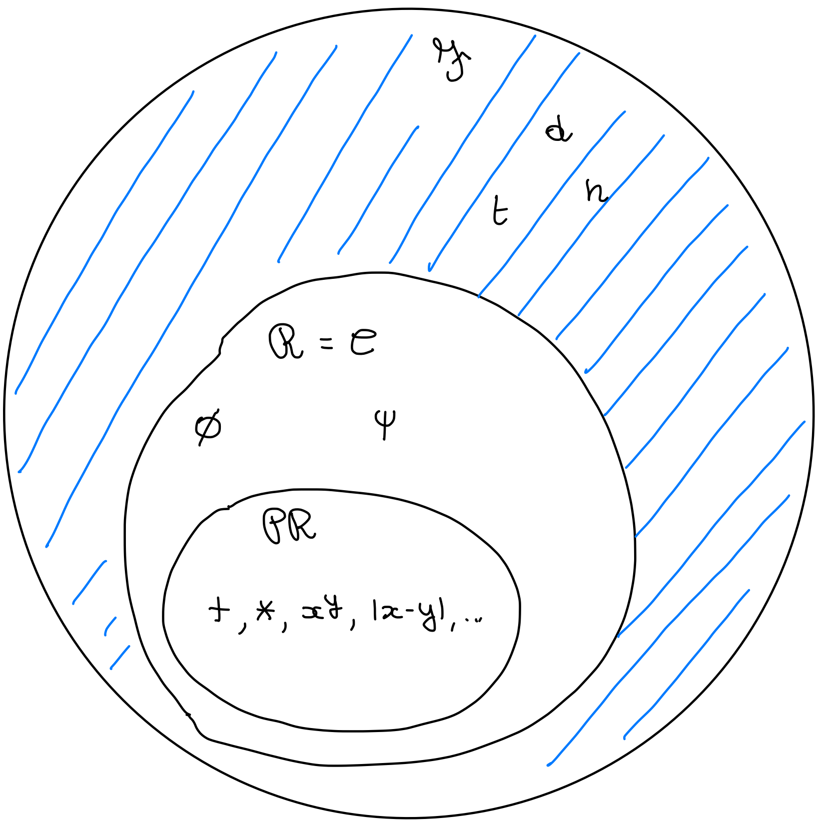
COMPUTABILITY (24/11/2025)

RECURSIVE and RECURSIVELY ENUMERABLE SETS



$$d(x) = \begin{cases} \varphi_x(x)+1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$h(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$t(x) = \begin{cases} 1 & \text{if } W_x = \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

given $X \subseteq \mathbb{N}$ " $x \in X$ " ?

set of programs

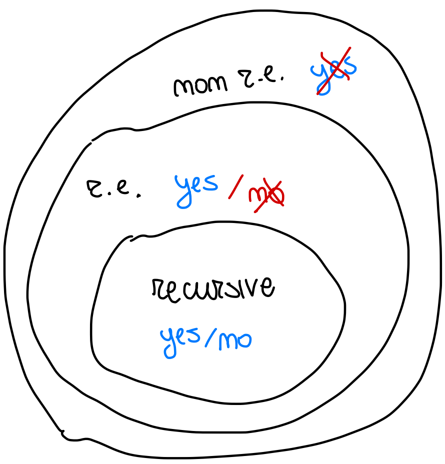
$X = \{ x \mid \varphi_x = fact \}$

$X = \{ x \mid P_x$ has linear complexity $\}$

$X = \{ x \mid P_x$ does not access a given register $\}$

$X = \{ x \mid P_x$ executes each instruction for some input $\}$

⋮



decidable properties / recursive sets

answer: yes/no

semi-decidable properties / recursively enumerable sets (r.e.)

answer yes , ~~no~~

× RECURSIVE Set

A set $A \subseteq \mathbb{N}$ is <u>recursive</u> if the characteristic function

$$\chi_A : \mathbb{N} \to \mathbb{N}$$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{is computable}$$

($\Leftrightarrow$ "$x \in A$" is decidable)

<u>Examples</u> :

