Riassunto:

$T \leq \mathbb{R}^n$ $\quad B_T = \{u_1, \dots, u_k\}$ base ortonormale di T

$T^\perp \leq \mathbb{R}^n$ $\quad B_{T^\perp} = \{u_{k+1}, \dots, u_n\}$ " " " $T^\perp$

allora

$$B = \{u_1, \dots, u_k, u_{k+1}, \dots, u_n\} \text{ è base ortonormale di } \mathbb{R}^n$$

$\forall v \in \mathbb{R}^n \qquad v = \sum_{i=1}^{n} (v \cdot u_i)\, u_i \Rightarrow$

$$v = \underbrace{\sum_{i=1}^{k} (v \cdot u_i)\, u_i}_{T} + \underbrace{\sum_{i=k+1}^{n} (v \cdot u_i)\, u_i}_{T^\perp}$$

$v = u + w$ con $u \in T$, $w \in T^\perp$ $\qquad T \oplus T^\perp = \mathbb{R}^n$

$$P_T(v) = u = \sum_{i=1}^{k} (v \cdot u_i)\, u_i$$

Proiezione ortogonale

$$\boxed{P_T(v) = \sum_{i=1}^{k} (v \cdot u_i)\, u_i \quad \text{con } B_T = \{u_1, \dots, u_k\} \text{ base ortonormale di } T}$$

Procedimento di ortonormalizzazione di Gram-Schmidt G-S

Sia $B=\{v_1,\dots,v_k\}$ una base di un sottospazio $T \le \mathbb{R}^n$ allora esiste $\mathcal{C}=\{u_1,\dots,u_k\}$ base ortonormale di T tale che

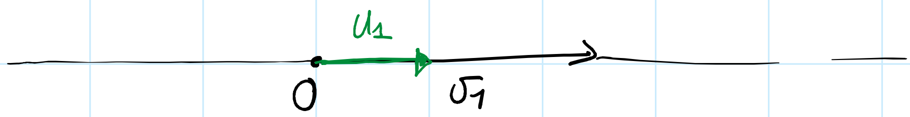
$$\boxed{\langle v_1,\dots,v_i\rangle = \langle u_1,\dots,u_i\rangle \quad \forall i=1,\dots,k.}$$

La base $\mathcal{C}$ si trova con il metodo G-S

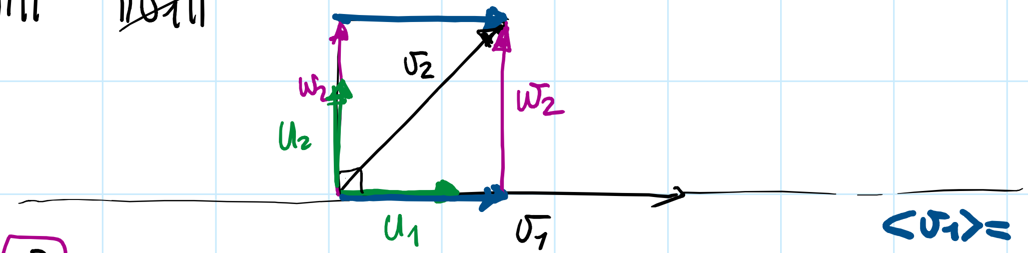
Passo 1:

$B=\{v_1,\dots,v_k\}$ $\quad i=1$ $\quad \boxed{\langle v_1\rangle = \langle u_1\rangle}$

$$\boxed{u_1=\frac{v_1}{\|v_1\|}}$$



$$\|u_1\| = \left\|\frac{v_1}{\|v_1\|}\right\| = \frac{1}{\cancel{\|v_1\|}}\cancel{\|v_1\|} = 1 \implies u_1 \text{ è versore}$$



