Es $\quad f:[0,1]\to\mathbb{R} \qquad f\in C[0,1]$

$$\|f\|_1 = \int_0^1 |f(t)|\,dt$$

norm on the space $C([0,1])$

distance $d_1(f,g)=\|f-g\|_1$

$\|f\|_1=0 \iff f=0$

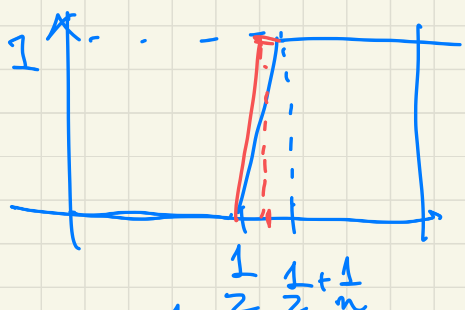
$\int_0^1 |f(t)|\,dt = 0$

$\|\lambda f\|_1 = |\lambda|\,\|f\|_1$

$\|f+g\|_1 \le \|f\|_1 + \|g\|_1$

$|f(t)+g(t)| \le |f(t)| + |g(t)|$

$f_n(t)$

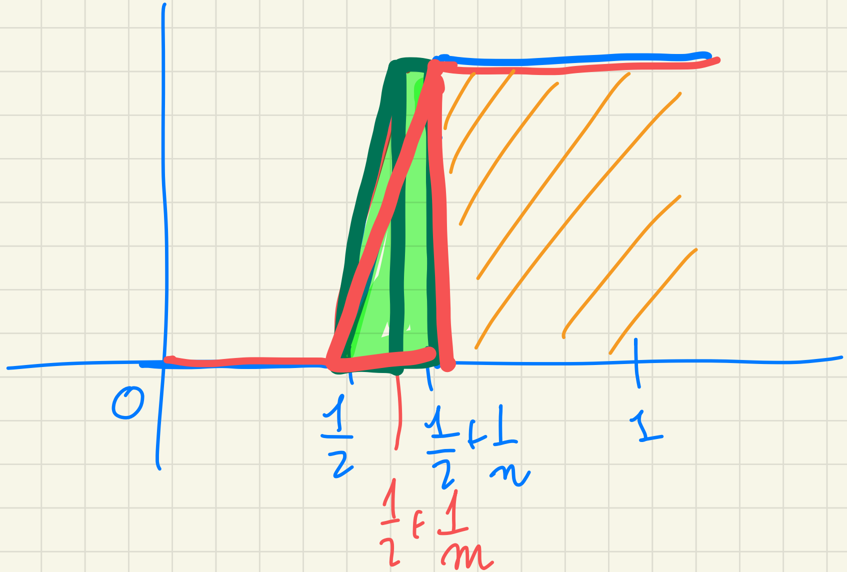


$\frac{1}{m} < \frac{1}{n}$

$n, m \qquad m > n$

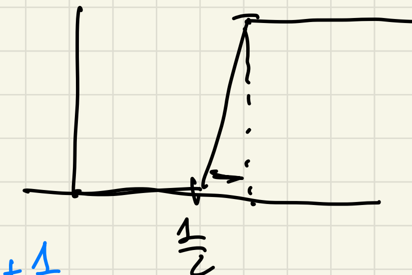
$$f_n(t) = \begin{cases} 0 & t \le \frac{1}{2} \\ n\left(t-\frac{1}{2}\right) & \frac{1}{2} < t \le \frac{1}{2}+\frac{1}{n} \\ 1 & t \ge \frac{1}{2}+\frac{1}{n} \end{cases}$$

$$\|f_n - f_m\|_1 = \int_0^1 |f_n(t) - f_m(t)|\,dt$$



$m > n$

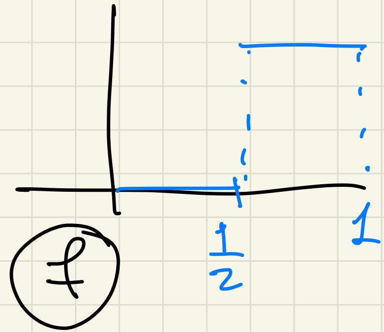
f_n is a Cauchy sequence



$$\|f_n - f_m\|_1 = \int_0^1 |f_n(t) - f_m(t)|\,dt = \int_{1/2}^{\frac{1}{2}+\frac{1}{n}} |f_n(t) - f_m(t)| =$$