$\mathbb{N}$ recursive $\quad \chi_{\mathbb{N}}(x) = 1 \quad \forall x \in \mathbb{N}$

$\emptyset$ " $\quad \chi_{\emptyset}(x) = 0 \quad \forall x \in \mathbb{N}$

$\mathbb{P}$ " $\quad \chi_{\mathbb{P}}(x) = \overline{sg}(rm(2,x))$

$\vdots$

<u>OBSERVATION</u> : All finite sets are recursive

<u>dim</u>

let $A = \{x_0, x_1, \ldots, x_n\}$

$$\chi_A(x) = \overline{sg}\Big(\underbrace{\prod_{i=0}^{m} |x - x_i|}_{\substack{0 \text{ if } x = x_i \text{ for some } i \\ \neq 0 \text{ otherwise}}}\Big)$$

$$\boxed{\begin{aligned} K &= \{x \in \mathbb{N} \mid \varphi_x(x)\downarrow\} \\ &= \{x \in \mathbb{N} \mid x \in W_x\} \end{aligned}}$$

NOT RECURSIVE

$$\chi_K(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases} \quad \text{not computable}$$

<u>OBSERVATION</u> : Let $A, B \subseteq \mathbb{N}$ sets, if A, B are recursive then

(i) $\overline{A} = \mathbb{N} \setminus A$

(ii) $A \cup B$ are recursive

(iii) $A \cap B$

<u>proof</u> (ii) $\chi_{A \cup B}(x) = \begin{cases} 1 & \text{if } x \in A \text{ or } x \in B \\ 0 & \text{otherwise} \end{cases}$

$$= sg(\chi_A(x) + \chi_B(x))$$

(same proof as for decidable predicates)

* <u>REDUCTION</u>

problems $\mathcal{A}$ and $\mathcal{B}$

$\mathcal{A}$ reduces to $\mathcal{B}$ every instance of $\mathcal{A}$ can be transformed into an instance of $\mathcal{B}$ in a simple way

equivalent

<u>Def</u> : Given $A, B \subseteq \mathbb{N}$

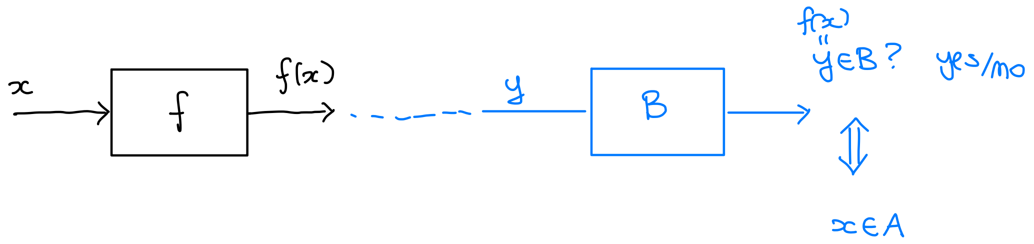
we say that the problem "$x \in A$" reduces to "$x \in B$"

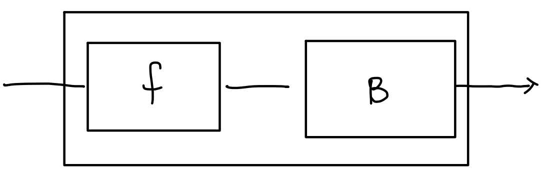
or, for short, A <u>reduces</u> to B, write $A \leq_m B$

if there is a <u>total</u> <u>computable</u> function $f : \mathbb{N} \to \mathbb{N}$ s.t.

$\forall x \in \mathbb{N}$

$$x \in A \quad \text{iff} \quad f(x) \in B$$





program for A

OBSERVATION : Let $A, B \subseteq \mathbb{N}$ $\quad A \leq_m B$

(i) if B is recursive then A is recursive

(ii) if A is not recursive then B is not recursive

proof

(i) let B be recursive i.e.

