MORREY INEQUALITY $n < p < +\infty$

Let $n < p < +\infty$. Then $\exists\, C(n,p)$ such that $\forall f \in W^{1,p}(\mathbb{R}^n)$

(M1) $\|f\|_\infty \le C(n,p)\, \|f\|_{W^{1,p}}$ $\quad \left(W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \right)$

(M2) $|f(x) - f(y)| \le C\, |x-y|^{1-\frac{n}{p}} \cdot \|\, |Df|\, \|_{L^p}$ for a.e. x, y

(So $f \in W^{1,p}(\mathbb{R}^n) \Rightarrow f$ is in $L^\infty(\mathbb{R}^n)$ and – up to a representative – $f \in C(\mathbb{R})$ and Hölder of exponent $1 - \frac{n}{p}$).

NB the contrary is not true! bounded continuous functions which are Hölder in general have not weak derivative.

proof. We start with $\phi \in C^\infty_c(\mathbb{R}^n)$ since $W^{1,p}(\mathbb{R}^n) = \overline{C^\infty_c(\mathbb{R}^n)}^{\|\cdot\|_{1,p}}$.

1) $\forall \phi \in C^\infty_c(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$

$$(*) \quad \int_{B(x,r)} |\varphi(x)-\varphi(y)|\,dy \le \frac{r^n}{n} \underbrace{\int_{B(x,r)} \frac{|D\phi(y)|}{|x-y|^{n-1}}\,dy}$$

observe that $\frac{1}{|x-y|^{n-1}} \in L^{p'}(B(x,r))$ $\quad p' = \frac{p}{p-1}$

$\iff (n-1)\frac{p}{p-1} < n \iff p > n$

$\frac{|D\phi(y)|}{|x-y|^{n-1}} \in L^1(B(x,r))$

just integration on spheres formula

$$\int_{B(x,r)} |\varphi(x)-\varphi(y)|\,dy = \int_0^r \int_{\partial B(0,1)} s^{n-1} \underbrace{|\varphi(x)-\varphi(x+sw)|}_{\left|\int_0^s D\phi(x+tw)\cdot w\,dt\right| \overset{C.S.}{\le} \int_0^s |D\phi(x+tw)|\,dt} \,d\mathcal{H}^{n-1}(w)\,ds =$$

$$\leq \int_0^r s^{n-1} \int_{\partial B(0,1)} \int_0^s |D\phi(x+tw)| \, dt \, d\mathcal{H}^{n-1}(\omega) \, ds =$$ (EXCHANGE THE ORDER of INTEGRALS)

$$= \int_0^r s^{n-1} \int_0^s \int_{\partial B(0,1)} |D\phi(x+tw)| \, d\mathcal{H}^{n-1}(\omega) \, dt = \int_0^r \underbrace{\int_0^s \int_{\partial B(x,t)} |D\phi(z)| \frac{1}{t^{n-1}} d\mathcal{H}^{n-1}(z) \, dt}_{ds}$$

$$= \int_0^r s^{n-1} \int_{B(x,s)} \frac{|D\phi(z)|}{|x-z|^{n-1}} dz \leq \int_0^r s^{n-1} \int_{B(x,r)} \frac{|D\phi(z)|}{|x-z|^{n-1}} dz \, ds =$$

$$= \frac{r^n}{n} \int_{B(x,r)} \frac{|D\phi(z)|}{|x-z|^{n-1}} dz \qquad \square .$$

by Hölder from (1)

$$\int_{B(x,r)} |\varphi(y) - \phi(x)| dy \leq \frac{r^n}{n} \cdot \|D\phi\|_{L^p(B(x,r))} \cdot \left\| \frac{1}{|x-\cdot|^{n-1}} \right\|_{L^{p'}(B(x,r))}$$

$$\left(\int_{B(x,r)} \left(\frac{1}{|x-y|^{n-1}} \right)^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} = \left(\int_0^r \left(\frac{1}{s^{n-1}} \right)^{\frac{p}{p-1}} s^{n-1} \omega_n \right)^{\frac{p-1}{p}} = \left(\frac{\omega_n}{(n-1)\left(1-\frac{p}{p-1}\right)+1} r^{(n-1)\left(1-\frac{p}{p-1}\right)+1} \right) = \dots$$

$$= \left(\omega_n \frac{p-1}{p-n} r^{\frac{p-n}{p-1}} \right)^{\frac{p-1}{p}}$$

(E)

$$\int_{B(x,r)} |\varphi(x)-\varphi(y)|\,dy \le \frac{r^n}{n} \cdot \| \, |D\phi| \, \|_{L^p(B(x,r))} \cdot \omega_n^{1-\frac{1}{p}} \left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} |x-y|^{1-\frac{n}{p}}$$