$$\underbrace{\left(\frac{1}{n} - \frac{1}{m}\right)\cdot 1} + \underbrace{\frac{1}{2}\cdot\frac{1}{m}\cdot 1} - \underbrace{\frac{1}{2}\cdot\frac{1}{n}\cdot 1} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) \longrightarrow 0$$

the limit of f_n is



which is not continuous

$$\int_0^1 |f(t)|\,dt < +\infty.$$

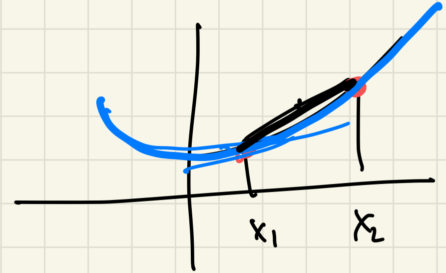
$\|\cdot\|_1$ $\quad$ $\mathcal{C}[0,1], \|\cdot\|_1$ is NOT BANACH (it is NOT COMPLETE)

$$\underbrace{L^1(0,1)}, \|\cdot\|_1$$

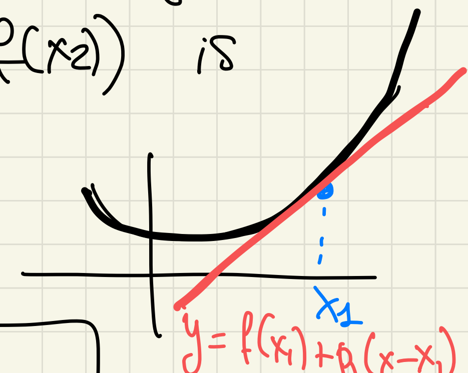
$$L^1(0,1) = \left\{ f : [0,1] \to \mathbb{R} \quad \int_0^1 |f(t)|\,dt < +\infty \right\} \supseteq \mathcal{C}[0,1]$$

JENSEN INEQUALITY (in $\mathbb{R}$)

$f : \mathbb{R} \to \mathbb{R}$ convex

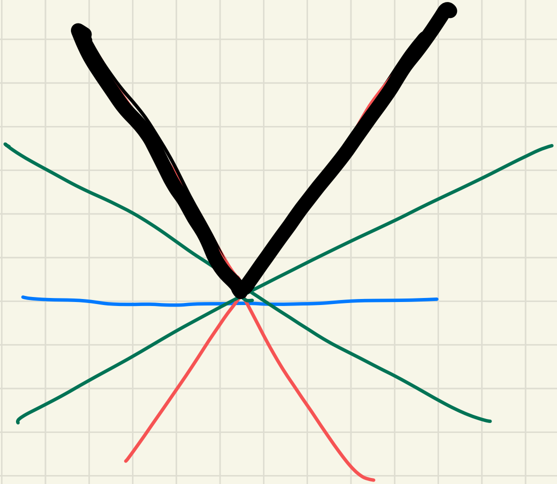


f convex means $\forall$ (x_1, $x_2 \in \mathbb{R}$ the segment in $\mathbb{R}^2$ connecting $(x_1, f(x_1))$ $(x_2, f(x_2))$ is above the graph of f



$f'' \geq 0$

$$\boxed{\begin{array}{l} \forall\, x_1 \in \mathbb{R} \quad \exists\, p_1 \in \mathbb{R} \qquad (p_1 = f'(x_1)) \\ f(x) \geq \boxed{f(x_1) + p_1 (x - x_1)} \quad \forall\, x \in \mathbb{R} \end{array}}$$



$$f(x) = |x|$$

$$p_0 = 0$$

$$f(x) \geq f(0) + 0 \cdot (x-0)$$

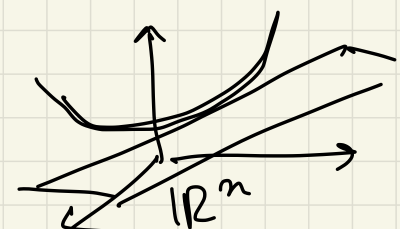
$$f(x) \geq f(0) + 1 \cdot (x-0)$$

$$f(x) \geq f(0) - 1\,(x-0)$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex

$\forall x_1 \in \mathbb{R}^n \; \exists p_1 \in \mathbb{R}^n \; (p_1 = Df(x_1))$ scalar product in $\mathbb{R}^n$

$$f(x) \geq \underbrace{f(x_1) + \overbrace{p_1 \cdot (x - x_1)}}$$



Jensen inequality

$f : \mathbb{R} \to \mathbb{R}$ convex

$\mu \in \mathcal{P}(\mathbb{R})$ μ probability measure $(\mu(\mathbb{R}) = 1)$

(a) $\rightarrow$ $\displaystyle\int_{\mathbb{R}} f(x)\, d\mu \geq f\Big(\overbrace{\int_{\mathbb{R}} x\, d\mu}^{\in \mathbb{R}}\Big)$ if $\displaystyle\int_{\mathbb{R}} x\, d\mu < +\infty$

(b) $\rightarrow$ $\boxed{\text{If } \displaystyle\int_{\mathbb{R}} x\, d\mu = +\infty \Rightarrow \int_{\mathbb{R}} f(x)\, d\mu = +\infty .}$

(b) $\exists\, p_0$ $\quad f(x) \geq f(0) + p_0 \cdot (x - 0)$ $\quad p_0 \neq 0$

$$\int_{\mathbb{R}} f(x)\, d\mu \geq \underbrace{\int_{\mathbb{R}} f(0)\, d\mu}_{} + \int_{\mathbb{R}} p_0\, x\, d\mu = f(0) + p_0 \int_{\mathbb{R}} x\, d\mu$$

If $p_0 = 0$ $\to$ we change point

take $x_1 \in \mathbb{R}$ $\quad p_1 \neq 0 \quad p_1 \geq 0$

$$f(x) \geq f(x_1) + p_1 (x - x_1)$$

$$\int_{\mathbb{R}} f(x)\, d\mu \geq \int_{\mathbb{R}} f(x_1)\, d\mu + \int_{\mathbb{R}} p_1 x\, d\mu - \int_{\mathbb{R}} \underbrace{p_1 x_1}\, d\mu =$$

$$= f(x_1) + \underbrace{p_1 \int_{\mathbb{R}} x\, d\mu}_{= +\infty} - p_1 x_1$$

Assume $\int_{\mathbb{R}} y\, d\mu(y) < +\infty$. $\qquad \underline{\underline{x_0}} = \underbrace{\int_{\mathbb{R}} y\, d\mu(y)} \in \mathbb{R}$

$$\boxed{f(x) \geq f(x_0) + p_{x_0} (x - x_0) \qquad \forall x}$$

$$\underbrace{\int_{\mathbb{R}} f(x)\,d\mu(x)}_{\parallel} \geq \underbrace{\int_{\mathbb{R}} f(x_0)\,d\mu} + \int_{\mathbb{R}} p_{x_0}(x-x_0)\,d\mu(x)$$

$$x_0 = \int_{\mathbb{R}} y\,d\mu(y)$$

$$\int_{\mathbb{R}} f(x)\,d\mu(x) \geq f(x_0) + p_0\left[\int_{\mathbb{R}} x\,d\mu(x) - x_0\right]$$

$$\int_{\mathbb{R}} f(x)\,d\mu(x) \geq f\left(\int_{\mathbb{R}} y\,d\mu(y)\right) + p_0\underbrace{\cancel{\left[\int_{\mathbb{R}} x\,d\mu(x) - \int_{\mathbb{R}} y\,d\mu(y)\right]}}_{0}$$

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space

$M^p = \{ X : \Omega \to \mathbb{R}$, random variables such that $\mathbb{E}|X|^p = \int_{\mathbb{R}} |x|^p \, \underline{d\mathcal{L}_X(x)} < +\infty \}$

$\mathcal{L}_X \in \mathcal{P}(\mathbb{R})$

$p \geq 1$

Corollary of Jensen inequality:

If $X \in M^p$ $p > 1$ then $X \in M^q$ $\forall 1 \leq q < p$

(If $\mathbb{E}(|X|^p) < +\infty \implies \mathbb{E}|X|^q < +\infty$ $\forall 1 \leq q < p$

$M^1 \supseteq M^2 \supseteq M^3 \supseteq M^4 \supset \dots$

proof $\quad p > 1$

$1 \leq q < p \qquad f(x) = |x|^{p/q} \qquad \frac{p}{q} > 1$

$\underline{f \text{ is convex}}$

$f'(x) = \frac{p}{q} |x|^{\frac{p}{q}-1} \frac{x}{|x|}$

for $\mathbb{E}|X|^p = \int_{\mathbb{R}} |x|^p \, d\mathcal{L}_X(x) = \int_{\mathbb{R}} \underbrace{f(|x|^q)}_{(|x|^q)^{p/q} = |x|^p} \, d\mathcal{L}_X(x)$