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases} \qquad \text{computable}$$

since $A \leq_m B$ there is $f : \mathbb{N} \to \mathbb{N}$ total computable s.t

$\forall x \qquad x \in A \quad \text{iff} \quad f(x) \in B$

Then

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \ \ (\Leftrightarrow f(x) \in B) \\ 0 & \text{otherwise} \end{cases} = \chi_B(f(x))$$

↑ computable by composition

(ii) counternominal

NOTE : Knowing $A \leq_m B$ and χ_A tells us very little about χ_B

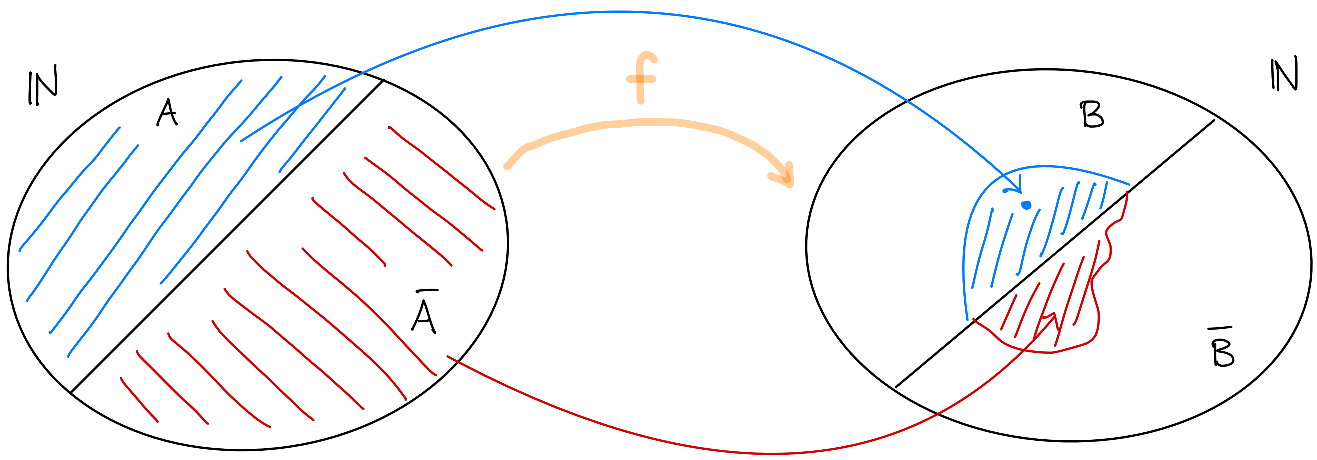
E.g., with $A = \mathbb{N}$, $B = K$

note that $A \leq_m B$ let $e \in \mathbb{N}$ s.t. $\varphi_e = id$, then

$f : \mathbb{N} \to \mathbb{N}$ $\quad f(x) = e \ \ \forall x$ is a reduction for $A \leq_m B$

in fact $e \in K$ $\ (\varphi_e(e) = e \downarrow)$, hence

$\forall x \qquad x \in A = \mathbb{N} \quad \text{iff} \quad f(x) \in B = K$



Example : $K = \{x \mid x \in W_x\} = \{x \mid \varphi_x(x)\downarrow\}$

$T = \{x \mid W_x = \mathbb{N}\} = \{x \mid \varphi_x(y)\downarrow \ \forall y\}$

$\boxed{K \leq_m T}$

assume that we have

$P_x \longrightarrow \boxed{\ T\ } \longrightarrow$ P_x is defined on all inputs ? yes/no

given P_x we can construct $P_{f(x)}$ s.t.

$P_x(x)\downarrow$ iff $P_{f(x)}$ is defined on all inputs

then we can combine T and f

$P_x \longrightarrow \boxed{\ f\ } \xrightarrow{P_{f(x)}} \boxed{\ T\ } \longrightarrow$ $P_{f(x)}$ is total? yes/no

$\Updownarrow$

$P_x(x)\downarrow$

How do we define f ?

