μ Borel σ-finite measure on $\mathbb{R}$ $\qquad \mu = \mu_{ac} + \mu_s$

$\mu_{ac} \ll \mathcal{L}$

$\mu_s \perp \mathcal{L}$

$\mu_{ac} \longrightarrow f_\mu$ density $\quad f_\mu \geq 0$

$\forall B \in \mathcal{B} \qquad \mu_{ac}(B) = \int_B f_\mu(x)\,dx = \int_{B \setminus \underline{\underline{N}}} f_\mu(x)\,dx$

$\mu \longleftrightarrow F$ cumulative distribution funct.

$f_\mu(x) = F'(x)$

(since F is monotone $\rightarrow F$ is differentiable for almost every $x \in \mathbb{R}$ that is F is differentiable $\forall x \in \mathbb{R} \setminus N \quad |N| = 0$

$\underline{F}$ cumulative distribution function associated to μ

$\downarrow$

Jump set of $F = \{ a \in \mathbb{R} \quad F(a) \neq \overbrace{\lim_{y \to a^-} F(y)}^{F(a^-)} \} = J$

(theorem: $|J| = 0$ J is at most COUNTABLE

$$F_J(x) = \sum_{\substack{a \in J \\ a \leq x}} \left(F(a) - F(a^-) \right) = \sum_{\substack{a \in J \\ a \leq x}} \mu(\{a\})$$

PIECEWISE CONSTANT

$\boxed{F - F_J \quad \text{is continuous}}$ $F_J' = 0$ a.e.

$$f_\mu = (F - F_J)' = F' - \cancel{F_J'}$$

μ_s^J (the jump singular part of μ) is associated with the cumulative distribution function F_j

μ_{ac} has density given by $f_\mu = (F - F_j)'$
$= F'$

$$\underbrace{F_{ac}(x) = \int_{-\infty}^{x} f_\mu(y)\,dy}_{} \qquad (\mu \text{ is finite})$$

$= \mu_{ac}(-\infty, x]$

In general $F_{ac}(x) \leq F(x) - F_j(x)$

It is not true in general that

F continuous

$$F(x) = \int_{-\infty}^{x} F'(y)\, dy$$

$f_\mu = F'(x) \quad F_{ac}(x) = \int_{-\infty}^{x} f_\mu(y)\, dy \le F(x) - F_j(x)$

(there are continuous functions which are differentiable a.e. such that they are not the integral of their derivative).

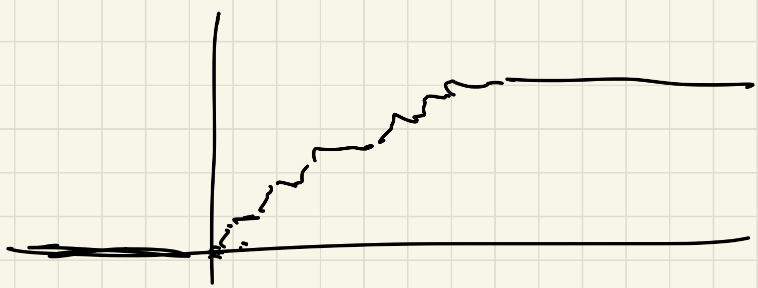
Devil staircase (Cantor function)

$F : \mathbb{R} \to \mathbb{R}$ $\quad F(x) = 0 \quad x \leq 0$

$F(x) = 1 \quad x \geq 1$

F monotone non decreasing, continuous (NO JUMPS)

$F'(x) = 0$ for a.e. x



Cumulative distribution function of a measure μ SINGULAR w.r.t. $\mathcal{L}$.

$\mathbb{R}^2$ σ-algebra of Borel sets $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$

$\shortparallel$

$\mathbb{R} \times \mathbb{R}$ $A \in \mathcal{B}(\mathbb{R}^2)$ $A = \underline{\underline{B \times C}}$

$B \in \mathcal{B}(\mathbb{R})$

$C \in \mathcal{B}(\mathbb{R})$

μ Borel measure on $\mathbb{R}^2$ $\mu : \underbrace{\mathcal{B}(\mathbb{R}^2)} \longrightarrow [0, +\infty]$

$(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu)$

$(\Omega, \underline{\mathcal{F}}, \mathbb{P})$ a probability space

$X : \Omega \to \mathbb{R}$

X, Y 2 random variables $Y : \Omega \to \mathbb{R}$ measurable

$$\boxed{(X,Y) : \Omega \longrightarrow \mathbb{R}^2 \qquad \omega \longmapsto \big(X(\omega), Y(\omega)\big)}$$

$$\boxed{(X,Y):\Omega \to \mathbb{R}^2 \text{ is measurable}}$$

$A \in \mathcal{B}(\mathbb{R}^2)$ $\quad A = B \times C$

$$(X,Y)^{-1}(B\times C) = \{\omega \mid X(\omega)\in B,\ Y(\omega)\in C\}$$

$$(X,Y)(\omega) \longrightarrow (X(\omega), Y(\omega)) \in B \times C$$

$$= \underbrace{\{\omega \mid X(\omega)\in B\}}_{\in \mathcal{F}} \cap \underbrace{\{\omega \mid Y(\omega)\in C\}}_{\in \mathcal{F}} \in \mathcal{F}$$

X is measurable $\qquad$ Y is meas.

