Corollary of Morrey (RADEMACHER THEOREM)

Let $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ $p > n$ (p also $+\infty$).
Then f is differentiable a.e and the a.e. derivative coincides with the weak derivative.

IN PARTICULAR: EVERY locally Lipschitz function is ALMOST EVERYWHERE DIFFERENTIABLE (Rademacher).

Def f is differentiable a.e if for a.e $x \in \mathbb{R}^n$ $\exists\, \underset{\overset{\shortparallel}{Df(x)}}{a} \in \mathbb{R}^n$ s. that
$$\lim_{y \to x} \frac{f(y) - f(x) - a\cdot(y-x)}{|y-x|} = 0$$

<u>Obs</u> $f \in W^{1,\infty}_{loc}(\mathbb{R}^n) \Rightarrow f \in W^{1,p}_{loc}(\mathbb{R}^n)$ $\forall p$

<u>proof</u> Let $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ $p < +\infty$ ($p = +\infty$ is obtained)

$\frac{\partial f}{\partial x_i} \in L^p_{loc}(\mathbb{R}^n)$. We choose a representative of them.

$$v(y) := f(y) - f(x) - \sum_i \frac{\partial f}{\partial x_i}(x)\,(y-x) = \underbrace{f(y)-f(x)}_{\in W^{1,p}_{loc}(\mathbb{R}^n)} - \underbrace{Df(x)\cdot(y-x)}_{\in W^{1,p}_{loc}(\mathbb{R}^n)}$$

$v \in W^{1,p}_{loc}(\mathbb{R}^n)$

We aim to prove that $\lim_{y\to x} \frac{|v(y)-v(x)|}{|y-x|} = 0$ for a.e. x

(this implies $Df(x) = a$, f differentiable in x)
the weak gradient coincides with the differential

Apply (H) to v $\quad |y-x| \leq r$

$$|v(y)-v(x)| \leq c(n,p)\, 2^{n+1} \|Dv\|_{L^p(B(x,2r))}\, r^{-\frac{n}{p}} \cdot |x-y| =$$

$$= \tilde{C}\, 4^{n+1} \underbrace{\left[\frac{1}{\omega_n 2^n r^n} \int_{B(x,2r)} |Df(y)-Df(x)|^p \, dy\right]^{1/p}} |x-y|$$

$$0 \leq \frac{|v(y)-v(x)|}{|y-x|} \leq \tilde{C} \underbrace{\left[\frac{1}{\omega_n (2r)^n} \int_{B(x,2r)} |Df(y)-Df(x)|^p \, dy\right]^{1/p}}_{\to 0 \text{ a.e. } x \text{ for Lebesgue theorem.}}$$

Thm $f \in W^{1,\infty}_{loc}(\mathbb{R}^n) \Longleftrightarrow f$ is locally Lipschitz in $\mathbb{R}^n$.

$\forall\, U \subset\subset \mathbb{R}^n \ \exists\, C_U$ s.that
$|f(x)-f(y)| \le C_U |x-y|$

proof $\xrightarrow{}$ $f \in W^{1,\infty}_{loc}(\mathbb{R}^n) \longrightarrow f \in W^{1,p}_{loc}(\mathbb{R}^n) \quad p < +\infty \implies f \in \mathcal{C}(\mathbb{R}^n)$
and $|f(x)-f(y)| \le C_U |x-y| \quad \forall\, x,y \in U$.
(by Morrey inequality, $p \to +\infty$).

$\longleftarrow$ f locally Lipschitz

$\forall i \quad \forall x \quad v_i^h(x) = \dfrac{f(x+he_i)-f(x)}{h} \qquad h \in \mathbb{R}$

$v_i^h \in L^\infty_{loc}(\mathbb{R}^n) \quad \forall i, \forall h$

Indeed $\quad |v_i^h(x)| = \left|\dfrac{f(x+e_i h)-f(x)}{h}\right| \leq C_U \qquad x \in U \subset\subset \mathbb{R}^n$

$x+the_i \in U$

(h sufficiently small).

