Definizione: dato $T \leq \mathbb{R}^n$ l'ortogonale di T è

$$T^\perp = \{ v \in \mathbb{R}^n \mid v \cdot t = 0 \ \forall \ t \in T \}$$

Domanda 15: $\forall \ T \leq \mathbb{R}^n$ si ha $T^\perp \leq \mathbb{R}^n$.

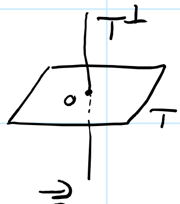
Domanda 16:

Proposizione: dato $T \leq \mathbb{R}^n$ si ha

$$T \oplus T^\perp = \mathbb{R}^n$$

Dimostrazione:

① Dimostriamo $T \cap T^\perp = \{\vec{0}\}$



Sia $v \in T \cap T^\perp$ e dimostriamo che $v = \vec{0}$

$$v \perp v \Rightarrow v \cdot v = 0 \Rightarrow \|v\|^2 = 0 \rightarrow v = \vec{0}$$

② Dimostriamo $\dim T + \dim(T^\perp) = n$

Perché $T + T^\perp \leq \mathbb{R}^n$ $\qquad T \oplus T^\perp = \mathbb{R}^n$

$\dim T + \dim T^\perp = n$

Se $\dim T = k$ $\quad B_T = \{ t_1, \dots, t_k \}$ dimostriamo

$\dim T^\perp = n - k$

$$T^\perp = \{ v \in \mathbb{R}^n \mid t \cdot v = 0 \ \forall\, t \in T \} =$$

$$= \left\{ v \in \mathbb{R}^n \;\middle|\; \left(\sum_{i=1}^{k} a_i t_i \right) \cdot v = 0 \quad \forall\, a_1, \dots, a_k \in \mathbb{R} \right\} =$$

$$= \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{array}{lll} t_1 \cdot v = 0 & a_1 = 1 & a_2 = 0 \dots = a_k = 0 \\ t_2 \cdot v = 0 & a_1 = 0 & a_2 = 1 \quad a_3 = \dots = a_k = 0 \\ \vdots & & \\ t_k \cdot v = 0 & a_k = 1 & a_1 = \dots = a_{k-1} = 0 \end{array} \right\}$$

$$\begin{cases} t_1 \cdot v = 0 \\ t_2 \cdot v = 0 \\ \vdots \\ t_k \cdot v = 0 \end{cases}$$

Sistema lineare di k equazioni in n incognite perché $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$t_1 \cdot v = t_1^t v = 0$$

La matrice associata al sistema è

$$A = \begin{pmatrix} t_1^t \\ t_2^t \\ \vdots \\ t_k^t \end{pmatrix}$$

$\dim T^\perp =$
$= \dim \operatorname{Ker} A = n - rg(A) = n - k$

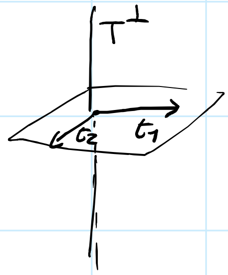
perché le righe sono lin. indip. in quanto $B_T = \{ t_1, \dots, t_k \}$ è base di T.

$\square$

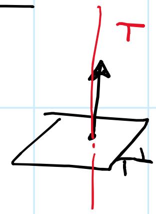
Esempi:

Nota bene $T = \langle t_1, \dots t_k \rangle$ allora

$$T^\perp = \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{cases} t_1 \cdot v = 0 \\ \vdots \\ t_k \cdot v = 0 \end{cases} \right\}$$

