EMBEDDINGS FOR $W^{k,p}(\mathbb{R}^n)$

$W^{2,p}(\mathbb{R}^n)$

1) $p > n$. $\quad f \in W^{2,p}(\mathbb{R}^n) \xRightarrow{\text{(Morrey)}} f, \frac{\partial f}{\partial x_i}$ are bounded $f \in W^{1,\infty}(\mathbb{R}^n)$

$\left[f, \frac{\partial f}{\partial x_i} \in W^{1,p}(\mathbb{R}^n)\right]$ and $f, \frac{\partial f}{\partial x_i}$ are Hölder continuous

$\Downarrow$

$f \in C^{1, 1-\frac{n}{p}}(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$

2) $p = n$ $\quad f \in W^{2,n}(\mathbb{R}^n) \xRightarrow{\text{(GNS)}} f, \frac{\partial f}{\partial x_i} \in L^p(\mathbb{R}^n) \quad n \leq p < +\infty$

$\left(f, \frac{\partial f}{\partial x_i} \in W^{1,n}(\mathbb{R}^n)\right) \quad \Rightarrow f \in W^{1,p}(\mathbb{R}^n) \quad \forall \quad n \leq p < +\infty$

$\Rightarrow f \in L^{\infty}(\mathbb{R}^n) \cap C^{0,\alpha}(\mathbb{R}^n) \quad \forall \alpha \leq 1$

3) $p < n$ $\Rightarrow$ $f \in W^{2,p}(\mathbb{R}^n) \Rightarrow$ GNS $f, \frac{\partial f}{\partial x_i} \in L^{p^*}(\mathbb{R}^n)$

$\left(f, \frac{\partial f}{\partial x_i} \in W^{1,p}(\mathbb{R}^n)\right.$

$\Rightarrow f \in W^{1,p^*}(\mathbb{R}^n)$

(a) $p^* > n$ $\left(\Leftrightarrow \frac{np}{n-p} > n \Leftrightarrow p > \frac{n}{2}\right) \overset{(M)}{\Longrightarrow} f \in L^\infty(\mathbb{R}^n) \cap C^{0,1-\frac{n}{p^*}}$

$p > \frac{n}{2}$

$f \in L^\infty(\mathbb{R}^n) \cap C^{0,2-\frac{n}{p}}$

$0 < 2 - \frac{n}{p} < 1$!

(b) $p^* = n$ $\left(p = \frac{n}{2}\right)$ $f \in L^q(\mathbb{R}^n)$ $\forall q < +\infty$

$p = n/2$

(c) $p^* < n$ $f \in L^{(p^*)^*}(\mathbb{R}^n)$ $(p^*)^* = \frac{np^*}{n-p^*} = \frac{np}{n-2p}$

$p < \frac{n}{2}$

And so on for $W^{k,p}(\mathbb{R}^n)$ ....

what we may say for $U \subseteq \mathbb{R}^n$ open bdd?

(obs: $f \in L^p(U) \Rightarrow \tilde{f} = \begin{cases} f & U \\ 0 & \mathbb{R}^n \setminus U \end{cases}$ $\tilde{f} \in L^p(U)$)

$\boxed{\text{EXTENSION}}$

Theorem Let $U \subseteq \mathbb{R}^n$ open bounded and of class C^1 then $\forall$ V bdd open set such that $U \subset\subset V \subseteq \mathbb{R}^n$ $\forall$ $p \in [1, +\infty]$ there exists

$$E_V : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$$
$$f \longmapsto E_V(f)$$

such that
(1) $E_V(f)(x) = f(x)$ for a.e $x \in U$
(2) $\text{supp}\, E_V(f) \subseteq V$
(3) $\exists\, C(U,V) = C > 0$ $\|E_V(f)\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|f\|_{W^{1,p}(U)}$

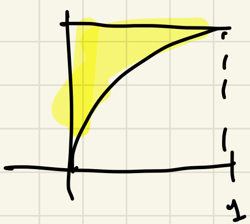
(E_V is continuous!)

WE ARE NOT PROVING IT

Some remarks

1) It is necessary to have U of class at least lipschitz to get this extension theorem

Example $U = \{ (x,y) \in \mathbb{R}^2 \quad x \in (0,1), \quad x^{\frac{1}{3}} < y < 1 \}$



U open, bdd but the boundary is Hölder!

