(GNS) $\qquad p^* = \frac{p \cdot n}{n-p} \qquad f \in W^{1,p}(\mathbb{R}^n) \Rightarrow f \in L^{p^*}(\mathbb{R}^n)$

$$\|f\|_{p^*} \overset{(GNS)}{\leq} C(n,p) \, \| \, |Df| \, \|_{L^p}.$$

proof ① Observe that we may reduce to $\phi \in C_c^\infty(\mathbb{R}^n)$

Assume we prove GNS) for $\phi \in C_c^\infty(\mathbb{R}^n)$.

Then $f \in W^{1,p}(\mathbb{R}^n) \to \exists \, \phi_k \in C_c^\infty(\mathbb{R}^n) \quad \phi_k \to f$ in $W^{1,p}$

$$\|\phi_k - \phi_h\|_{p^*} \underset{(GNS)}{\leq} C(n,p) \, \| \underbrace{|D\phi_k - D\phi_h|}_{} \|_p$$

$D\phi_k$ is Cauchy in $L^p(\mathbb{R}^n)$

$\Rightarrow \phi_k$ is Cauchy in $L^{p^*}(\mathbb{R}^n) \Rightarrow \phi_k \to f$ in $L^{p^*}(\mathbb{R}^n)$ and we pass to the limit in (GNS).

② **Lemma** Let $u_1 \dots u_n \in C_c^\infty(\mathbb{R}^n)$ $\quad u_i$ does not depend on x_i, $u_i \geq 0$. Then $\int_{\mathbb{R}^n} u_1 \dots u_n \, dx \leq \prod_{i=1}^n \|u_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$

by induction on $n > 1$

$n = 2$ Fubini Tonelli $\int_{\mathbb{R}^2} u_1(x_2)\, u_2(x_1)\, dx_1 x_2 = \|u_1\|_{L^1(\mathbb{R})} \cdot \|u_2\|_{L^1(\mathbb{R})}$

$n > 2$

$$\int_{\mathbb{R}^{n+1}} u_1 \dots u_{n+1}\, dx = \int_{\mathbb{R}^n} u_{n+1} \left[\int_{\mathbb{R}} u_1 \dots u_n\, dx_{n+1}\right] \leq \|u_{n+1}\|_{L^n(\mathbb{R}^n)} \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} u_1 \dots u_n\, dx_{n+1}\right)^{\frac{n}{n-1}} dx\right]^{\frac{n-1}{n}}$$

Holder $p_1 = n$ $p_2 = \frac{n}{n-1}$

$$\int_{\mathbb{R}} u_1 \dots u_n \leq (\text{generalized Holder}) \leq \prod_{i=1}^{n} \left[\int_{\mathbb{R}} u_i^n\right]^{1/n}$$

$$\leq \|u_{n+1}\|_{L^n(\mathbb{R}^n)} \left[\int_{\mathbb{R}^n} \prod_{i=1}^{n} \left[\int_{\mathbb{R}} u_i^n\, dx_{n+1}\right]^{\frac{1}{n-1}}\right]^{\frac{n-1}{n}} \leq$$

use inductive assumption on

$$v_i = \left[\int_{\mathbb{R}} u_i^n\, dx_{n+1}\right]^{\frac{1}{n-1}}$$

$$\leq \|u_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^{n} \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}} u_i^n\right]^{\frac{1}{n-1} \frac{n-1}{n}} = \prod_{i=1}^{n+1} \|u_i\|_{L^n(\mathbb{R}^n)} . \quad \square .$$

③ start for $p=1$ $\quad 1^* = \dfrac{n}{n-1}$ $\quad \dfrac{1}{1^*} = 1 - \dfrac{1}{n}$

(1^* coincides with the Hölder conjugate).

$\phi \in C_c^\infty(\mathbb{R}^n)$ $\quad \phi(x) = \int_{-\infty}^{x_i} \frac{\partial \phi}{\partial x_i}(.x\ t\ .)\,dt$ $\quad \forall i \to |\phi(x)| \le \int_{-\infty}^{+\infty} \left|\frac{\partial \phi}{\partial x_i}\right| dt$

$$|\phi(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \underbrace{\left(\int_{-\infty}^{+\infty} \left|\frac{\partial \phi}{\partial x_i}\right| dt \right)^{1/n-1}}_{u_i}$$

(does not depend on x_i!