$f(|x|^q) \geq f(\underline{\underline{z_0}}) + \underline{\underline{p_0}} (|x|^q - \underline{\underline{z_0}}) \quad \forall x$

p_0 depends on $z_0 \qquad z_0 = \int_{\mathbb{R}} |y|^q \mathcal{L}_X(y) < +\infty$

$$f(|x|^q) = |x|^p \geq f\left[\int_{\mathbb{R}} |y|^q \, d\mathcal{L}_X(y)\right] + p_0 \cdot \left[|x|^q - \int_{\mathbb{R}} |y|^q \, d\mathcal{L}_X(y)\right]$$

integrate w.r. to $\mathcal{L}_X$

$$\int_{\mathbb{R}} |x|^p \, d\mathcal{L}_X(x) = \int_{\mathbb{R}} f(|x|^q) \, d\mathcal{L}_X(x) \geq$$

$$\geq \int_{\mathbb{R}} f\left[\left(\int_{\mathbb{R}} |y|^q \, d\mathcal{L}_X(y)\right)\right] d\mathcal{L}_X(x) + \underbrace{\cancel{p_0 \left(\int_{\mathbb{R}} |x|^q \, d\mathcal{L}_X(x) - \int_{\mathbb{R}} \int_{\mathbb{R}} |y|^q \, d\mathcal{L}_X(y) \, d\mathcal{L}_X(x) \right)}}_{0}$$

$$= \left[\int_{\mathbb{R}} |y|^q \, d\mathcal{L}_X(y)\right]^{p/q}$$

$$\boxed{\mathbb{E}(|X|^p) \geq \mathbb{E}(|X|^q)^{p/q}}$$

Moreover $\boxed{\left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \geq \left(\mathbb{E}|X|^q\right)^{1/q} \qquad \forall\, q < p}$

$$M^p = \{ X : \Omega \to \mathbb{R} \quad \mathbb{E}(|X|^p) < +\infty \}$$

$$\|X\|_p = \left(\mathbb{E}(|X|^p)\right)^{1/p}$$

Let check it is a norm

(1) $\|X\|_p = 0 \iff \underbrace{\mathbb{E}|X|^p = 0}$

$\Updownarrow$

$X = 0$ with probability 1 (ALMOST SURELY)

$\lambda \in \mathbb{R}$

(2) $\|\lambda X\|_p = \mathbb{E}(|\lambda X|^p)^{\frac{1}{p}} =$

$= \left(\mathbb{E}|\lambda|^p |X|^p\right)^{1/p} = \left(|\lambda|^p\right)^{1/p} \mathbb{E}(|X|^p)^{1/p}$

$= |\lambda| \left(\mathbb{E}|X|^p\right)^{1/p} = |\lambda|\, \|X\|_p$

③ If $X, Y \in M^p$ $\quad \|X+Y\|_p \overset{?}{\le} \|X\|_p + \|Y\|_p$

$$\underbrace{\left[\mathbb{E}(|X+Y|^p)\right]^{1/p}} \le \underbrace{\left(\mathbb{E}(|X|^p)\right)^{1/p}} + \underbrace{\left(\mathbb{E}(|Y|^p)\right)^{1/p}}$$

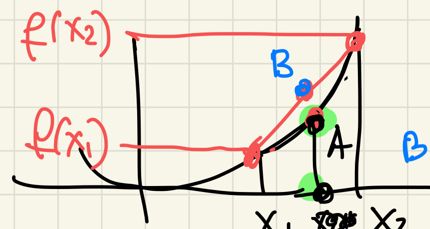
$$\sqrt{\mathbb{E}(|X+Y|)^2} \le \sqrt{\mathbb{E}|X|^2} + \sqrt{\mathbb{E}|Y|^2}$$

[In order to prove this, we prove before the so called Hölder inequality.

Obs $\quad X, Y \in M^p \Rightarrow \underbrace{X+Y \in M^p}$ $\quad$ (why this is true?)

