Prodotto scalare usuale in $V = \mathbb{R}^n$

Definizione:

Siano $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ il prodotto scalare tra v e w

$$v \cdot w = \sum_{i=1}^{n} x_i y_i \in \mathbb{R} \qquad (x_1 \dots x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i$$

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto v \cdot w$$

Esempio: $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $v \cdot w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} = 1 \cdot (-3) + 2 \cdot 1 =$
$= -3 + 2 = -1$

$$w \cdot v = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1$$

$$v \cdot w = v^t w = (1 \ \ 2)\begin{pmatrix} -3 \\ 1 \end{pmatrix} = 1 \cdot (-3) + 2 \cdot 1 = -1$$

Proprietà:

1) Simmetria $\quad v \cdot w = w \cdot v \quad \forall v, w \in \mathbb{R}^n$

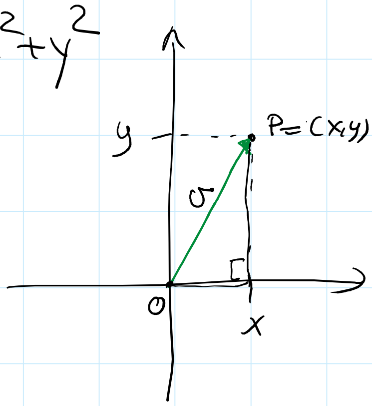
2) Bilinearità: $\quad (a_1 v_1 + a_2 v_2) \cdot w = a_1 v_1 \cdot w + a_2 v_2 \cdot w \quad \forall v_1, v_2, v, w, w_1, w_2 \in \mathbb{R}^n$
$\quad v \cdot (a_1 w_1 + a_2 w_2) = a_1 v \cdot w_1 + a_2 v \cdot w_2 \quad \forall a_1, a_2 \in \mathbb{R}$

3) Positività $\quad v \cdot v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i^2 \geq 0 \quad \forall v \in \mathbb{R}^n$

4) Non degenere $\quad v \cdot v = 0$? $\quad \sum_{i=1}^{n} x_i^2 = 0 \iff x_i = 0 \ \forall i = 1, \dots, n$
$\quad v \cdot v = 0 \iff v = \vec{0}$

Norma = lunghezza di un vettore.

$n=2$ $\quad \mathbb{R}^2 \quad v=\begin{pmatrix} x \\ y \end{pmatrix} \quad v\cdot v = x^2+y^2$



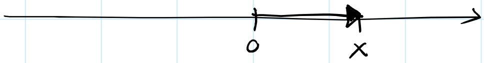
$v = P - O$

tv

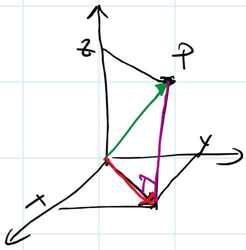
$$\|v\| = \sqrt{x^2+y^2} = \sqrt{v\cdot v}$$

Norma: $v\in\mathbb{R}^n \qquad \|v\| = \sqrt{v\cdot v}$

$n=1 \qquad v=(x) \qquad \|v\| = \sqrt{x^2} = |x|$



$n=2 \qquad v=\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \|v\| = \sqrt{x^2+y^2}$



$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$n=3 \qquad v=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \|v\| = \sqrt{x^2+y^2+z^2}$

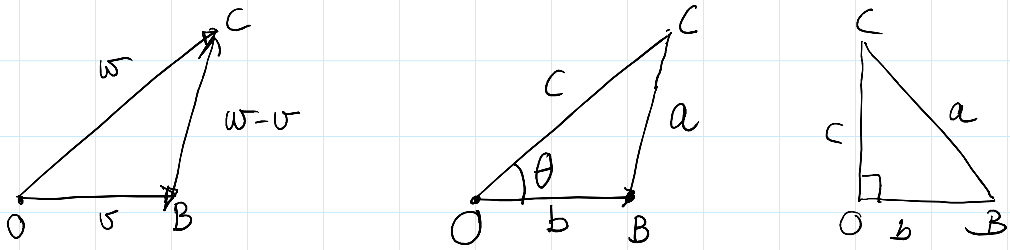
Proprietà:

1) $\|av\| = \sqrt{(av)\cdot(av)} = \sqrt{a^2\, v\cdot v} = |a|\sqrt{v\cdot v} = |a|\,\|v\|$