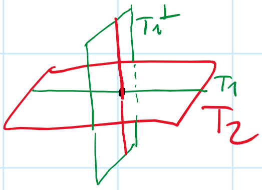


$$(T^\perp)^\perp = T$$

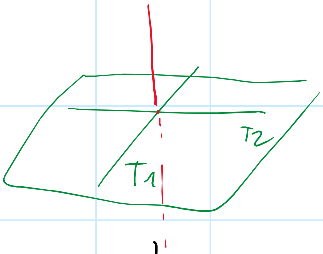
T	$T^\perp$
1) $T=\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ $\quad \mathbb{R}^3$ $\dim T = 1$	$T^\perp$ $\quad t_1 \cdot v = \boxed{z=0}$ $\dim T^\perp = 2$ 
2) $T=\left\langle \underset{t_1}{\begin{pmatrix} 1 \\ 3 \\ 8 \\ -5 \end{pmatrix}}, \underset{t_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 7 \end{pmatrix}} \right\rangle$ $\dim T = 2$	$T^\perp \begin{cases} x_1+3x_2+8x_3-5x_4=0 \\ x_2+7x_4=0 \end{cases}$ $\quad \begin{matrix} t_1\cdot v=0 \\ t_2 \cdot v = 0 \end{matrix}$ $\dim T^\perp = 2$
3) $T \begin{cases} 5x_1+8x_4=0 \end{cases}$ $\quad t_1\cdot v=0$ $\quad \mathbb{R}^4$ $\dim T = 3$	$T^\perp = \left\langle \overset{t_1}{\begin{pmatrix} 5 \\ 0 \\ 0 \\ 8 \end{pmatrix}} \right\rangle$
4) $T \begin{cases} x-4z=0 \\ x+y+7z=0 \end{cases}$ $\quad \mathbb{R}^3$ $\dim T = 1$	$T^\perp = \left\langle \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\rangle$ $\dim T^\perp = 2$

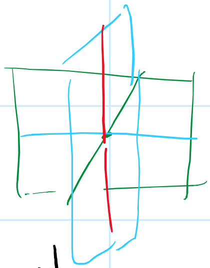
Proprietà:

1) $T \leq \mathbb{R}^n \Longrightarrow T^\perp \leq \mathbb{R}^n$
2) $T \oplus T^\perp = \mathbb{R}^n \quad \forall\, T \leq \mathbb{R}^n$
3) $(\mathbb{R}^n)^\perp = \{\vec{0}\}$ $\qquad \{\vec{0}\}^\perp = \mathbb{R}^n$ $\qquad 0=0$ equazione cartesiana di $\mathbb{R}^n$
4) Se $T_1 \leq T_2$ allora $T_1^\perp \geq T_2^\perp$



5) $$(T_1 + T_2)^\perp = T_1^\perp \cap T_2^\perp$$





$$(T_1 \cap T_2)^\perp = T_1^\perp + T_2^\perp$$

6) $$(V^\perp)^\perp = V \qquad \forall\, V \le \mathbb{R}^n.$$

Osservazione: Se $\mathcal{C}_U = \{u_1, \dots, u_k\}$ base ortonormale di $U \le \mathbb{R}^n$ e $\mathcal{C}_{U^\perp} = \{u_{k+1}, \dots, u_n\}$ base ortonormale di $U^\perp$ allora

$u_i \perp u_j \quad \forall\, i,j$

$$\mathcal{C} = \{u_1, \dots, u_k, u_{k+1}, \dots, u_n\} \text{ base ortonormale}$$

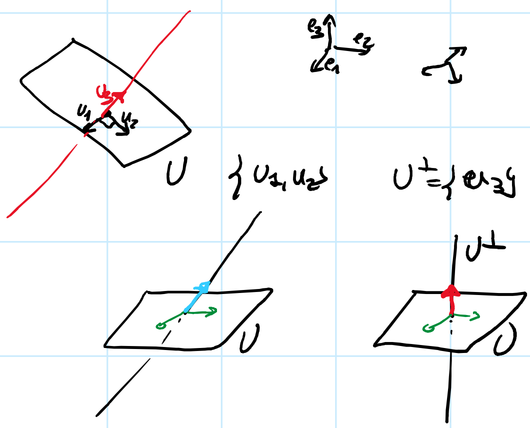
di $\mathbb{R}^n$.