$P_x \rightsquigarrow$

```
def P_f(x)(y):
    return P_x(x)
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if $P_x(x)\downarrow$ $\rightsquigarrow$ $P_{f(x)}(y)\downarrow \ \forall y$

if $P_x(x)\uparrow$ $\rightsquigarrow$ $P_{f(x)}(y)\uparrow \ \forall y$

Formally

$$g(x, y) = \varphi_x(x) = \psi_U(x, x) \qquad \text{computable}$$

By the smn theorem there is $f: \mathbb{N} \to \mathbb{N}$ total computable such that

$\forall x, y$

$$\varphi_{f(x)}(y) = g(x, y) = \varphi_x(x)$$

We claim that f is the reduction function for $K \leq_m T$

i.e.

$$\forall x \qquad x \in K \quad \text{iff} \quad f(x) \in T$$

* if $x \in K$ then $f(x) \in T$

assume $x \in K$, i.e. $\varphi_x(x)\downarrow$. Then $\forall y$ $\varphi_{f(x)}(y) = \varphi_x(x)\downarrow$

which means $\varphi_{f(x)} \in T$

* if $x \notin K$ then $f(x) \notin T$

assume that $x \notin K$ i.e. $\varphi_x(x)\uparrow$ then $\forall y$

$$\varphi_{f(x)}(y) = \varphi_x(x)\uparrow$$

which implies $\varphi_{f(x)} \notin T$

Therefore f is the reduction function for $K \leq_m T$

and since K is know to be not recursive, then T is not recursive.

<u>Example</u> (Input problem)

Let $n \in \mathbb{N}$ be fixed. Consider $A_n = \{x \mid \varphi_x(n)\downarrow\}$

$K \leq_m A_n$

given P_x $\rightsquigarrow$ $\boxed{\begin{array}{l} \text{def } P_{f(x)}(y): \\ \quad \text{return } P_x(x) \end{array}}$

if $P_x(x)\downarrow$ then $P_{f(x)}(y)\downarrow$ $\forall y$
in particular $P_{f(x)}(n)\downarrow$

if $P_x(x)\uparrow$ then $P_{f(x)}(y)\uparrow$ $\forall y$
in particular $\varphi_{f(x)}(n)\uparrow$

Define $g: \mathbb{N}^2 \to \mathbb{N}$

$g(x,y) = \varphi_x(x) = \Psi_U(x,x)$ computable

hence by smn theorem there is $f: \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x,y$ $\varphi_{f(x)}(y) = g(x,y) = \varphi_x(x)$

We show that f is a reduction function for $K \leq_m A_n$

* if $x \in K$ then $f(x) \in A_n$

if $x \in K$ then $\varphi_x(x)\downarrow$. Therefore $\forall y$ $\varphi_{f(x)}(y) = \varphi_x(x)\downarrow$

hence, in particular $\varphi_{f(x)}(n)\downarrow$ hence $f(x) \in A_n$

* if $x \notin K$ then $f(x) \notin A_n$

If $x \notin K$ then $\varphi_x(x)\uparrow$. Therefore $\forall y$ $\varphi_{f(x)}(y) = \varphi_x(x)\uparrow$

hence, in particular, $\varphi_{f(x)}(n)\uparrow$ hence $f(x) \in A_n$

$\Rightarrow$ $K \leq_m A_n$ and, since K is not recursive, we conclude A_n not recursive

* **EXERCISE**: show that $A_n \leq_m K$ (easy)

" " $T \not\leq_m K$ (less easy)

EXERCISE: $ONE = \{x \mid \varphi_x = 1\}$

$$P_x \longrightarrow \boxed{ONE} \longrightarrow$$ P_x correct implementation of constant 1

EXERCISE: (OUTPUT PROBLEM)

Let $n \in \mathbb{N}$ fixed. Consider $B_n = \{x \mid n \in E_x\}$

$$P_x \longrightarrow \boxed{B_n} \longrightarrow$$ P_x produces n as output

EXERCISE: URM with programs where only forward jumps are allowed

I_i : $J(m, n, t)$ $\quad t > i$

show that all functions are total, hence $\mathcal{C}' \subsetneq \mathcal{C}$

What if I can only jump backward

I_i : $J(m, n, t)$ $\quad t < i$?

[home]