$\boxed{\text{by (E) } L^\infty \text{ bound.}}$

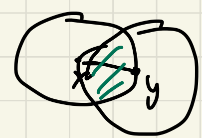
$$|\phi(x)| \le \left[\int_{B(x,r)} |\varphi(x)-\varphi(y)| + \int_{B(x,r)} |\varphi(y)| \right] \frac{1}{\omega_n r^n} \le$$

$$\le \text{by (E)} \le \frac{1}{n} \| \, |D\phi| \, \|_{L^p} \, \omega_n^{-\frac{1}{p}} \left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} r^{1-\frac{n}{p}} + \frac{1}{\omega_n r^n} \|\phi\|_{L^p} \left(\omega_n r^n\right)^{1-\frac{1}{p}}$$

taking $r=1$ $\quad |\varphi(x)| \le C(n,p) \, \| \, |D\phi| \, \|_{L^p(\mathbb{R}^n)} + C(n,p) \, \|\phi\|_{L^p(\mathbb{R}^n)}$

$$\le \bar{C}(n,p) \, \|\phi\|_{W^{1,p}(\mathbb{R}^n)} \qquad (M1)$$

$\boxed{\text{by (E) the Hölder estimate}}$



$x, y \quad |x-y| = r \qquad W = B(x,r) \cap B(y,r) \qquad |W| \ge \omega_n \left(\frac{r}{2}\right)^n \ge \frac{1}{2^n} |B(x,r)|$

$B\left(\frac{x+y}{2}, r/2\right) \subseteq W$

$$|\varphi(x)-\varphi(y)| \le \frac{1}{|W|}\int_W |\varphi(x)-\varphi(z)|\,dz + \frac{1}{|W|}\int_W |\varphi(y)-\varphi(z)|\,dz \le$$

$$\le \frac{2^n}{r^n\omega_n}\int_{B(x,r)} |\varphi(x)-\varphi(z)|\,dz + \frac{2^n}{r^n\omega_n}\int_{B(y,r)} |\varphi(y)-\varphi(z)|\,dz \le$$

(E) applied to x, y

$$\le \frac{2^n}{\cancel{r^n}\cancel{\omega_n}} \cdot 2 \cdot \frac{\cancel{r^n}}{n}\,\omega_n^{\cancel{1}-\frac{1}{p}}\,\| D\phi \|_{L^p(B(x,2r))}\; r^{1-\frac{n}{p}} \le$$

$$\le C(n,p)\,\|\,|D\phi|\,\|_{L^p(B(x,2r))}\;|x-y|^{1-\frac{n}{p}} \qquad |x-y| = r!$$

(M2)

Let $f \in W^{1,p}(\mathbb{R}^n)$ $\quad \phi_k \to f$ in $W^{1,p}(\mathbb{R}^n)$ $\Rightarrow \|\phi_k\|_{W^{1,p}} \le C$

by (M1), (M2) ϕ_k are equibounded in L^∞ and equi-Hölder continuous by A.A $\exists$ $\phi_{k_n} \to \tilde{f}$ locally uniformly in $\mathbb{R}^n$

$$|\tilde{f}(x) - \tilde{f}(y)| \leq C(n,p)\, |x-y|^{1-\frac{n}{p}}\, C$$

$\tilde{f} = f$ a.e. since $\phi_{k_n} \to f$ in L^p and then a.e.

f is (up to a representative) Hölder continuous of exponent $1-\frac{n}{p}$.

