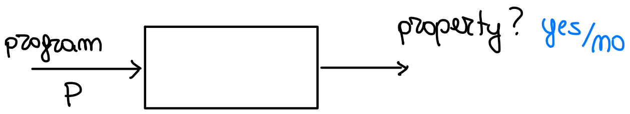
COMPUTABILITY (25/11/2025)

RICE'S THEOREM



> every property of programs which concerns to I/O behaviour is <u>undecidable</u>

"P is terminating on every input"
"P has some fixed $n \in \mathbb{N}$ in output"
"P has dead code"
"P computes function f"
⋮

undecidable

"the length of P is ≤ 10"
"the program P is not using jumps"

decidable

What is a behavioural property of programs?

$$A \subseteq \mathbb{N}$$

set of programs
(property of programs

$$T = \{ n \mid P_n = \gamma^{-1}(n) \text{ is terminating on every input} \}$$
$$= \{ n \mid \varphi_n \text{ is total} \}$$

$$ONE = \{ n \mid P_n \text{ is a correct implementation of } \mathbb{1} \}$$
$$= \{ n \mid \varphi_n = \mathbb{1} \}$$

$A \subseteq \mathbb{N}$ (program property) is <u>behavioural</u> property if for all programs $n \in \mathbb{N}$ the fact that $n \in A$ or $n \notin A$ only depends on φ_n

<u>Def</u> (saturated / extensional set): $A \subseteq \mathbb{N}$ is saturated (extensional)

if $\forall m, n$

if $m \in A$ and $\varphi_m = \varphi_n$ then $n \in A$

$\Updownarrow$

$$A = \{ n \mid \varphi_n \text{ satisfies a property} \}$$

$$= \{ n \mid \varphi_n \in \mathcal{A} \}$$

for suitable $\mathcal{A} \subseteq \mathcal{F}$

$\uparrow$ property of functions

<u>Examples</u>

* $T = \{ n \mid P_n \text{ is terminating on every input} \}$

$= \{ n \mid \varphi_n \text{ is total} \}$

$= \{ n \mid \varphi_n \in \mathcal{T} \}$ $\qquad \mathcal{T} = \{ f \mid f \text{ is total} \}$

* $ONE = \{ n \mid \varphi_n \text{ is a correct implementation of } \mathbb{1} \}$

$= \{ n \mid \underbrace{\varphi_n = \mathbb{1}}_{\varphi_n \in \{\mathbb{1}\}} \}$

* $LEN10 = \{ n \mid P_n = \gamma^{-1}(n) \text{ has } \leq 10 \text{ lines} \}$

$m \in LEN10$

and $\varphi_m = \varphi_n$

$n \notin LEN10$

e.g. $m = \gamma(Z(1))$ $\qquad \varphi_m(x) = 0 \quad \forall x$

$\boxed{m \in LEN10}$

$\boxed{\varphi_m = \varphi_n}$

$$n = \gamma \left(\begin{array}{l} Z(1) \\ S(2) \\ S(2) \\ \vdots \\ S(2) \end{array} \right\} 10 \text{ times} \Bigg) \qquad \varphi_n(x) = 0 \quad \forall x$$

$\boxed{n \notin LEN10}$

* NODEAD = { n | P_n has no dead code }

$n = \gamma(Z(1))$

$n \in$ NODEAD

$\varphi_n = \varphi_m = 0$

$$m = \gamma\left(\begin{array}{l} Z(1) \\ J(1,1,END) \\ S(1) \leftarrow \text{never executed} \\ END: \end{array}\right)$$

* $K = \{ n \mid \varphi_n(n)\downarrow \}$

$= \{ n \mid \varphi_n \in \mathcal{K} \}$

where $\mathcal{K} = \{ f \mid f(?)\downarrow \}$ apparently not extensional

to conclude K not saturated we need $m, n \in \mathbb{N}$

$m \in K$ $\quad \varphi_m(m)\downarrow$

$\qquad\qquad\qquad \shortparallel$ $\quad$ and $\quad \underline{\varphi_m = \varphi_n}$

$n \notin K$ $\quad \varphi_n(n)\uparrow$

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assume that we are able to show that there is a program $m \in \mathbb{N}$ s.t.

$$\varphi_m(x) = \begin{cases} 1 & \text{if } x = m \\ \uparrow & \text{otherwise} \end{cases}$$

(*)

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we have

① $\boxed{m \in K}$ $\quad \varphi_m(m) = 1\downarrow$

② for a computable functions there are infinitely many programs

hence there is $n \neq m$ s.t.