$\|2v\| = 2\|v\| \qquad \|-2v\| = 2\|v\|$

2) $\|v\| = 0 \iff v = \vec{0}$

$w - v + v = w = C - O$



$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

$c = \|w\|$ $\qquad \|v\| = \sqrt{v \cdot v}$

$b = \|v\|$ $\qquad \|v\|^2 = v \cdot v$

$a = \|w - v\|$

$$a^2 = \|w - v\|^2 = (w - v) \cdot (w - v) = w \cdot w - v \cdot w - \underbrace{w \cdot v}_{v \cdot w} + v \cdot v =$$

$$= \|w\|^2 - 2 v \cdot w + \|v\|^2 = c^2 + b^2 - 2 v \cdot w =$$

$$= c^2 + b^2 - 2bc\cos\theta$$

$$v \cdot w = bc\cos\theta = \|v\| \, \|w\| \cos\theta \qquad \boxed{v \cdot w = \|v\| \, \|w\| \cos\theta}$$

Se $v, w \in \mathbb{R}^n \setminus \{\vec{0}\}$ $\qquad \boxed{\cos\theta = \frac{v \cdot w}{\|v\| \, \|w\|}}$

Dimostriamo $\left| \frac{v \cdot w}{\|v\| \, \|w\|} \right| \leq 1 \qquad |v \cdot w| \leq \|v\| \, \|w\|$

Domanda 13:

Disuguaglianza di Cauchy-Schwartz $\qquad$ CS

Dati $v, w \in \mathbb{R}^n$

$$|v \cdot w| \leq \|v\| \, \|w\|$$

e vale l'uguaglianza se e solo se i vettori v e w sono linearmente dipendenti.

Dimostrazione:

1° caso $\{\vec{0}, w\}$ $\quad \{v, \vec{0}\}$ sono lin. dipendenti

se $v = \vec{0}$ $\quad \{\vec{0}, w\}$

$$|\vec{0} \cdot w| = 0 \overset{?}{=} \|\vec{0}\| \, \|w\| = 0 \cdot \|w\|$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \qquad 0 = 0$$

Se $w = \vec{0}$ $\quad \{v, \vec{0}\}$ $\qquad |v \cdot \vec{0}| = 0 = \|v\| \cdot \|\vec{0}\| = \|v\| \cdot 0$

2° caso $\quad v, w \in \mathbb{R}^n \setminus \{\vec{0}\}$ $\qquad w = x v$ con $x \in \mathbb{R}$

Condizione di lineare dipendenza

$$w - xv = \vec{0}$$

$$\|w - xv\|^2 = (w - xv)\cdot(w - xv) = \|w\|^2 + x^2\|v\|^2 - 2x\, v\cdot w =$$

$$= \boxed{\|v\|^2 x^2 - 2(v\cdot w)\, x + \|w\|^2 \geq 0 \qquad \forall\, v, w \in \mathbb{R}^n \setminus \{\vec{0}\}\ \boxed{\forall\, x \in \mathbb{R}}}$$

$$\Longrightarrow \quad \Delta \leq 0 \qquad 4(v\cdot w)^2 - 4\|v\|^2\|w\|^2 \leq 0$$

$$\frac{\Delta}{4} \leq 0 \qquad (v\cdot w)^2 - \|v\|^2\|w\|^2 \leq 0$$

$$(v\cdot w)^2 \leq \|v\|^2\|w\|^2$$

$$\boxed{|v\cdot w| \leq \|v\|\,\|w\|}$$

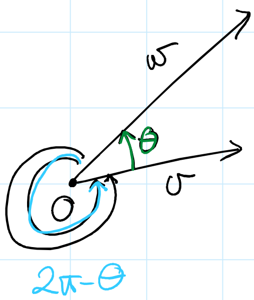
$\{v, w\}$ lin dip.

$\Updownarrow$

$\|w - xv\| = 0$ per un opportuno $x \in \mathbb{R} \iff \Delta = 0 \iff$ $|v\cdot w| = \|v\|\,\|w\|$.

Osservazione:

$\mathbb{R}^3$



$$\cos\theta = \frac{v\cdot w}{\|v\|\,\|w\|}$$

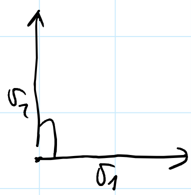
$$\cos(2\pi - \theta) = \cos\theta$$

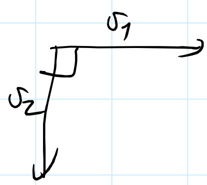
$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = v_2$$

$$v_1 \cdot v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 - 1 + 0 = 0 \qquad \cos\theta = \frac{v_1\cdot v_2}{\|v_1\|\,\|v_2\|} = \frac{0}{\sqrt{2}\cdot\sqrt{2}} = 0$$

$$\|v_1\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\|v_2\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$





Definizione:

Dati $v, w \in \mathbb{R}^n$, $v \, e \, w$ si dicono **ortogonali** se $v \cdot w = 0$ $\boxed{v \perp w}$

"v è ortogonale a w"

Definizione:

un vettore $v \in \mathbb{R}^n$ si dice **versore** se $\|v\| = 1$

Definizione:

una base $\mathcal{B} = \{v_1, \dots, v_n\}$ di $\mathbb{R}^n$ si dice **ortogonale** se $v_i \perp v_j \quad \forall \, i \neq j$ (cioè $v_i \cdot v_j = 0$)

Una base $\mathcal{C} = \{u_1, \dots, u_n\}$ di $\mathbb{R}^n$ si dice **ortonormale** se $u_i \perp u_j \quad \forall \, i \neq j$ e $\|u_i\| = 1 \quad \forall \, i = 1, \dots, n$.