$\forall (t,s)\in\mathbb{R}^2$ $\quad \{\omega \mid X(\omega)\le t,\ Y(\omega)\le s\} = \{\omega\ X(\omega)\le t\}\cap\{\omega\ Y(\omega)\le s\}$

joint law $\mathcal{L}_{(X,Y)} = (X,Y)\# \mathbb{P}$ is a Borel meas. on $\mathbb{R}^2$)

$$\mathcal{L}_{(X,Y)}(A \times B) = \mathbb{P}\{\omega,\ X(\omega)\in A,\ Y(\omega)\in B\}$$

$$\mathcal{L}_{(X,Y)}(\mathbb{R}\times\mathbb{R}) = \mathcal{L}_{(X,Y)}(\mathbb{R}^2) = 1$$

cumulative distribution function $(x,y)\in\mathbb{R}^2$

$$\underline{F_{(X,Y)}(x,y)} = \mathcal{L}_{(X,Y)}\big((-\infty,x]\times(-\infty,y]\big) =$$

$$= \mathbb{P}(\omega \mid X(\omega)\le x,\ Y(\omega)\le y)$$

$$\mathcal{L}_{(X,Y)}(A \times \mathbb{R}) = \mathbb{P}\{\omega \mid X(\omega) \in A, \underline{\underline{Y(\omega) \in \mathbb{R}}}\} =$$

$$= \mathbb{P}(\omega \mid X(\omega) \in A) = \mathcal{L}_X(A)$$

$$\mathcal{L}_{(X,Y)}(\mathbb{R} \times B) = \mathcal{L}_Y(B)$$

the first marginal of $\mathcal{L}_{(X,Y)}$ is $\mathcal{L}_X$

the second marginal of $\mathcal{L}_{(X,Y)}$ is $\mathcal{L}_Y$

$\mu : \mathcal{B}(\mathbb{R}^2) \longrightarrow [0, +\infty]$ measure

first marginal of μ $\qquad \mu(\cdot \times \mathbb{R}) : \mathcal{B}(\mathbb{R}) \longrightarrow [0, +\infty]$

second marginal of μ $\qquad \mu(\mathbb{R} \times \cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow [0, +\infty]$

$\mathcal{P}(\mathbb{R})$ = probability measure on $\mathbb{R}$ =

= $\{\mu$ Borel measures on $\mathbb{R}$ $\mu(\mathbb{R})=1\}$

= laws of all possible random variables taking values in $\mathbb{R}$.

$p \geq 1$

$\mathcal{P}_p(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$

$\mathcal{P}_p(\mathbb{R})$ = {probability measures on $\mathbb{R}$ with finite p-moment}

$= \{\mu$ Borel measure, $\mu(\mathbb{R})=1$ and $\int_{\mathbb{R}} |x|^p \, d\mu < +\infty\}$

$\mathcal{P}_1(\mathbb{R})$ = laws of all random variables X $\mathbb{E}(X) < +\infty$

$\mathcal{P}_2(\mathbb{R})$ = " " " " " X $E(\underline{X^2}) < +\infty$

BACK TO ABSTRACT DEFINITIONS

Def of METRIC SPACE

A a set : distance on A (metric on A)

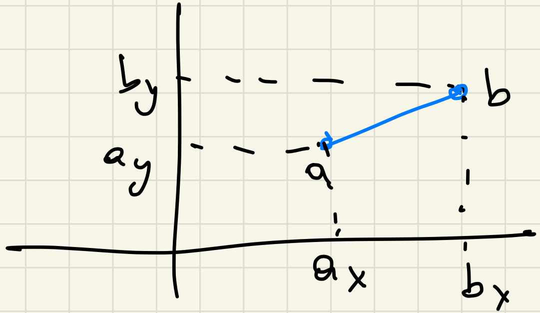
$$d : A \times A \longrightarrow [0, +\infty)$$

$$a, b \longrightarrow d(a,b) = \text{distance between } a, b.$$

d is a distance if it satisfies :