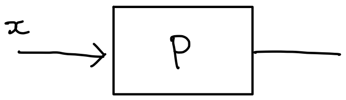
$\boxed{\varphi_m = \varphi_n}$

③ $\boxed{n \notin K}$ $\quad \varphi_n(n) = \varphi_m(n)\uparrow$

$\varphi_n = \varphi_m$ $\qquad m \neq n$

K not saturated

What about assumption (*) ?



if $x = P$ then 1

if $x \neq P$ then $\uparrow$

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def P(x):
    if x = " def P(x):
               if x = -
                  \
                   \   "
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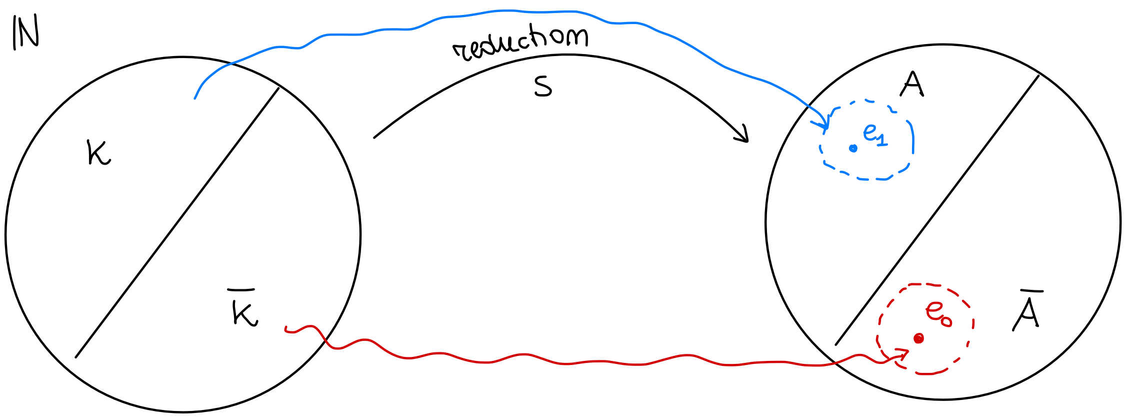
RICE'S THEOREM

Let $A \subseteq \mathbb{N}$ if A is <u>saturated</u> and $A \neq \emptyset$, $A \neq \mathbb{N}$

then A is not <u>recursive</u>

<u>proof</u>

we prove that $K \leq_m A$ (since K is not recursive we deduce that A is not recursion)



Let $e_0 \in \mathbb{N}$ be s.t. $\varphi_{e_0}(x)\uparrow \quad \forall x$ (a program for function which is always undefined)

① Assume $e_0 \notin A$

take $e_1 \in A$ (it exists because $A \neq \emptyset$)

define

$$g(x,y) = \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \varphi_{e_0}(y) & \text{if } x \notin K \end{cases}$$

$$= \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \varphi_{e_1}(y) \cdot 1\!\!1(\varphi_x(x))$$

$$= \varphi_{e_1}(y) \cdot 1\!\!1(\psi_U(x,x)) \qquad \text{computable}$$

By smn there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ s.t. $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \varphi_{e_0}(y) & \text{if } x \notin K \end{cases}$$

I claim that s is the reduction function for $K \leq_m A$

* if $x \in K$ then $s(x) \in A$

Assume $x \in K$. Then $\forall y$ $\varphi_{s(x)}(y) = g(x,y) = \varphi_{e_1}(y)$.

Hence $\varphi_{s(x)} = \varphi_{e_1}$. Since $e_1 \in A$ and A saturated we deduce $s(x) \in A$

* if $x \notin K$ then $s(x) \notin A$

Assume $x \notin K$. Then $\forall y$ $\varphi_{s(x)}(y) = g(x,y) = \varphi_{e_0}(y)$.