$f(x) = |x|^p$ is convex $p \ge 1$

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$$



$A = \left(\frac{x_1+x_2}{2}, f\left(\frac{x_1+x_2}{2}\right)\right.$

$B\left(\frac{x_1+x_2}{2}, \frac{f(x_1)+f(x_2)}{2}\right)$

$$\left|\frac{x_1+x_2}{2}\right|^p \le \frac{|x_1|^p + |x_2|^p}{2}$$

$$\frac{|x_1+x_2|^p}{2^p} \le \frac{|x_1|^p + |x_2|^p}{2}$$

$$\underbrace{|x_1+x_2|^p \le 2^{p-1}\left(|x_1|^p+|x_2|^p\right)}_{\forall\, x_1, x_2 \in \mathbb{R}}$$

X, Y random v.

$$|X+Y|^p \le 2^{p-1}\left(|X|^p + |Y|^p\right)$$

$$\left(\underbrace{\mathbb{E}\left(|X+Y|^p\right)}\right)^{1/p} \le \left(2^{p-1}\left(\underbrace{\mathbb{E}(|X|^p)} + \underbrace{\mathbb{E}(|Y|^p)}\right)\right)^{1/p}$$

$X, Y \in M^p \longrightarrow X+Y \in M^p.$

YOUNG INEQUALITY

$p > 1$ $\quad$ $q =$ conjugate of p

$q > 1$

$$\boxed{\frac{1}{p} + \frac{1}{q} = 1}$$

(p, q)

$p = 2 \to q = 2$

$p = 4 \to q = \frac{4}{3}$

$p = \frac{3}{2} \to q = 3$

$\forall\, a, b \geq 0.$

$$\boxed{a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}}$$

(YOUNG INEQUALITY)

$p = q = 2$ $\qquad a \cdot b \leq \frac{a^2}{2} + \frac{b^2}{2}$

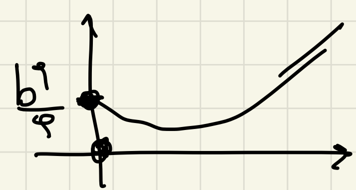
$(a-b)^2 = a^2 + b^2 - 2ab \geq 0$

$\frac{a^2 + b^2}{2} \geq \frac{\cancel{2}ab}{\cancel{2}}$

Fix $b > 0$ $\quad$ p, q conjugate

$$a \longmapsto \underbrace{\frac{a^p}{p}} + \frac{b^q}{q} \underbrace{- ab}$$

$a \geq 0$



o

$$f(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab \qquad f(0) = \frac{b^q}{q} \geq 0$$

$$\lim_{a \to +\infty} f(a) = +\infty$$

$p > 1$

$$a^p \left(\frac{1}{p} + \cancel{\frac{1}{a^p} \cdot \frac{b^q}{q}} - \cancel{\frac{b}{a^{p-1}}} \right)$$

$a^p \to +\infty$, $\to 0$, $\to 0$

f has a minimum

$$f'(a) = a^{p-1} - b \rightarrow a = b^{\frac{1}{p-1}} \text{ is the POINT of MINIMUM}$$

$$f\left(b^{\frac{1}{p-1}}\right) = \frac{b^{\frac{p}{p-1}}}{p} + \frac{b^q}{q} - b^{\frac{1}{p-1}} b^1$$

$$q = \frac{p}{p-1}$$

$$\frac{1}{q} + \frac{1}{p} = 1$$

$$f\Big(\underbrace{b^{\frac{1}{p-1}}}_{\text{POINT of MINIMUM}}\Big) = \underbrace{\min f}_{\text{MINIMUM of } f} = \frac{b^{\frac{p}{p-1}}}{p} + \frac{b^q}{q} - b^{1+\frac{1}{p-1}} =$$

$$= \underbrace{b^q\left[\frac{1}{p} + \frac{1}{q} - 1\right] = 0}$$

$$f(a) \geq 0 \quad \forall a \qquad \frac{a^p}{p} + \frac{b^q}{q} - ab \geq 0.$$

<u>Hölder inequality</u> $\quad X \in M^p, \; Y \in M^q \qquad \frac{1}{p} + \frac{1}{q} = 1$

$$\mathbb{E}(X \cdot Y) \leq \mathbb{E}(|X|^p)^{\frac{1}{p}} \cdot \left(\mathbb{E}(|Y|^q)\right)^{1/q}$$

$p = q = 2 \qquad X, Y \in M^2 \qquad \mathbb{E}(X \cdot Y) \leq \sqrt{\mathbb{E}|X|^2}\,\sqrt{\mathbb{E}(|Y|^2)}$

proof $\quad a = \dfrac{|X|}{(\mathbb{E}(|X|^p))^{1/p}} \qquad b = \dfrac{|Y|}{\mathbb{E}(|Y|^q)^{1/q}}$