$v_2 = \boxed{(v_2 \cdot u_1)\, u_1} + \boxed{w_2}$

$P_{\langle v_1\rangle}(v_2)$

$\langle v_1\rangle = \langle u_1\rangle$

Passo 2:

$$\boxed{w_2 = v_2 - (v_2 \cdot u_1)\, u_1}$$

$$\boxed{u_2 = \frac{w_2}{\|w_2\|}}$$

Passo 3:

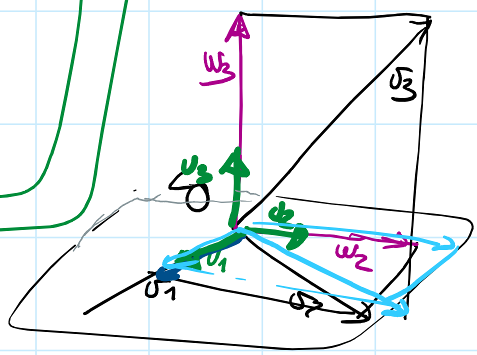
$$w_3 = v_3 - \left[(v_3 \cdot u_1)\, u_1 + (v_3 \cdot u_2)\, u_2 \right] \qquad \in \langle v_1, v_2, v_3 \rangle$$

$$u_3 = \frac{w_3}{\|w_3\|}$$

Passo i-esimo: $i \geq 2$ supp. $u_1, \dots, u_{i-1}$

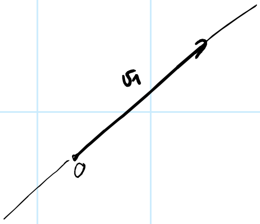
$$w_i = v_i - \left(\sum_{j=1}^{i-1} (v_i \cdot u_j)\, u_j \right)$$

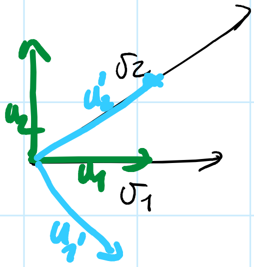
$$u_i = \frac{w_i}{\|w_i\|}$$



Esempio: applicare G-S a $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\qquad\qquad\qquad\qquad\qquad\qquad v_1 \qquad v_2 \qquad v_3$





Passo 1: $\boxed{u_1 = \dfrac{v_1}{\|v_1\|}}$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \|v_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ $\quad \boxed{u_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}}$

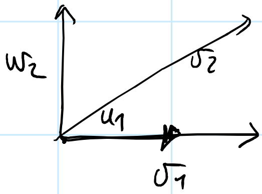
Passo 2: $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$w_2 = v_2 - (v_2 \cdot u_1)\, u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} =$$

$$\left[1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} + 0 \cdot \frac{1}{\sqrt{3}} \right]$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \frac{2}{3} \\ 1 - \frac{2}{3} \\ 0 - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$



$$w_2 \cdot v_1 = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} + \frac{1}{3} - \frac{2}{3} = 0$$

$w_2 \perp v_1$ $\qquad \|a v\| = |a|\,\|v\|$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\cancel{3}}{\sqrt{6}} \frac{1}{\cancel{3}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \qquad \|w_2\| = \left\| \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \right\| = \left\| \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\| =$$

$$= \frac{1}{3} \left\| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\| = \frac{1}{3} \sqrt{1^2+1^2+4} = \frac{\sqrt{6}}{3}$$

$\boxed{u_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}}$ $\quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad u_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$

Passo 3: $w_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)\, u_2$

$$w_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} - \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\right]\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} - \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}\right]\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix} =$$

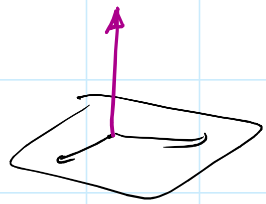
$$\left(v_3 \cdot \tfrac{1}{\sqrt{3}} v_1\right)\left(\tfrac{1}{\sqrt{3}} v_1\right)$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{3}\underbrace{\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right]}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{6}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}\right]\begin{pmatrix}1\\1\\-2\end{pmatrix} =$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}\frac{6-2-1}{6}\\[4pt] \frac{0-2-1}{6}\\[4pt] \frac{0-1+1}{3}\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$w_3 \cdot v_1 = 0$