$$\|\phi\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} = \int_{\mathbb{R}^n} |\phi(x)|^{\frac{n}{n-1}} dx \le \int_{\mathbb{R}^n} \left[\prod_{i=1}^{n} \int_{-\infty}^{+\infty} \left|\frac{\partial \phi}{\partial x_i}\right| dt \right]^{\frac{1}{n-1}} dx \le \ \ldots \ \le$$

$$\le \prod_{i=1}^{n} \left[\int_{\mathbb{R}^{n-1}} \left[\int_{-\infty}^{+\infty} \left|\frac{\partial \phi}{\partial x_i}\right| dt \right] dx \right]^{\frac{1}{n-1}} = \prod_{i=1}^{n} \left[\int_{\mathbb{R}^n} \left|\frac{\partial \phi}{\partial x_i}\right| \right]^{\frac{1}{n-1}} \le$$

$$\le \| \, |D\phi| \, \|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}$$

$$\|\phi\|_{L^{n/n-1}} \le \| \, |D\phi| \, \|_{L^1}$$

(4) $p>1$ Take $\gamma>1$ $|\phi(x)|^\gamma \in C^1_c(\mathbb{R}^n)$

$$|D|\phi(x)|^\gamma| = \gamma |\phi(x)|^{\gamma-1} |D\phi(x)|$$

apply (GNS) to $|\phi(x)|^\gamma$ for $p=1$

$$\left[\int_{\mathbb{R}^n} |\phi(x)|^{\gamma \frac{n}{n-1}}\, dx\right]^{\frac{n-1}{n}} = \|\phi\|^\gamma_{\frac{n\gamma}{n-1}} \le \gamma \int_{\mathbb{R}^n} |\phi(x)|^{\gamma-1} \underbrace{|D\varphi(x)|}_{\in L^p} dx \le$$

$$\le (\text{Holder}) \le \gamma \cdot \||D\phi|\|_p \cdot \|\phi\|^{\gamma-1}_{\left(\frac{p}{p-1}\right)\cdot \gamma-1}$$

$$\frac{n\gamma}{n-1} = \frac{p}{p-1}(\gamma-1) \Rightarrow \left(\frac{p}{p-1} - \frac{n}{n-1}\right)\gamma = \frac{p}{p-1}$$

$$\boxed{\gamma = \frac{p(n-1)}{n-p}}$$

$$\frac{p}{p-1}\cdot(\gamma-1) = \frac{n\gamma}{n-1} = \frac{p}{n-p} = p^*.$$

$$\|\phi\|_{p^*} \le \frac{p(n-1)}{n-p} \||D\phi|\|_p . \qquad \square.$$

$\boxed{\text{Case } p = n}$ $\quad \|\phi\|^{\gamma}_{\frac{n\gamma}{n-1}} \le \gamma \cdot \| |D\phi| \|_{n} \, \|\phi\|^{\gamma-1}_{\left(\frac{n}{n-1}\right)(\gamma-1)}$

- choose $\gamma = n$ $\quad \|\phi\|^{n}_{n^2/n-1} \le n \underbrace{\| |D\phi| \|_n \, \|\phi\|^{n-1}_{n}} \le n\bar{C} \|\phi\|^{n}_{W^{1,n}}$

$W^{1,n}(\mathbb{R}^n) \hookrightarrow L^{n^2/n-1}(\mathbb{R}^n)$

- choose $\gamma = n+1$ $\quad \|\phi\|^{n+1}_{\frac{n(n+1)}{n-1}} \le (n+1) \| |D\phi| \|_n \, \|\phi\|^{n}_{n^2/n-1} \le (n+1)\tilde{C} \|\phi\|^{n+1}_{W^{1,n}}$

$W^{1,n}(\mathbb{R}^n) \hookrightarrow L^{n(n+1)/n-1}(\mathbb{R}^n)$

...

IN CONCLUSION:

- $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^{q}(\mathbb{R}^n)$ $\quad \forall q < +\infty$ continuously
- $W^{1,n}(\mathbb{R}^n) \not\hookrightarrow L^{\infty}(\mathbb{R}^n)$

ex $f(x) = \lg\left(\lg\left(1 + \frac{1}{|x|}\right)\right)\psi(x)$ $\quad \psi \in C^\infty_c(\mathbb{R}^n)$ $\quad \psi = \begin{cases} 0 & |x|>2 \\ 0 \le \psi \le 1 & \\ 1 & |x|<1 \end{cases}$

$f(x) \in W^{1,n}(\mathbb{R}^n)$
$f(x) \notin L^{\infty}(\mathbb{R}^n)$

COROLLARY 1) $U \subseteq \mathbb{R}^n$ $\quad W_0^{1,p}(U) \subseteq W_0^{1,p}(\mathbb{R}^n)$

since $W_0^{1,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{1,p}} \subseteq \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{1,p}}$.