$u_i \cdot u_j = 0$

$1 \le i \le k \qquad u_i \in U$

$k+1 \le j \le n \qquad u_j \in U^\perp$

$U \oplus U^\perp$



Definizione: $V = U \oplus W$ $\quad v = u + w$

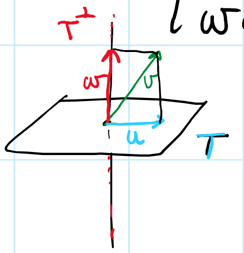
$$P_U^W : V \longrightarrow V \qquad P_U^W(v) = u.$$

Definizione: dato $T \leq \mathbb{R}^n$ l'endomorfismo di proiezione ortogonale su T è

$\mathbb{R}^n = T \oplus T^\perp$

$$P_T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad P_T(v) = u \qquad v = u + w \quad \begin{cases} u \in T \\ w \in T^\perp \end{cases}$$

$$P_T(u) = u \qquad \forall\, u \in T$$
$$P_T(w) = \vec{0} \qquad \forall\, w \in T^\perp$$



$v = u + w$

Proposizione: se $\mathcal{C} = \{u_1, \dots, u_n\}$ e base ortonormale di $\mathbb{R}^n$ allora dato $v \in \mathbb{R}^n$

$$\boxed{v = \sum_{i=1}^{n} (v \cdot u_i)\, u_i}$$

cioè
$$(v)_{\mathcal{C}} = \begin{pmatrix} v \cdot u_1 \\ v \cdot u_2 \\ \vdots \\ v \cdot u_n \end{pmatrix}.$$

Dimostrazione: $\mathcal{C}=\{u_1,\dots,u_n\}$ base ortonormale $\iff$

$v=\sum_{i=1}^{n} a_i u_i$ $\qquad u_i\cdot u_j = \begin{cases} 0 & \text{se } i\neq j \\ 1 & \text{se } i=j \end{cases}$ $\qquad u_i\cdot u_i=\|u_i\|^2=1$

$$v\cdot u_1 = \left(\sum_{i=1}^{n} a_i u_i\right)\cdot u_1 = \sum_{i=1}^{n} a_i\,(u_i\cdot u_1) = a_1\underbrace{(u_1\cdot u_1)}_{1} + a_2\underbrace{(u_2\cdot u_1)}_{0}+\dots$$

$$\dots + a_n\underbrace{(u_n\cdot u_1)}_{0} = a_1$$

$$v\cdot u_k = \left(\sum_{i=1}^{n} a_i u_i\right)\cdot u_k = a_k\, u_k\cdot u_k = a_k \qquad \forall k=1,\dots,n.$$

Esempio: calcoliamo le coordinate del vettore $v=\begin{pmatrix}1\\0\end{pmatrix}$

nella base $\mathcal{C}=\left\{\underset{u_1}{\begin{pmatrix}\sqrt{3}/2\\1/2\end{pmatrix}},\ \underset{u_2}{\begin{pmatrix}-1/2\\\sqrt{3}/2\end{pmatrix}}\right\}$.

$$v\cdot u_1=\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\sqrt{3}/2\\1/2\end{pmatrix}= \left(1\cdot\frac{\sqrt{3}}{2}\right)+0\cdot\frac{1}{2}=\frac{\sqrt{3}}{2}$$

$$(v)_{\mathcal{C}}=\begin{pmatrix}v\cdot u_1\\v\cdot u_2\end{pmatrix}=\begin{pmatrix}\sqrt{3}/2\\-1/2\end{pmatrix}$$

$$v\cdot u_2=\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}-1/2\\\sqrt{3}/2\end{pmatrix}= 1\cdot\left(-\frac{1}{2}\right)+0\cdot\frac{\sqrt{3}}{2}=-\frac{1}{2}$$

$$v=\frac{\sqrt{3}}{2}u_1-\frac{1}{2}u_2$$

Proiezioni ortogonali su sottospazi di dimensione 1.