$w_3 \cdot v_2 = 0$



$$w_3 \cdot v_1 = \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{2}\Big[\underbrace{1\cdot 1 + (-1)\cdot 1}_{1-1} + 0\cdot 1\Big] = 0$$

$$w_3 \cdot v_2 = \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}\left[1\cdot 1 + (-1)\cdot 1 + 0\cdot 0\right] = 0$$

$$u_3 = \frac{w_3}{\|w_3\|} = \qquad \boxed{w_3 = \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}} \qquad \|w_3\| = \frac{1}{2}\left\|\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$= \sqrt{2}\,\frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2\\-\sqrt{2}/2\\0\end{pmatrix}$$

$$\mathcal{C} = \left\{\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}, \begin{pmatrix}\sqrt{2}/2\\-\sqrt{2}/2\\0\end{pmatrix}\right\}$$

Esercizio 1. Si considerino $\mathbb{R}^4$ con coordinate x_1, x_2, x_3, x_4 e i sottospazi:

$$U : \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases} \qquad W : \begin{cases} 2x_1 - x_2 - x_4 = 0 \\ -4x_1 + 2x_2 + 2x_4 = 0 \end{cases}$$

(a) Determinare una $\mathcal{B}_U$ di U, una base $\mathcal{B}_W$ di W, una base $\mathcal{B}_{U\cap W}$ di $U \cap W$ e una base $\mathcal{B}_{U+W}$ di $U + W$. I sottospazi U e W sono in somma diretta? (4 pti)

(b) Completare la base $\mathcal{B}_{U\cap W}$ di $U \cap W$ ad una base $\mathcal{C}$ di W.

Verificare che il vettore $v = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}$ appartiene al sottospazio W e determinare una base di $\mathbb{R}^4$ che contenga v. (2 pti)

~~(c)~~ Determinare una base ortonormale di U e calcolare la proiezione ortogonale del vettore $v' = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}$ sul sottospazio U. (2 pti)

$\mathbb{R}^4 \qquad U: \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases} \qquad \mathcal{C}_U = \{u_1, u_2\}$ base ortonormale di U.

$P_U \begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}$.

Determinare $B_U = \{v_1, v_2\}$ base di U.

$\begin{cases} x_1 = -x_2 + x_4 \\ x_3 = -x_4 \end{cases} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad U = \left\{ \begin{pmatrix} -x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \middle| \; x_2, x_4 \in \mathbb{R} \right\}$

$B_U = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ Applichiamo G-S alla base B_U

Passo 1: $\boxed{u_1 = \dfrac{v_1}{\|v_1\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}}$ $\qquad v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \|v_1\| = \sqrt{2}$

$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Passo 2: $w_2 = v_2 - (v_2 \cdot u_1) u_1 =$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} - \underbrace{\left[\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \right]}_{-\frac{1}{\sqrt{2}}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1/2 \\ 1/2 \\ -2/2 \\ 2/2 \end{pmatrix} \qquad w_2 \cdot v_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2} + \frac{1}{2} + 0 + 0 = 0$$

$$u_2 = \frac{w_2}{\|w_2\|} = \qquad w_2 = \frac{1}{2}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix} \qquad \|w_2\| = \frac{1}{2}\sqrt{1+1+4+4} = \frac{\sqrt{10}}{2}$$

$$= \frac{\cancel{2}}{\sqrt{10}}\,\frac{1}{\cancel{2}}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix} = \boxed{\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix} = u_2}$$