$\Downarrow$

apply Young

$$a \cdot b = \frac{|X|\,|Y|}{\underbrace{(\mathbb{E}|X|^p)^{1/p}\,\mathbb{E}(|Y|^q)^{1/q}}} \leq \frac{|X|^p}{p\,\mathbb{E}(|X|^p)} + \frac{|Y|^q}{q\,\mathbb{E}(|Y|^q)}$$

$$\frac{a^p}{p} \qquad \frac{b^q}{q}$$

COMPUTE EXPECTED VALUE

$$\frac{\mathbb{E}(|X \cdot Y|)}{\mathbb{E}(|X|^p)^{1/p}(\mathbb{E}|Y|^q)^{1/q}} \leq \frac{\cancel{\mathbb{E}|X|^p}}{p\,\cancel{\mathbb{E}|X|^p}} + \frac{\cancel{\mathbb{E}(|Y|^q)}}{q\,\cancel{\mathbb{E}(|Y|^q)}} = \frac{1}{p} + \frac{1}{q} = 1$$

$$\mathbb{E}(X\cdot Y) \le \mathbb{E}\,|X\cdot Y| \le \left(\mathbb{E}|X|^p\right)^{1/p}\cdot\left(\mathbb{E}|Y|^q\right)^{1/q}.$$

$$X\in M^p \quad Y\in M^q \implies X\cdot Y\in M^1$$

COROLLARY

: $X, Y \in M^p$

$$\underbrace{\|X+Y\|_p}_{} = \left(\mathbb{E}|X+Y|^p\right)^{1/p} \le \underbrace{\left(\mathbb{E}|X|^p\right)^{1/p} + \left(\mathbb{E}|Y|^p\right)^{1/p}}_{\|X\|_p + \|Y\|_p}$$

$\|\cdot\|_p$ this is a norm

$$|X+Y|^p = \underbrace{\overbrace{|X+Y|}}_{\in M^p}\cdot\underbrace{\overbrace{|X+Y|^{p-1}}}_{\in M^{\frac{p}{p-1}}}$$

$$|X+Y|^{p-1}\in M^{\frac{p}{p-1}}$$

$$\mathbb{E}\left(|X+Y|^{\cancel{p-1}}\right)^{\frac{p}{\cancel{p-1}}} < +\infty$$

$$\frac{1}{p}+\frac{p-1}{p}=1$$

$$\underbrace{\mathbb{E}|X+Y|^p} = \mathbb{E}\,\underbrace{[|X+Y|]}_{\in\, M^p}\,\underbrace{|X+Y|^{p-1}}_{\in\, M^q} \le$$ apply Hölder inequality.

$$q = \frac{p}{p-1}$$

$$\frac{1}{p}+\frac{1}{q}=1$$

$$\|X+Y\|_p = (\mathbb{E}|X+Y|^p)^{1/p} \le (\mathbb{E}|X|^p)^{1/p} + \mathbb{E}(|Y|^q)^{1/q}$$

Recap

1) $p \geq 1$ $\qquad \underline{\underline{M^p}} = \{ X : \Omega \to \mathbb{R}$ random v.
$\mathbb{E}(|X|^p) < +\infty \}$

$M^1 \supseteq M^2 \supseteq \ldots$

2) on M^p I may define a norm
$\|X\|_p := \left[\mathbb{E}(|X|^p)\right]^{1/p}$ this is a norm
$\downarrow$
distance $\quad d_p(X,Y) = \mathbb{E}(|X-Y|^p)^{1/p} = \|X-Y\|_p$
Convergence $\quad X_n \longrightarrow X$ in M_p $\quad \mathbb{E}(|X_n - X|^p) \longrightarrow 0$

$p = 1$ $X_n \to X$ in MEAN $\quad \mathbb{E}\,|X_n - X| \underset{n \to +\infty}{\longrightarrow} 0$

$p = 2$ $X_n \to X$ in MEAN SQUARE $\quad \mathbb{E}\,|X_n - X|^2 \underset{n \to +\infty}{\longrightarrow} 0$

$(M^p, \|\cdot\|_p)$ is a BANACH SPACE.
(every Cauchy sequence is converging)