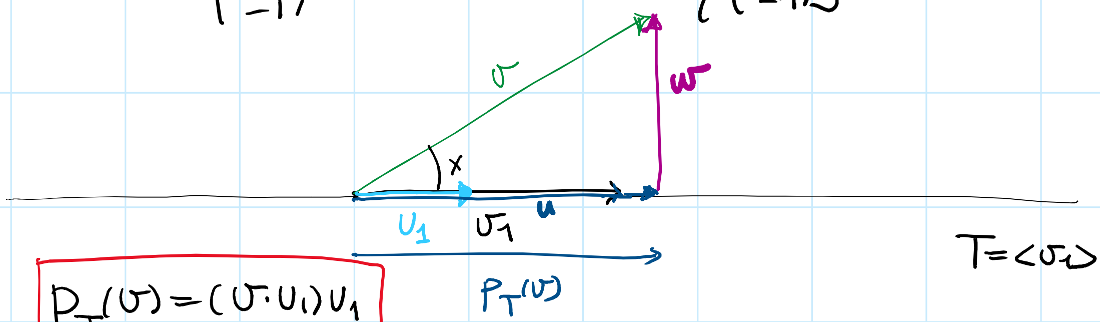
$T\leq\mathbb{R}^n \qquad \mathcal{C}_T=\{u_1\} \qquad \dim T=1$

$T^{\perp}\leq\mathbb{R}^n \qquad \mathcal{C}_{T^{\perp}}=\{u_2,\dots,u_n\}$ base ortonormale di $T^{\perp}$

$$v=\underbrace{(v\cdot u_1)u_1}_{T}+\underbrace{\left[(v\cdot u_2)u_2+\dots+(v\cdot u_n)u_n\right]}_{T^{\perp}}$$

$$\boxed{P_T(v) = (v\cdot u_1)\,u_1 \quad \text{con} \quad \{u_1\} \text{ base ortonormale di } T}$$

$$T = \left\langle \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \right\rangle \qquad B_T = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}} \right\} \qquad \|v_1\| = \sqrt{1^2+5^2+(-1)^2} = \sqrt{27}$$



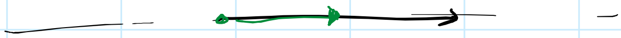
$$\boxed{P_T(v) = (v\cdot u_1)\,u_1}$$

$T = \langle v_1 \rangle$

$v = u + w \qquad u \in T, \quad w \in T^\perp$

$$\|u\| = \|v\|\,|\cos x| = \cancel{\|v\|}\,\frac{|u_1\cdot v|}{\cancel{\|v\|}} \qquad \cos x = \frac{u_1\cdot v}{\underset{\overset{\shortparallel}{1}}{\|u_1\|}\,\|v\|} = \frac{u_1\cdot v}{\|v\|}$$

$$v_1 = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \qquad \|v_1\| = \sqrt{27}$$



Normalizzazione:

$$\boxed{u_1 = \frac{v_1}{\|v_1\|}} = \begin{pmatrix} 1/\sqrt{27} \\ 5/\sqrt{27} \\ -1/\sqrt{27} \end{pmatrix} \quad \text{è versore}$$

$$\|a\,w\| = |a|\,\|w\|$$

$$\|u_1\| = \left\| \frac{1}{\|v_1\|}\, v_1 \right\| = \frac{1}{\|v_1\|}\,\|v_1\| = 1$$

$$P_T(v) = (v\cdot u_1)\,u_1 = \underbrace{\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{27} \\ 5/\sqrt{27} \\ -1/\sqrt{27} \end{pmatrix} \right]} \begin{pmatrix} 1/\sqrt{27} \\ 5/\sqrt{27} \\ -1/\sqrt{27} \end{pmatrix} =$$

$$= \left(\frac{x}{\sqrt{27}} + \frac{5y}{\sqrt{27}} - \frac{z}{\sqrt{27}}\right)\begin{pmatrix} \frac{1}{\sqrt{27}} \\ \frac{5}{\sqrt{27}} \\ -\frac{1}{\sqrt{27}} \end{pmatrix} = \begin{pmatrix} \frac{x+5y-z}{27} \\ \frac{5x+25y-5z}{27} \\ \frac{-x-5y+z}{27} \end{pmatrix}$$

$$A_{\mathcal{E}_3,\mathcal{E}_3,P_T} = P = \begin{pmatrix} \frac{1}{27} & \frac{5}{27} & -\frac{1}{27} \\ \frac{5}{27} & \frac{25}{27} & -\frac{5}{27} \\ -\frac{1}{27} & -\frac{5}{27} & \frac{1}{27} \end{pmatrix}$$