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$e_U = \left\{ \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}, \begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix} \right\} = \{u_1, u_2\}$$

$$\boxed{P_U(v') = (v'\cdot u_1)\,u_1 + (v'\cdot u_2)\,u_2} \qquad v' = \begin{pmatrix}0\\1\\-3\\4\end{pmatrix}$$

$$P_U\begin{pmatrix}0\\1\\-3\\4\end{pmatrix} = \Bigg[\underbrace{\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}}_{1/\sqrt{2}}\Bigg]\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix} + \Bigg[\underbrace{\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix}}\Bigg]\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix} + \left(\frac{1}{\sqrt{10}} + \frac{6}{\sqrt{10}} + \frac{8}{\sqrt{10}}\right)\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix} =$$

$$v' = u + (v' - u)$$

$$= \begin{pmatrix}-1/2\\1/2\\0\\0\end{pmatrix} + \begin{pmatrix}\cancel{15}^{3}/\cancel{10}_{2}\\\cancel{15}^{3}/\cancel{10}_{2}\\-\cancel{30}^{3}/\cancel{10}\\\cancel{30}^{3}/\cancel{10}\end{pmatrix} = \underset{U}{\begin{pmatrix}1\\2\\-3\\3\end{pmatrix}}$$

$$\overset{v'}{\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}} = \underset{U}{\overset{u}{\begin{pmatrix}1\\2\\-3\\3\end{pmatrix}}} + \underset{U^\perp}{\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}}$$

$$\boxed{U: \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}} \qquad \begin{matrix} 1+2-3=0 \ \checkmark \\ -3+3=0 \ \checkmark \end{matrix} \implies \begin{pmatrix}1\\2\\-3\\3\end{pmatrix} \in U$$

$$U^\perp = \left\langle \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle$$

1) Determinare tutti i vettori $v \in \mathbb{R}^4$ tali che

$$P_U(v) = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$$

2) Determinare la matrice associata rispetto alla base canonica della simmetria ortogonale di asse U (direzione $U^\perp$).

3) Completare la base $\mathcal{C}_U = \{u_1, u_2\}$ a base ortonormale di $\mathbb{R}^4$.

$$U = \left\langle \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \right\rangle$$

Svolg:

1) $$\left\{ v \in \mathbb{R}^4 \;\middle|\; P_U(v) = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} \right\} = P_U^{-1}\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\right\}$$

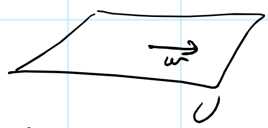
$P_U^{-1}\{w\}$ — NO $\emptyset$ se $w \notin \operatorname{Im} P_U = U$ — SI

ma noi abbiamo $w = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} \in U$

$\bar{v} + \ker P_U$ con $P_U(\bar{v}) = w$

$\forall u \in U$ $P_U(u) = u$, $\bar{v} = w$

$$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + U^\perp$$



① $$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle$$

$$U: \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

2) $S = 2P - I_4$

$$\boxed{P_U(v) = (v \cdot u_1)\,u_1 + (v \cdot u_2)\,u_2}$$

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad u_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ -2/\sqrt{10} \\ 2/\sqrt{10} \end{pmatrix}$$

$$P_U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \right] \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ -2/\sqrt{10} \\ 2/\sqrt{10} \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ -2/\sqrt{10} \\ 2/\sqrt{10} \end{pmatrix} =$$

$$= \frac{-x_1+x_2}{\sqrt{2}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} + \frac{x_1+x_2-2x_3+2x_4}{\sqrt{10}} \begin{pmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ -2/\sqrt{10} \\ 2/\sqrt{10} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{x_1-x_2}{2} \\ \frac{-x_1+x_2}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{x_1+x_2-2x_3+2x_4}{10} \\ \frac{x_1+x_2-2x_3+2x_4}{10} \\ \frac{-x_1-x_2+2x_3-2x_4}{5} \\ \frac{x_1+x_2-2x_3+2x_4}{5} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{5x_1-5x_2+x_1+x_2-2x_3+2x_4}{10} \\ \frac{-5x_1+5x_2+x_1+x_2-2x_3+2x_4}{10} \\ \frac{-x_1-x_2+2x_3-2x_4}{5} \\ \frac{x_1+x_2-2x_3+2x_4}{5} \end{pmatrix} = \begin{pmatrix} \frac{6x_1-4x_2-2x_3+2x_4}{10} \\ \frac{-4x_1+6x_2-2x_3+2x_4}{10} \\ \frac{-x_1-x_2+2x_3-2x_4}{5} \\ \frac{x_1+x_2-2x_3+2x_4}{5} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \end{pmatrix} \qquad P^t = P$$