Hence $\varphi_{s(x)} = \varphi_{e_0}$. Since $e_0 \notin A$ and A saturated we deduce $s(x) \notin A$

So indeed $K \leq_m A$ and since K not recursive, we have A not recursive.

(2) Assume $e_0 \in A$

if we let $B = \bar{A}$ then

$$\left.\begin{array}{l} e_0 \notin B \\ B \neq \emptyset \quad (\text{since } A \neq \mathbb{N}) \\ B \neq \mathbb{N} \quad (\;\;'' \;\; A \neq \emptyset) \\ B \text{ saturated } (\text{since } A \text{ is}) \end{array}\right\} \text{by (1)} \quad \underset{\bar{A}}{\overset{B}{\shortparallel}} \text{ is not recursive}$$

Since $B = \bar{A}$ not recursive we conclude A not recursive. $\square$

* Output problem

$B_n = \{x \mid n \in E_x\}$ ↖ programs which has n in output

→ B_n is saturated, in fact

$B_n = \{x \mid \varphi_x \in \mathcal{B}_n\}$

where $\mathcal{B}_n = \{f \mid n \in cod(f)\}$

→ $B_n \neq \emptyset$

e.g. let $e_1 \in \mathbb{N}$ s.t. $\varphi_{e_1}(x) = x$ then $n \in E_{e_1} = \mathbb{N}$, so $e_1 \in B_n$

→ $B_n \neq \mathbb{N}$

e.g. let $e_0 \in \mathbb{N}$ s.t. $\varphi_{e_0}(x)\uparrow \ \forall x$ then $n \notin E_{e_0} = \emptyset$, so $e_0 \in B_n$

Hence, by Rice's theorem, B_n is not recursive.

EXERCISE:

$O = \{x \mid P_x$ has infinitely many outputs$\}$

$= \{x \mid E_x$ is infinite$\}$

* O saturated since $O = \{x \mid \varphi_x \in \mathcal{O}\}$

where $\mathcal{O} = \{f \mid cod(f)$ is infinite$\}$

* $O \neq \emptyset$

if e_1 is an index for the identity (as above) $E_{e_1} = \mathbb{N}$ infinite

hence $e_1 \notin O$

* $O \neq \mathbb{N}$

if e_0 an index of the function always undefined (as above)

$E_{e_0} = \emptyset$ not infinite hence $e_0 \notin O$

Hence by Rice's theorem, O is not recursive.

EXERCISE

$A = \{ x \mid x \in W_x \cap E_x \}$

is A saturated?

$A = \{ x \mid \varphi_x \in \mathcal{A} \}$

$\mathcal{A} = \{ f \mid ? \in dom(f) \cap cod(f) \}$

don't know what to put here

probably not saturated, I will not use Rice's theorem, instead I use reduction

$K \leq_m A$

i.e. I need a function $s : \mathbb{N} \to \mathbb{N}$ total and computable s.t.

$\forall x$

$x \in K$ iff $s(x) \in A$

$\Updownarrow$

$s(x) \in W_{s(x)} \leadsto \varphi_{s(x)}(s(x))\downarrow$

and

$s(x) \in E_{s(x)} \leadsto \exists y \ \varphi_{s(x)}(y) = s(x)$

not necessarily $s(x)$

we define

$$g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$

$$= y \cdot \mathbb{1}(\varphi_x(x)) = y \cdot \mathbb{1}(\psi_U(x,x)) \quad \text{computable}$$

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

s is the reduction function for $K \leq_m A$

→ if $x \in K$ then $\varphi_{s(x)}(y) = y \quad \forall y \quad \Rightarrow s(x) \in \underbrace{W_{s(x)}}_{= \mathbb{N}} \cap \underbrace{E_{s(x)}}_{= \mathbb{N}} = \mathbb{N}$

and thus $s(x) \in A$.

→ if $x \notin K$ then $\varphi_{s(x)}(y) \uparrow \quad \forall y \quad \Rightarrow s(x) \notin \underbrace{W_{s(x)}}_{= \emptyset} \cap \underbrace{E_{s(x)}}_{= \emptyset} = \emptyset$

and thus $s(x) \notin A$

Hence $K \leq_m A$ and therefore A not recursive $\square$