$\phi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \phi \cdot v_i^h(x)\,dx = \int_{\mathbb{R}^n} \frac{\phi(x-e_i h)-\phi(x)}{h} f(x)\,dx$$

$$\xrightarrow[h\to 0^+]{} \int_{\mathbb{R}^n} -\frac{\partial \phi}{\partial x_i}(x) f(x)\,dx$$

v_i^h are in $L^\infty_{loc}(\mathbb{R}^n) \rightarrow v_i^{h_k} \to \bar{v}_i$ weak* in $L^\infty_{loc}(\mathbb{R}^n)$

$\forall U \subset\subset \mathbb{R}^n \ \forall g \in L^1(U) \quad \int_U v_i^{h_k} g\,dx \to \int_U \bar{v}_i g\,dx$

$\forall \phi \in C_c^\infty(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} \phi\, v_i^{h_k}(x)\,dx \to \int -\frac{\partial \phi}{\partial x_i} f\,dx$

$\Rightarrow v_i = \dfrac{\partial f}{\partial x_i}$ in the weak sense

$\Rightarrow f \in W^{1,\infty}_{loc}(\mathbb{R}^n)$

Theorem $f \in W^{1,\infty}(\mathbb{R}^n) \iff f$ is lipschitz continuous and bounded in $\mathbb{R}^n$

$$\| \underbrace{|Df|}_{\text{weak derivative} = \text{a.e. derivative}} \|_\infty = \text{lipschitz constant of } f := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|} = C$$

PROOF. OF THE EQUALITY

$$|f(x)-f(y)| \le \int_0^1 |Df(x+t(y-x))\cdot(y-x)|\, dy \le \| |Df| \|_\infty \cdot |y-x|$$

$$C = \text{lipschitz constant of } f \le \| |Df| \|_\infty$$

Assume $\exists\, \varepsilon > 0$ $\quad C + \varepsilon \le \| |Df| \|_\infty$ $\quad$ Let $x \in \mathbb{R}^n$ $\quad |Df(x)| \ge C + \frac{\varepsilon}{2}$

$$\frac{|f(y)-f(x)-Df(x)\cdot(y-x)|}{|y-x|} \le \frac{\varepsilon}{4} \quad \text{for } |y-x| \text{ sufficiently small}$$

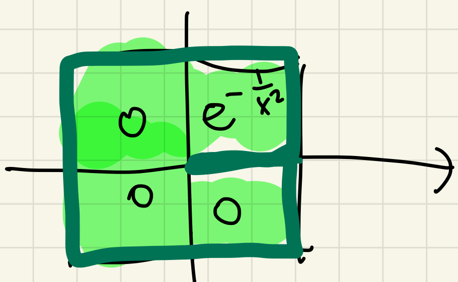
$$-\frac{\varepsilon}{4} \le \frac{f(y)-f(x)}{|y-x|} - \frac{Df(x)\cdot(y-x)}{|y-x|} \le \frac{|f(y)-f(x)|}{|y-x|} - |Df(x)| \le \frac{|f(x)-f(y)|}{|y-x|} - C - \frac{\varepsilon}{2}$$

$\Rightarrow \dfrac{|f(x)-f(y)|}{|x-y|} \geq C + \dfrac{\varepsilon}{2}$ so C cannot be the lipschitz constant of f.

CAREFUL: In general $W^{1,\infty}(U) \neq C^{0,1}(U)$ $\quad U \subseteq \mathbb{R}^n$

$\boxed{\text{The only implication is: } f \in W^{1,\infty}(U) \text{ then } f \text{ is locally lipschitz continuous in } U.}$

$$U = (-1,1)\times(-1,1) \setminus \left([0,1)\times\{0\}\right)$$



$$f(x,y) = \begin{cases} e^{-\frac{1}{x^2}} & y>0,\ x>0 \\ 0 & y \leq 0 \\ & y \in (-1,1)\ x \leq 0 \end{cases}$$

$$\sup_{\substack{(x_1,y_1)\neq(x_2,y_2) \\ (x_i,y_i)\in U}} \frac{|f(x_1,y_1)-f(x_2,y_2)|}{|(x_1,y_1)-(x_2,y_2)|} \geq \left(\begin{array}{l} \text{take} \\ (x_1,y_1)=(x,y) \\ (x_2,y_2)=(x,-y) \end{array}\right) = \sup_{\substack{x\in(0,1) \\ y\in(0,1)}} \frac{e^{-1/x^2}}{2y} = +\infty$$

$U \subseteq \mathbb{R}^n$ open set

In general what can be said is the following

$$f \in W^{1,\infty}(U) \Longleftrightarrow f \in \mathcal{C}(U) \cap L^{\infty}(U)$$

and

$$\sup_{\substack{x,y \in U \\ x \neq y}} \frac{|f(x)-f(y)|}{d_U(x,y)} \leq C$$

where $d_U(x,y) = \inf\limits_{\substack{\gamma \in \mathrm{lip}([0,1],U) \\ \gamma(0)=x \ \ \gamma(1)=y}} \mathcal{H}^1(\gamma)$ (geodesic distance in U)

d_U is equivalent to euclidean distance if U is $\mathcal{C}^1$.