(3) Hölder inequality

$X \in M^p \quad Y \in M^q \qquad \frac{1}{p} + \frac{1}{q} = 1$

$$\mathbb{E}(X \cdot Y) \leq \mathbb{E}(|X \cdot Y|) \leq \left(\mathbb{E}|X|^p\right)^{1/p} \left(\mathbb{E}(|Y|^q)\right)^{1/q} = \|X\|_p \, \|Y\|_q$$

$X, Y \in M^2$ $\quad \frac{1}{2} + \frac{1}{2} = 1 \quad p = q = 2$

$$\mathbb{E}(X \cdot Y) \leq \|X\|_2 \, \|Y\|_2 = \sqrt{\mathbb{E}|X|^2} \sqrt{\mathbb{E}|Y|^2} =$$

$$= \sqrt{\mathbb{E}|X|^2 \, \mathbb{E}(|Y|^2)}$$

$$\left[\mathbb{E}(X \cdot Y)\right]^2 \leq \left(\mathbb{E}(|X|^2)\right) \mathbb{E}(|Y|^2)$$

$X_n \to X$ in mean square $\quad \mathbb{E}(|X_n - X|^2) \to 0$
(in M_2)

$\Rightarrow X_n \to X$ in mean (in M_1) $\quad \mathbb{E}|X_n - X| \to 0$

$$\left[\mathbb{E}(X_n - X) \cdot Y\right]^2 \leq \left(\mathbb{E}(|X_n - X|^2)\right) \cdot \mathbb{E}(|Y|^2)$$

$$Y = \frac{X_n - X}{|X_n - X|} \qquad (X_n - X)\, Y = |X_n - X| \qquad |Y| = 1$$

$$\mathbb{E}\left(|X_n - X|\right) \leq \underbrace{\mathbb{E}\left(|X_n - X|^2\right)}_{\to\, 0} \cdot 1$$

$$M^2 = \{ X \quad \mathbb{E}(|X|^2) < +\infty \}$$

Hölder ineq. : it makes sense to compute the product $X \cdot Y$ with $X, Y \in M^2$.

$$\omega \longmapsto X(\omega) \cdot Y(\omega)$$

$$X, Y \in M^2 \longrightarrow X \cdot Y \in M^1$$

* we define a scalar product $M^2 \times M^2 \longrightarrow \mathbb{R}$

$$X, Y \longrightarrow \langle X, Y \rangle$$

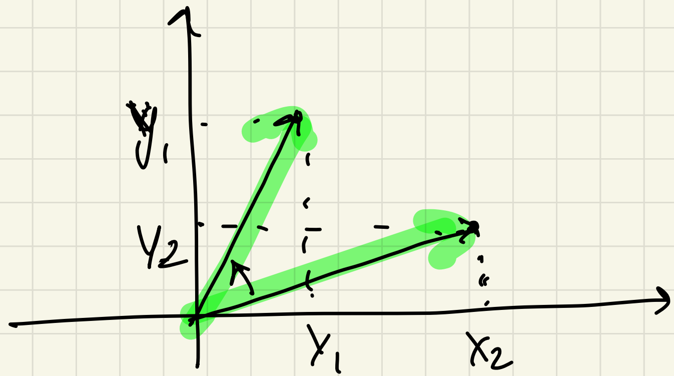
$$\langle X, Y \rangle = \mathbb{E}(X \cdot Y)$$

$\mathbb{R}^n$ we have the scalar product

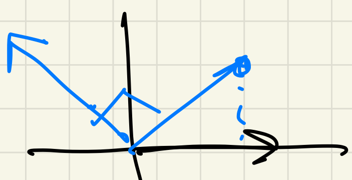
$$\langle (x_1 \dots x_n), (y_1 \dots y_n) \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

$(x_1 \dots x_n) \in \mathbb{R}^n$

$(y_1 \dots y_n) \in \mathbb{R}^n$



$(x_1, x_2 \dots x_n) \in \mathbb{R}^n$
$(y_1 \dots y_n) \in \mathbb{R}^n$
are orthogonal if the scalar product is 0



Also in M^2 we have a scalar product

(related to the norm

$$\langle X, X \rangle = \mathbb{E}(|X|^2) = \|X\|_2^2)$$

$$\langle X, Y \rangle = \mathbb{E}(X \cdot Y)$$

Definition

$X \perp Y$ (X, Y are 2 orthogonal random variables if

iff $\mathbb{E}(X \cdot Y) = 0$ $\qquad$ ($\cancel{\mathbb{E}(|X \cdot Y|) = 0}$)