1) $d(a,a) = 0$
2) $d(a,b) = d(b,a)$
3) $d(a,b) \leq d(a,c) + d(c,b)$

$\mathbb{R}^2$

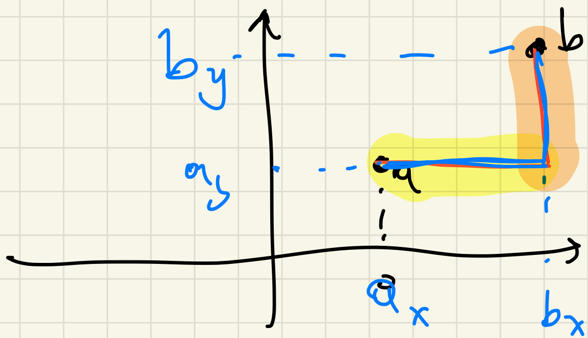


$$d_1(a,b) = |b-a|$$

length of the interval a, b

$$= \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}$$

$\mathbb{R}^2$



$$d_2(a,b) = |b_x - a_x| + |b_y - a_y|$$

$\underbrace{\mathcal{P}(\mathbb{R})}$ $\qquad \mu, \nu \in \mathcal{P}(\mathbb{R})$

$$\underbrace{d_{TV}}(\mu, \nu) = 2 \sup_{A \in \mathcal{B}(\mathbb{R})} |\mu(A) - \nu(A)|$$

$\downarrow$
total variation distance

$$d_{TV}(\mu, \nu) \leq d_{TV}(\mu, \rho) + d_{TV}(\rho, \nu)$$

$$\parallel$$

$$2 \sup_{A} |\mu(A) - \nu(A)| = 2 \sup_{A} \underbrace{|\mu(A) - \rho(A) + \rho(A) - \nu(A)|}$$

$$\leq 2 \sup_{A} |\mu(A) - \rho(A)| + |\rho(A) - \nu(A)| \leq$$

$$\leq 2 \sup_{B \in \mathcal{B}(\mathbb{R})} |\mu(B) - \rho(B)| + 2 \sup_{C \in \mathcal{B}(\mathbb{R})} |\rho(C) - \nu(C)|$$

(A, d) $\qquad d : A \times A \to [0, +\infty)$ distance

is a metric space

$\boxed{(A,d) \text{ is COMPLETE if every Cauchy sequence in } A \text{ has a limit in } A.}$

GIVEN A DISTANCE, I HAVE A NOTION OF CONVERGENCE

② $\underbrace{(a_n) \in A}_{a_1 \in A \;\; a_2 \in A \;\ldots}$ $\qquad a_n \to a$ in A $\quad$ if $\underbrace{\lim_{n \to +\infty} d(a_n, a) = 0}$

(ex $\mu_n \in \mathcal{P}(\mathbb{R})$ $\qquad \mu_n \xrightarrow{TV} \mu$ $\quad$ if

$$\lim_{n \to +\infty} \sup_{A \in \mathcal{B}(\mathbb{R})} |\mu_n(A) - \mu(A)| = 0 \Rightarrow \boxed{\begin{array}{c} \forall A \in \mathcal{B}(\mathbb{R}) \\ \mu_n(A) \to \mu(A) \end{array}}$$

Def of CAUCHY SEQUENCE

$(a_n)_{n\in\mathbb{N}}$ $\quad \forall n, a_n \in A$ $\quad a_n \in A$ is a Cauchy sequence if

$$\lim_{\substack{m\to+\infty \\ n\to+\infty}} d(a_n, a_m) = 0$$

Obs if $a_n \to a$ then $(a_n)_n$ is a Cauchy sequence

$$d(a_n, a_m) \le d(a_n, a) + d(a, a_m)$$

$$0 \le \lim_{\substack{n\to+\infty \\ m\to+\infty}} d(a_n, a_m) \le \lim_{n\to+\infty} d(a_n, a) + \lim_{m\to+\infty} d(a, a_m)$$

$$= 0$$

Ex $\quad \mathbb{Q}$ = rational numbers

$d(a,b) = |b-a|$

$(\mathbb{Q}, d)$ is <u>NOT</u> complete

$a_0 = 1 \quad a_1 = 1,4 \quad a_2 = 1,41 \quad a_3 = 1,414$

$a_4 = 1,4142 \quad a_5 = 1,41423 \ldots$

$\sqrt{2} = \ldots\ldots$

a_n = the first n-elements of the number $\sqrt{2}$

B be a linear space on $\mathbb{R}$ (vectorial space)

B is a set such that

$\underbrace{\forall\, a,b \in B \quad a+b \in B}$ $\quad (0 \in B \quad 0+a=a \;\} \forall a \in B)$

$\forall\, a \in B \quad \forall\, \lambda \in \mathbb{R} \quad \lambda \cdot a \in B$