$\operatorname{Im} P_T = T = \left\langle \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \right\rangle$

$P^2 = P$

Nota bene:

Le matrici di proiezioni ortogonali rispetto alla base canonica sono sempre simmetriche $P = P^t$

$u_1 = \begin{pmatrix} 1/\sqrt{27} \\ 5/\sqrt{27} \\ -1/\sqrt{27} \end{pmatrix}$ 3×1

$P_T(v) = (v \cdot u_1)(u_1) =$ $\quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$= u_1 \, (u_1 \cdot v) = u_1 (u_1^t v) =$

$\boxed{3\times \cancel{1} \ \cancel{1}\times 3}\ 3\times 1$

$= (u_1 u_1^t)\, v$

$$P = u_1 u_1^t = \frac{1}{\sqrt{27}} v_1 \ \frac{1}{\sqrt{27}} v_1^t =$$

$$= \frac{1}{27}\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}(1 \ \ 5 \ \ -1) = \frac{1}{27}\begin{pmatrix} 1 & 5 & -1 \\ 5 & 25 & -5 \\ -1 & -5 & 1 \end{pmatrix}$$

$S_T = 2P - \mathbb{I}_3$ simmetria ortogonale di asse T

Oss: $\mathbb{R}^3 = T \oplus T^\perp$

$v = u + w$ $\quad u \in T$, $w \in T^\perp$

$P_T(v) = u$ $\quad P_{T^\perp}(v) = w = v - u$

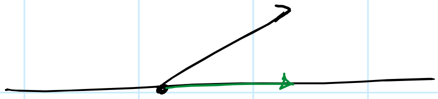
$P \quad\quad P' = \mathbb{I}_3 - P$

P $\qquad$ $P' = \mathbb{I}_3 - P$

$P' = A_{\mathcal{E}_3, \mathcal{E}_3, P_{T^\perp}}$

Esempio: determinare la matrice $A_{\mathcal{E}_3, \mathcal{E}_3, P_U} = P'$ delle proiezione ortogonale su $\boxed{U: x+5y-z=0}$ (cioè $U = T^\perp$).

$P' = \mathbb{I}_3 - P$

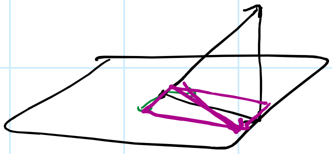


$T = \langle u_1 \rangle$

$T^\perp = \langle u_2, u_3 \rangle$

basi ortonormali

$\mathcal{C} = \{u_1, u_2, u_3\}$



$$v = \underbrace{(v \cdot u_1)\, u_1}_{T} + \underbrace{[(v \cdot u_2)\, u_2 + (v \cdot u_3)\, u_3]}_{T^\perp}$$

Definizione: $B = \{v_1, \dots, v_n\}$ base ortonormale di $\mathbb{R}^n$

$H = T_B^{\mathcal{E}} = (u_1 \dots u_n)$ allora vi dimostro che

$H^{-1} = H^t$

Una matrice H invertibile tale che

$H^{-1} = H^t$ si chiama matrice ortogonale.

$$\underbrace{\begin{pmatrix} U_1^t \\ U_2^t \\ \vdots \\ U_n^t \end{pmatrix}}_{H^t} \underbrace{\begin{pmatrix} U_1 \cdots U_n \end{pmatrix}}_{H} = \underbrace{\begin{pmatrix} U_1 \cdot U_1 & U_1 \cdot U_2 & U_1 \cdot U_3 & \cdots & U_1 \cdot U_n \\ \vdots & & & & \\ U_n \cdot U_1 & U_n \cdot U_2 & & & U_n \cdot U_n \end{pmatrix}}_{= \mathbb{I}_n}$$

$U_i \cdot U_j = 0$ se $i \neq j$

$U_i \cdot U_i = 1$