If X, Y are independent

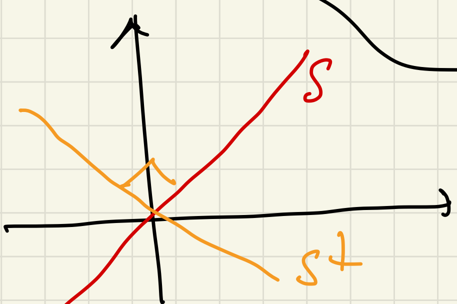
$(X - \mathbb{E}(X)) \perp Y \qquad X \perp (Y - \mathbb{E}(Y))$

Def $S \subseteq M^2$ $\quad S$ subspace of M^2

$S^\perp$ = orthogonal subspace =

$= \{ X \in M^2$ such that X is orthogonal to every element in $S \}$

$= \{ X \in M^2$ such that $\forall Y \in S$ $\mathbb{E}(X \cdot Y) = 0 \}$



in $\mathbb{R}^2$ $\quad S$ = line passing through 0

$S^\perp$ = perpendicular line to S passing through 0.

Ex. $S = \{ X \in M^2, \mathbb{E}(X) = 0 \}$

S is a subspace of M^2

$X, Y \in S \qquad \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 0 + 0 = 0$

$0 \in S \qquad \mathbb{E}(\lambda \cdot X) = \lambda \underbrace{\mathbb{E}(X)} = \lambda \cdot 0 = 0$

$S^\perp = \{ Y \in M^2 \quad Y$ is orthogonal to <u>every</u> <u>X in S</u> $\}$

$Y \in S^\perp \iff \underline{\forall} X \in S \quad (\forall X \; \mathbb{E}(X) = 0)$

$\mathbb{E}(X \cdot Y) = 0$

If Y is constant $\quad Y(\omega) = c \quad \forall \omega$

$$\mathbb{E}(Y \cdot X) = \mathbb{E}(c \cdot X) = c\,\mathbb{E}(X) = 0$$

$$\forall X \quad \mathbb{E}(X) = 0$$

Constant random variables are in $S^\perp$

Assume $\left[Y \in S^\perp \right] \Leftrightarrow \left[\underbrace{\overset{\forall X \in S}{\cancel{\forall X \quad \mathbb{E}(X) = 0}}} \Rightarrow \mathbb{E}(Y X) = 0 \right]$

$$\underbrace{Y - \mathbb{E}(Y) \in S}$$

$$\mathbb{E}(Y - \mathbb{E}(Y)) = \mathbb{E}(Y) - \mathbb{E}(\mathbb{E}(Y)) =$$

$$= \mathbb{E}(Y) - \mathbb{E}(Y) = 0$$

$$\Downarrow$$

$$\mathbb{E}(Y \cdot \overbrace{(Y - \mathbb{E}(Y))}^{\in S}) = 0$$

$$0 = \mathbb{E}(Y \cdot (Y - \mathbb{E}(Y))) =$$

$$= \mathbb{E}(|Y|^2 - Y\,\mathbb{E}(Y)) =$$

$$= \mathbb{E}(|Y|^2) - \mathbb{E}(Y\,\underline{\mathbb{E}(Y)}) =$$

$$= \mathbb{E}(|Y|^2) - (\mathbb{E}(Y)) \cdot (\mathbb{E}(Y))$$

$$\mathbb{E}(|Y|^2) - (\mathbb{E}(Y))^2 = 0$$

$$\mathbb{E}((Y - \mathbb{E}(Y))^2) = \mathbb{E}(Y^2 - 2\underline{Y}\,\mathbb{E}(Y) + (\mathbb{E}(Y))^2) =$$

$$= \mathbb{E}(|Y|^2) - \underline{2(\mathbb{E}(Y))^2} + \underline{(\mathbb{E}(Y))^2} = \mathbb{E}(|Y|^2) - (\mathbb{E}(Y))^2 = 0$$

$|Y - \mathbb{E}(Y)| = 0$ with probability 1

$\Rightarrow Y = \mathbb{E}(Y)$ Y is constant

$S = \{ X \in M^2 \quad \mathbb{E}(X) = 0 \}$

$S^\perp = \{$ constant random variables $\}$

A Banach space with a scalar product is called HILBERT SPACE

(M^2 is a Hilbert space)