Let γ to be fixed

$\frac{\partial f_\gamma}{\partial x} = 0$ a.e. (PROVE IT!)

$$f_\gamma(x,y) = y^{-\gamma} = \frac{1}{y^\gamma} \in W^{1,p}(U) \iff \int_0^1 \int_{x^{1/3}}^1 \frac{1}{y^{(\gamma+1)p}} \, dy \, dx < +\infty$$

$$\int_0^1 \int_{x^{1/3}}^1 y^{-(\gamma+1)p} \, dy \, dx = \int_0^1 \frac{1}{1-(\gamma+1)p} \left(1 - x^{\frac{1-(\gamma+1)p}{3}} \right) dx < +\infty$$

$$\iff 1 + \frac{1-(\gamma+1)p}{3} > 0$$

Take $p = 3 > 2 = \text{dimension}$ $\quad 4 - (\gamma+1)3 > 0 \iff \gamma < \frac{1}{3}$

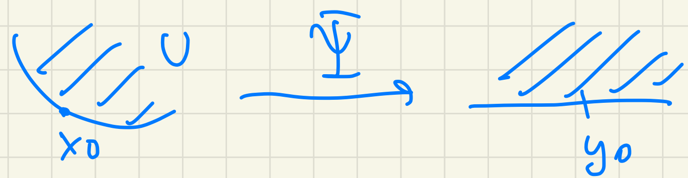
So $\gamma = \frac{1}{6}$ ok $f(x,y) = y^{-1/6} \in W^{1,3}(U)$

but it is not possible to have an extension of f to $W^{1,3}(\mathbb{R}^2)$ (since $W^{1,3}(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2)$ by Morrey in.)

ARGUMENT TO PROVE THE EXTENSION THM

"EXTENSION BY REFLECTION"

first of all we localize and we flatten the boundary



$f \in W^{1,p}(\mathbb{R}^n_+) \longrightarrow \tilde{f}(x', x_n) = \begin{cases} f(x', x_n) & x_n > 0 \\ f(x', -x_n) & x_n < 0 \end{cases}$

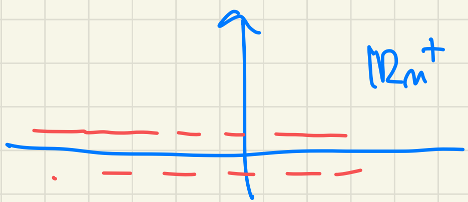
$\mathbb{R}^n_+ = \{(x_1 \dots x_n) \ \ x_n > 0\}$

$\|\tilde{f}\|_{L^p(\mathbb{R}^n)} \leq 2\|f\|_{L^p(\mathbb{R}^n_+)}$ easy and

$$\frac{\partial \tilde f}{\partial x_i} = \begin{cases} \dfrac{\partial f}{\partial x_i}(x', x_n) & x_n > 0 \\ \dfrac{\partial f}{\partial x_i}(x', -x_n) & x_n < 0 \end{cases}$$

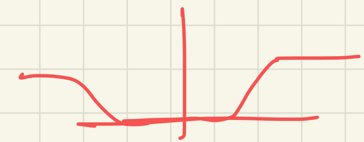
$$\frac{\partial \tilde f}{\partial x_n} = \begin{cases} \dfrac{\partial f}{\partial x_n}(x', x_n) & x_n > 0 \\ -\dfrac{\partial f}{\partial x_n}(x', -x_n) & x_n < 0 \end{cases}$$

one has to prove that it is true that $\frac{\partial \tilde f}{\partial x_i}$ is the weak derivative of $\tilde f$



$\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\psi(t) = \begin{cases} 0 & |t| < \frac{1}{2} \\ 1 & |t| > 1 \end{cases}$$



$\psi \in C^\infty(\mathbb{R})$

(then send $k \to +\infty$)

$$(x', x_n) \longmapsto [\varphi(x', x_n) + \varphi(x', -x_n)]\, \xi\left(\frac{x_n}{k}\right) \in C_c^\infty(\mathbb{R}^n_+)$$

$$(x', x_n) \longmapsto [\varphi(x', x_n) - \varphi(x', -x_n)]\, \xi\left(\frac{x_n}{k}\right) \in C_c^\infty(\mathbb{R}^n_+)$$

Obs U bdd of class C^1

$f \in W^{1,\infty}(U) \to f \in C(U)$ and Lipschitz in every connected component of U.

