^{1,p}(\mathbb{R}^n)$.

$$\forall\, \phi \in C_c^\infty(\mathbb{R}^n) \qquad \int_{\mathbb{R}^n} \phi\,(\tau_h f - f) \overset{\text{Hölder}}{\leq} \|\phi\|_{L^{p'}} \cdot \|\tau_h f - f\|_{L^p} \leq$$

$$\leq \|\phi\|_{L^{p'}} \cdot C\,|h|$$

$$\downarrow$$

$h = t e_i$

$$\frac{\|\phi\|_{L^{p'}}\, C \cancel{t}}{\cancel{t}} \geq \int_{\mathbb{R}^n} \phi \frac{(\tau_h f - f)}{t}\, dx = \int_{\mathbb{R}^n} \frac{\phi(x-te_i)-\phi(x)}{t}\, f(x)\,dx \xrightarrow{t\to 0^+} -\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i}\, f\,dx$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^n) \quad \underbrace{\left| - \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} f \right|}_{\substack{\| \\ \left(\frac{\partial}{\partial x_i} T_f\right)(\phi)}} \leq C \, \|\phi\|_{L^{p'}(\mathbb{R}^n)}$$

$\frac{\partial}{\partial x_i} T_f : C_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ can be extended to a linear continuous operator in $L^{p'}(\mathbb{R}^n)$

$$\boxed{p > 1 \Rightarrow p' < +\infty} \Rightarrow \left(\frac{\partial}{\partial x_i} T_f\right) \in L^p(\mathbb{R}^n) \quad \exists g_i \in L^p(\mathbb{R}^n)$$

For $p = 1$ NOT TRUE

$$\int_{\mathbb{R}^n} - \frac{\partial \phi}{\partial x_i} f \, dx = \int_{\mathbb{R}^n} g_i \, \phi \, dx \qquad g_i = \frac{\partial f}{\partial x_i} \in L^p(\mathbb{R}^n)$$

$$f \in W^{1,p}(\mathbb{R}^n)$$