$$S = 2P - I_4$$

$$P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

4) $\mathcal{C}_U = \{u_1, u_2\}$ completa a $\mathcal{C} = \{\underbrace{u_1, u_2}_{U}, \underbrace{u_3, u_4}_{U^\perp}\}$

base ortonormale di $\mathbb{R}^4$.

$B_{U^\perp} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ $\quad$ G-S $\quad \{u_3, u_4\}$

$\quad\quad\quad\;\; v_3 \quad\;\; v_4$

1° metodo

2° metodo $\quad \overline{B} = \{u_1, u_2, w_3, w_4\}$ base di $\mathbb{R}^4$ Fare G-S su $\overline{B}$.

1° metodo $\quad U = \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases} = u_1, u_2 \mathcal{C}_4 \ \{3, 4\} \quad u\ u \quad\quad U \oplus U^\perp = \mathbb{R}^4$

$U^\perp = \langle u_3, u_4 \rangle \quad \{u_3, u_4\}$ base ortonormale del $U^\perp$

$\overline{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ $\quad$ applicate G-S $\quad \{u_3, u_4\}$.

$\quad\quad v_3 \quad\quad v_4$

$$u_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{pmatrix} \qquad w_4 = v_4 - (v_4 \cdot u_3)\, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1 \\ 2/3 \end{pmatrix}$$

$$u_4 = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

2° metodo $\quad \{u_1, u_2, v_3, v_4\}$ base di $\mathbb{R}^4$ $\quad$ G-S su questa.

$$\left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ -2/\sqrt{10} \\ 2/\sqrt{10} \end{pmatrix}, \underset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$ $\quad$ applichiamo G-S

$$w_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 =$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \frac{1}{2}\left[\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\right]\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} - \frac{1}{10}\left[\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\\2\end{pmatrix}\right]\begin{pmatrix}1\\1\\-2\\2\end{pmatrix} =$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - 0\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + \frac{1}{5}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix} = \begin{pmatrix}1/5\\1/5\\3/5\\2/5\end{pmatrix} \qquad u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{15}}\begin{pmatrix}1\\1\\3\\2\end{pmatrix}$$

$$w_4 = v_4 - (v_4 \cdot u_1)u_1 - (v_4 \cdot u_2)u_2 - (v_4 \cdot u_3)u_3 =$$

$$= \begin{pmatrix}0\\0\\0\\1\end{pmatrix} - \frac{1}{2}\left[\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\right]\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} - \frac{1}{10}\left[\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\\2\end{pmatrix}\right]\begin{pmatrix}1\\1\\-2\\2\end{pmatrix} - \frac{1}{15}\left[\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\3\\2\end{pmatrix}\right]\begin{pmatrix}1\\1\\3\\2\end{pmatrix} =$$

$$= \begin{pmatrix}0\\0\\0\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix} - \frac{2}{15}\begin{pmatrix}1\\1\\3\\2\end{pmatrix} = \frac{1}{15}\begin{pmatrix}-3-2\\-3-2\\+6-6\\15-6-4\end{pmatrix} = \frac{1}{15}\begin{pmatrix}-5\\-5\\0\\5\end{pmatrix} = \begin{pmatrix}-1/3\\-1/3\\0\\1/3\end{pmatrix}$$

$$u_4 = \frac{w_4}{\|w_4\|} = \begin{pmatrix}-1/\sqrt{3}\\-1/\sqrt{3}\\0\\1/\sqrt{3}\end{pmatrix}$$