If f is Lipschitz continuous in $\overline{U}$, then it admits several Lipschitz extensions (which maintain the same Lipschitz constant)

$$\tilde{f}(y) = \sup_{x \in U} \left[f(x) - \| |Df| \|_{L^\infty(U)} |x-y| \right] = \max_{x \in \overline{U}} \left[f(x) - \| |Df| \|_{L^\infty} |x-y| \right]$$

$$\overline{f}(y) = \inf_{x \in U} \left[f(x) + \| |Df| \|_{L^\infty(U)} |x-y| \right] = \min_{x \in \overline{U}} \left[f(x) + \| |Df| \|_{L^\infty(U)} |x-y| \right]$$

maximal and minimal extension

<u>Obs</u> U of class $\mathcal{C}^2$ (bold)

$\forall V \supset\supset U \quad \exists E_V : W^{2,p}(U) \longrightarrow W^{2,p}(\mathbb{R}^n)$

1) $E_V(f) = f$ on U

2) $\|E_V(f)\|_{W^{2,p}} \leq C\|f\|_{W^{2,p}(U)}$

3) $supp(E_V(f)) \subseteq V$

in this case extension by reflection has to be refined.

Flatten out the bdry with a $\mathcal{C}^2$ diffeomorphism and then

$$\tilde{f}(x', x_n) = \begin{cases} f(x', x_n) & x_n > 0 \\ 4 \cdot f(x', -\frac{x_n}{2}) - 3f(x', -x_n) & x_n < 0 \end{cases}$$

$W_0^{1,p}(U)$ and $W^{1,p}(U)$.

TRACE THEOREM. Let $U \subseteq \mathbb{R}^n$ open bounded of class C^1

$1 \leq p < +\infty$ $\quad \exists\, T : W^{1,p}(U) \longrightarrow L^p(\partial U) = \{ g : \partial U \to \mathbb{R}$ measurable $\int_{\partial U} |g|^p \, d\mathcal{H}^{n-1} < +\infty \}$

$$f \longmapsto Tf$$

such that

1) $\|Tf\|_{L^p(\partial U)} \leq C \|f\|_{W^{1,p}(U)}$
2) if $f \in C^1(\bar{U})$ then $Tf = f$ for $x \in \partial U$
3) DIVERGENCE THEOREM $\forall \Phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, $\forall f \in W^{1,p}(U)$

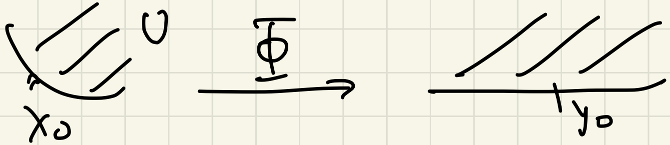
$$\int_U f \operatorname{div} \Phi + Df \cdot \Phi \, dx = \int_{\partial U} (Tf)\, \Phi \cdot \nu_x \, d\mathcal{H}^{n-1}(x).$$

Observe that $\begin{matrix} f_n \to f \text{ in } L^p(U) \\ Df_n \to Df \text{ in } L^p(U) \end{matrix} \Rightarrow Tf_n \to Tf$

4) $W_0^{1,p}(U) = \{ f \in W^{1,p}(U) \quad Tf = 0 \}$ So it is WEAKLY CLOSED in $W^{1,p}(U)$

($\subseteq$ simple by continuity $\quad \phi_n \in C^\infty_c(U)$ $\quad T\varphi_n = 0$ $\quad \phi_n \to f$ in $W^{1,p}(U)$ $\quad T\varphi_n \to Tf$)

(NO PROOF : idea localize and flatten out the boundary



$$\Phi(\partial U) \subseteq \{x_n = 0\} \quad \Phi(U) = \{x_n > 0\}$$

locally $\quad \mathbb{R}^n_+ = \{x_n > 0\}$

$$f \in W^{1,p}(\mathbb{R}^n_+) \xrightarrow{T} Tf = f(x',0) \in L^p(\mathbb{R}^{n-1})$$

$$f(x',0) = f(x',x_n) - \int_0^{x_n} \frac{d}{dt} f(x',t)\, dt \qquad x_n \in (0,1)$$

$$|f(x',0)|^p \le 2^{p-1} \left[|f(x',x_n)|^p + \left(\int_0^1 \left|\frac{\partial}{\partial t} f(x',t)\right| dt \right)^p \right] \le$$

$$\le_{\text{jensen}} \le 2^{p-1} \left[|f(x',x_n)|^p + \int_0^1 \left|\frac{\partial}{\partial t} f(x',t)\right|^p dt \right] \le$$

$$\le 2^{p-1} \left(|f(x',x_n)|^p + \int_0^1 |Df(x',t)|^p dt \right)$$

INTEGRATE w.r.t. dx', dx_n.