By (M1) if ϕ_k are Cauchy in $W^{1,p}$ then ϕ_k are Cauchy in L^∞ $\Rightarrow$ $\phi_k \to f$ in $L^\infty(\mathbb{R}^n)$ $\sup_{x \in \mathbb{R}^n} |\phi_k(x) - f(x)| \to 0$

So the convergence is <u>UNIFORM</u> in $\mathbb{R}^n$ and not up to subsequence $f \in L^\infty$ $\|f\|_\infty \leq C \|f\|_{W^{1,p}}$

$$|f(x) - f(y)| \leq C(n,p)\, \| |Df| \|_{W^{1,p}(\mathbb{R}^n)} |x-y|^{1-\frac{n}{p}}$$

Obs. by the theorem also flows:

1) $\forall p > n$ $f \in W^{1,p}(\mathbb{R}^n)$ the CONTINUOUS REPRESENTATIVE is the UNIFORM LIMIT of a sequence of $C_c^\infty(\mathbb{R}^n)$ functions

(so $\lim_{|x|\to+\infty} f(x) = 0$ $\forall f \in W^{1,p}(\mathbb{R}^n)$)

2) $\forall f \in W^{1,p}(\mathbb{R}^n)$ $|x-y| = r$

$$|f(x) - f(y)| \leq C(n,p) \, \| \, |Df| \, \|_{B(x,2r)} \, r^{1-\frac{n}{p}} \Rightarrow$$

$$\boxed{\frac{|f(x) - f(y)|}{|x-y|} \leq C(n,p) \frac{\| \, |Df| \, \|_{B(x,2r)}}{r^{n/p}} \quad \forall x,y \in \mathbb{R}^n, \ |x-y| = r.}$$

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(this is true also for $f \in W^{1,p}_{loc}(\mathbb{R}^n)$)

Therefore for $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ $\quad p > n$

$$\frac{|f(x)-f(y)|}{|x-y|} \leq \frac{2^{n+1}}{n\,\omega_n^{1/p}} \| |Df| \|_{L^p(B(x,2r))} \, r^{-\frac{n}{p}}$$

see next page

$f \in W^{1,\infty}_{loc}(\mathbb{R}^n) \Rightarrow f \in W^{1,p}_{loc}(\mathbb{R}^n) \quad \forall\, p < +\infty$ $\quad \| |Df| \|_{L^p(U)} \to \| |Df| \|_{L^\infty(U)}$ $\quad \forall\, U \subset\subset \mathbb{R}^n$

$$\boxed{\begin{gathered} f \in W^{1,\infty}_{loc}(\mathbb{R}^n) \to f \text{ is locally lipschitz continuous} \\ \frac{|f(x)-f(y)|}{|x-y|} \leq \frac{2^{n+1}}{n} \omega_n \| |Df| \|_{L^\infty(B(x,2r))} \end{gathered}}$$

Lemma Let $g \in L^p(U)$ $\forall p$ $\overline{p} \leq p \leq +\infty$ ($U \subseteq \mathbb{R}^n$)

then

$$\lim_{p \to +\infty} \|g\|_p = \|g\|_\infty .$$

proof ① $\|g\|_p = \left[\int_U |g(x)|^p \, dx\right]^{1/p} \leq \|g\|_\infty |U|^{1/p}$

$(|g(x)|^p \leq \|g\|_\infty^p)$

$$\limsup_{p \to +\infty} \|g\|_{L^p(U)} \leq \|g\|_{L^\infty(U)} \qquad (|U|^{1/p} \to 1)$$

② $\|g\|_p^p = \int_U |g(x)|^p \, dx \geq \int_{U \cap \{|g(x)| > c\}} |g(x)|^p dx \geq c^p |U \cap \{g(x) > c\}|$

$\|g\|_p \geq c \cdot |\{x \in U \ |g(x)| > c\}|^{1/p}$

Recall that

$$\|g\|_{L^\infty(U)} = \inf \{ c \in \mathbb{R},\ |\{x \in U \ |g(x)| > c\}| = 0 \}.$$

Let $c_n = \|g\|_{L^\infty(U)} - \frac{1}{n}$ $\qquad |\{x \in U \,|\, g(x) > c_n\}| > 0$!

$$\|g\|_{L^p(U)} \geq \left(\|g\|_{L^\infty(U)} - \frac{1}{n}\right) \underbrace{|\{x \in U \ |g(x)| > c_n\}|^{\frac{1}{p}}}_{\downarrow\, 1}$$

$$\liminf_{p \to +\infty} \|g\|_{L^p(U)} \geq \left(\|g\|_{L^\infty(U)} - \frac{1}{n}\right) \cdot 1 \qquad \forall n$$

$$\lim_{|p| \to +\infty} \|g\|_{L^p(U)} = \|g\|_{L^\infty(U)}.$$