Cioè

$$u_i \cdot u_j = \begin{cases} 0 & \text{se } i \neq j \\ u_i \cdot u_i = 1 & \text{se } i = j \end{cases} \quad \text{perché } u_i \cdot u_i = \|u_i\|^2 = 1^2 = 1.$$

Esempi:

1) $\mathcal{E} = \{e_1, \dots, e_n\}$ è base ortonormale

$$\underset{i \neq j}{e_i \cdot e_j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = 0 + \dots + 0 = 0 \qquad e_i \perp e_j \text{ se } i \neq j.$$

$$\text{se } i = j \quad e_i \cdot e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 + \dots + 0 + 1 + 0 \dots + 0 = 1 \qquad e_i \cdot e_i = \|e_i\|^2 = 1$$

2) $u_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$

$$\|u_1\|^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1 \quad \Rightarrow \|u_1\| = 1$$

$$\|u_2\|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + 0^2 = \frac{1}{2} + \frac{1}{2} = 1 \qquad \Rightarrow \|u_2\| = 1$$

$$u_1 \cdot u_2 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}\cdot\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{6}}\left(1\cdot 1 + 1\cdot(-1) + 1\,0\right) =$$
$$= \frac{1}{\sqrt{6}}(1-1) = 0$$

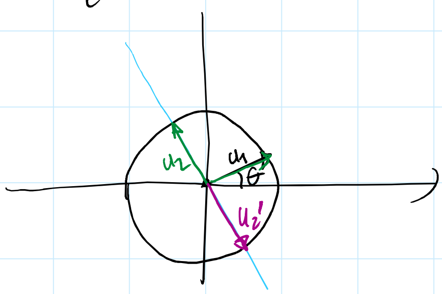
$$\|u_3\|^2 = \left\| \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} \right\|^2 = \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = \frac{6}{6} = 1$$

$$u_1 \cdot u_3 = \underbrace{\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{u_1} \cdot \underbrace{\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}_{u_3} = \frac{1}{\sqrt{18}} (1+1-2) = 0$$

$$u_2 \cdot u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{12}} \left(1 + (-1)\cdot 1 + 0\cdot(-2)\right) = 0$$

Esercizio: determinare tutte le basi ortonormali di $\mathbb{R}^2$.

$$B = \{u_1, u_2\} \qquad \|u_1\| = 1, \quad \|u_2\| = 1 \qquad u_1 \perp u_2$$



$$u_1 = \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix} \qquad u_2 = \begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix}$$

$$u_2' = \begin{pmatrix} \operatorname{sen}\theta \\ -\cos\theta \end{pmatrix}$$

Base ortonormale di $\mathbb{R}^2$ equiorientata con $\mathcal{E}$

$$B = \left\{ \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}, \begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix} \right\}$$

$$T_B^{\mathcal{E}} = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix}$$

$$\det T_B^{\mathcal{E}} = \cos^2\theta + \operatorname{sen}^2\theta = 1$$

Base ortonormali di $\mathbb{R}^2$ con orientamento opposto ad $\mathcal{E}$

$$B' = \left\{ \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}, \begin{pmatrix} \operatorname{sen}\theta \\ -\cos\theta \end{pmatrix} \right\}$$

$$T_{B'}^{\mathcal{E}} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ \operatorname{sen}\theta & -\cos\theta \end{pmatrix}$$

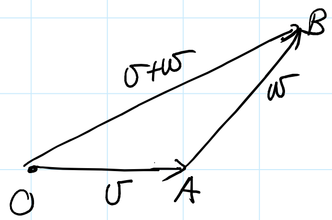
$$\det T_{B'}^{\mathcal{E}} = -\cos^2\theta - \operatorname{sen}^2\theta = -1$$

Domanda 14

Disuguaglianza triangolare:

$\forall\, v, w \in \mathbb{R}^n$

$$\|v+w\| \le \|v\| + \|w\|$$



Dimostrazione:

Essendo tutte le norme numeri reali positivi o nulli

$$\|v+w\| \le \|v\| + \|w\| \quad \text{se e solo se}$$

$$\|v+w\|^2 \le (\|v\| + \|w\|)^2 = \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$$

$$\|v+w\|^2 = (v+w)\cdot(v+w) = v\cdot v + 2v\cdot w + w\cdot w =$$

$$= \|v\|^2 + \|w\|^2 + 2\underbrace{v\cdot w}_{x} \le \|v\|^2 + \|w\|^2 + 2|v\cdot w| \underset{C\text{-}S}{\le}$$

$$\le \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| = (\|v\| + \|w\|)^2$$

$\boxed{x \in \mathbb{R} \qquad x \le |x|}$ $\qquad$ $\underset{C\text{-}S}{|v\cdot w| \le \|v\|\,\|w\|}$ $\qquad \square$

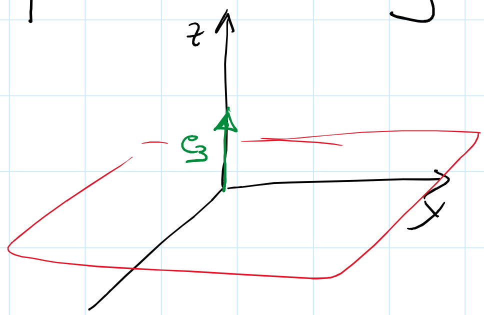
Definizione: dato $S \subseteq \mathbb{R}^n$ si dice *ortogonale di S*

$$S^\perp = \{ v \in \mathbb{R}^n \mid v \perp s \ \ \forall s \in S\} =$$

$$= \{ v \in \mathbb{R}^n \mid v\cdot s = 0 \ \ \forall s \in S\}$$

Esempi:

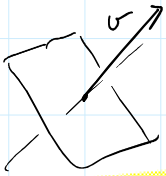
1) $S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



$\angle\ x$

$S^{\perp} = \left\{ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = 0 \right\}$

2) $S = \left\{ \begin{pmatrix} -18 \\ 31 \\ 5 \end{pmatrix} \right\}$ $\qquad S^{\perp} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -18 \\ 31 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$



$-18x + 31y + 5z = 0$

$\mathbb{R}^3$

1) $S = \left\langle \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\rangle$ $\quad \dim S = 1$	$S^{\perp}: \; 1 \cdot x + 5y + 2z = 0 \quad \dim S^{\perp} = 2$
2) $S: \; x + 5y + 2z = 0$	$S^{\perp} = \left\langle \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\rangle$
$\mathbb{R}^4 \; S: \; x_1 - x_2 + 3x_3 - x_4 = 0 \quad 3$	$S^{\perp} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 3 \\ -1 \end{pmatrix} \right\rangle \quad 1$
$S = \left\langle \begin{pmatrix} 1 \\ 5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\rangle$	$S^{\perp}: \begin{cases} x_1 + 5x_2 + x_4 = 0 \\ x_2 + 3x_3 = 0 \end{cases}$

Proprietà

Domanda 15:

Dato $V \leq \mathbb{R}^n$ allora $V^{\perp} \leq \mathbb{R}^n$.

Dimostrazione: $\qquad V^{\perp} = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \underline{x} \cdot v = 0 \quad \forall v \in V \right\}$

⓪ $\vec{0} \in V^{\perp}$

$$\vec{0} \cdot v = 0 \;\Rightarrow\; \vec{0} \in V^{\perp}$$
$$\forall v \in V$$

① $V^\perp$ è chiuso per la somma. $\underline{x}_1, \underline{x}_2 \in V^\perp$ cioè $\underline{x}_i \cdot v = 0$ $i=1,2$ $v \in V$

$$(\underline{x}_1 + \underline{x}_2) \cdot v = \underline{x}_1 \cdot v + \underline{x}_2 \cdot v = 0 + 0 = 0 \quad \forall v \in V; \text{ cioè}$$
$\underline{x}_1 + \underline{x}_2 \in V^\perp$

② $V^\perp$ è chiuso per prodotto per scalari $\underline{x} \in V^\perp$ cioè $\underline{x} \cdot v = 0$ $\forall v \in V$
$a \in \mathbb{R}$

$$(a\underline{x}) \cdot v = a(\underline{x} \cdot v) = a \cdot 0 = 0 \quad \forall v \in V \text{ quindi}$$
$a\underline{x} \in V^\perp$.

□.

Pr:

① $(\mathbb{R}^n)^\perp = \{\vec{0}\}$ $\{\vec{0}\}^\perp = \mathbb{R}^n$

$\left\{\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right\}^\perp$ $0=0$ equazione cart. di $\mathbb{R}^n$.

② $V_1 \geq V_2$ $V_1^\perp \leq V_2^\perp$