A NORM on B is a function $B \to [0,+\infty)$

$b \mapsto \|b\|$

such that

1) $\|0\| = 0 \qquad \|b\| = 0 \iff b = 0$

2) $\forall\, \lambda \in \mathbb{R} \qquad \|\lambda b\| = |\lambda|\,\|b\| \quad \forall\, b \in B$

3) $\forall\, a,b \in B \qquad \|a+b\| \leq \|a\| + \|b\|$

a norm INDUCES a DISTANCE on B $\quad d(a,b) = \|b-a\|$

NOT all the distances comes from norms

Def
$(B, \|\cdot\|)$ is a Banach space if it is complete as a metric space

$(B, \|\cdot\|) \rightarrow$ is also a metric space

$d(a,b) = \|b-a\| = \|a-b\|$

$b_n \in B \qquad b_n \longrightarrow b \iff d(b_n, b) \rightarrow 0$

$\lim_{n \to +\infty} \|b_n - b\| = 0$

$(\mathbb{R}, |\cdot|)$ is a Banach space

$(\mathbb{R}^2, |\cdot|) \qquad |(x,y)| = \sqrt{x^2+y^2} \qquad d(a,b) = |b-a| =$

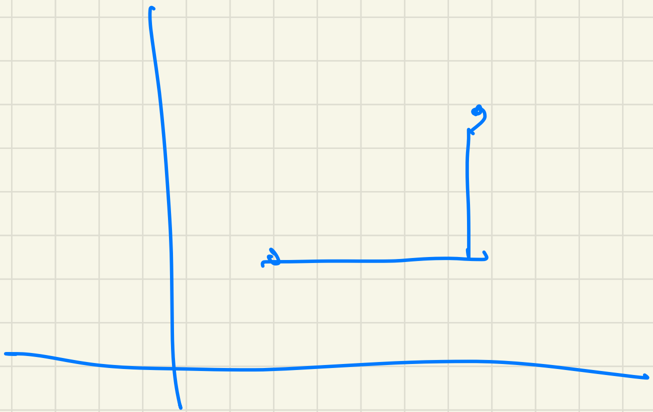
$= \sqrt{(b_x-a_x)^2+(b_y-a_y)^2}$

$$\mathbb{R}^2, \ \|\cdot\|_1$$

$$\|(x,y)\|_1 = |x| + |y| \qquad (x,y) \in \mathbb{R}^2$$

$$d_1(a,b) = \|(a_x,a_y) - (b_x,b_y)\|_1 =$$

$$= |a_x - b_x| + |a_y - b_y|$$



Ex $(\Omega, \mathcal{F}, \mathbb{P})$

$X : \Omega \to \mathbb{R}$ random variables

$\mathbb{E}(X) = \int_{\mathbb{R}} x \, d\mathcal{L}_X(x) < +\infty$ $\quad$ $\left(\mathcal{L}_X \in \mathcal{P}_1(\mathbb{R}) \right)$

$M^1 = \{ X : \Omega \to \mathbb{R}$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$
with $\mathbb{E}(|X|) < +\infty \}$ $\quad$ $\mathbb{E}(X) < +\infty \iff \mathbb{E}(|X|) < +\infty$

M^1 is a linear space $\quad X, Y \in M^1 \quad X + Y \in M^1$

$\lambda \in \mathbb{R} \quad \lambda X \in M^1 \quad \mathbb{E}(\lambda X) = \lambda \mathbb{E}(X) < +\infty$ $\quad$ $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

on $\mathcal{M}^1$ $\quad \|X\| = \mathbb{E}(|X|) = \int_{\mathbb{R}} \underbrace{|x|}\, \underbrace{d\mathcal{L}_X(x)}$

$\downarrow$

$\|\cdot\|$ is a norm

1) $\|X\| = 0 = \mathbb{E}(|X|) \Longleftrightarrow \quad X = 0$ $\quad X(\omega) = 0$ ~~fo~~ with probability 1

$X(\omega) = 0$ for $\omega \in \Omega'$

$\mathbb{P}(\Omega') = 1$

$\downarrow$

$X = 0$

2) $\|\lambda X\| = \mathbb{E}(|\lambda X|) = |\lambda|\, \mathbb{E}(|X|) = |\lambda|\, \|X\|$

3) $\|X+Y\| = \mathbb{E}(|X+Y|) \leq \mathbb{E}(|X| + |Y|) = \|X\| + \|Y\|$

$$d(X,Y) = \|X-Y\| = \mathbb{E}(|X-Y|)$$

is a distance on M^1

$X_n \longrightarrow X$ in this sense if

$\lim_{n\to+\infty} \mathbb{E}(|X_n - X|) = 0$ ($X_n \longrightarrow X$ in mean)

$(M^1, \|\cdot\|)$ is Banach.