$\forall f \in W_0^{1,p}(U) \qquad \|f\|_{p^*} \leq C(n,p) \, \| |Df| \|_{L^p}$

2) If U is bounded $\rightarrow f \in L^{p^*}(U) \Rightarrow f \in L^q(U) \;\; \forall q \leq p^*$ (by Hölder)

$q < p^*$ $\quad \|f\|_{L^q} \overset{\text{Hölder}}{\leq} \|f\|_{L^{p^*}} \cdot |U|^{\frac{p^*-q}{p^* q}}$ $\qquad \int |f|^q \underset{L^{p^*/q}}{\overset{\Uparrow}{\leq}} \|f^q\|_{p^*/q} \cdot |U|^{1-\frac{q}{p^*}}$

$= \|f\|_{p^*}^q \cdot |U|^{\frac{p^*-q}{p^*}}$.

$\Downarrow$

$\|f\|_q \leq C(n,p) \, \| |Df| \|_p \cdot |U|^{\frac{p^*-q}{p^* q}} \qquad \forall q < p^*$

3) For $p = q$ $\qquad \|f\|_p \leq C(n,p) \, \| |Df| \|_p \, |U|^{\frac{p^*-p}{p^* p}}$ (usually called POINCARÉ TYPE INEQUALITY)

in $W_0^{1,p}(U)$ $\quad \|f\|_{1,p}$ is equivalent to $\| |Df| \|_{L^p}$.

REMARK

Case $p=2$ Poincaré type inequality and first eigenvalue of the Laplacian (with Dirichlet bdry condition)

$$p=2 \qquad \int_U |f|^2\,dx \le \tilde{C} \int_U |Df|^2\,dx \qquad \forall f \in W_0^{1,2}(U)$$

$\rightarrow$ RAYLEIGHT QUOTIENT

by density

$$\inf_{\substack{f\in W_0^{1,2}(U)\\ f\neq 0}} \boxed{\frac{\int_U |Df|^2 dx}{\int_U |f|^2 dx}} = \frac{1}{C_U} \quad \left(\text{the minimal constant}\right) = \inf_{\substack{\phi\in C_c^\infty(U)\\ \phi\neq 0}} \frac{\int_U |D\phi|^2 dx}{\int_U |\phi|^2 dx}$$

~ . ~

if U is bdd f exists.

① If $\exists f \in W_0^{1,2}(U)$ which realizes the infimum, then f solves in the sense of distributions

$$-\Delta f = \frac{1}{C_U} f \quad \text{in } U$$

Assume $\exists f \in W_0^{1,2}(U)$ such that

$$M(\lambda)\quad I(f) = \frac{\int_U |Df|^2}{\int_U |f|^2} = \frac{1}{\tilde{c}} = \min_{g \in W_0^{1,2}(U)} \frac{\int_U |\nabla g|^2}{\int_U |g|^2} = \min_{g \in W_0^{1,2}(U)} I(g)$$

Let $\phi \in C_c^\infty(U)$ $\quad f + \varepsilon\phi \in W_0^{1,2}(U)$

By minimality of f $\quad I(f+\varepsilon\phi) \geq I(f) \quad \forall\, \varepsilon \in \mathbb{R}$

$$\frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} \geq 0 \quad \varepsilon > 0 \qquad \frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} \leq 0 \quad \varepsilon < 0$$

$$\frac{1}{\varepsilon}\left[I(f+\varepsilon\phi) - I(f)\right] = \frac{1}{\varepsilon \int_U |f|^2 \int_U |f+\varepsilon\phi|^2}\left[\int_U |Df+\varepsilon D\phi|^2 \int_U |f|^2 - \int_U |Df|^2 \int_U |f+\varepsilon\phi|^2\right]$$

$$= \frac{1}{\cancel{\varepsilon} \int_U |f|^2 \int_U |f|^2 + \varepsilon(f\cdot\phi\, 2 + |\phi|^2\varepsilon)}\Big[\cancel{\int_U |f|^2 \int_U |Df|^2} + \varepsilon^{\cancel{2}} \int_U |f|^2 \int_U |D\phi|^2 + 2\cancel{\varepsilon} \int |f|^2 \int_U Df\cdot D\phi \,+$$

$$- \cancel{\int_U |Df|^2 \int |f|^2} - 2\cancel{\varepsilon} \int_U |Df|^2 \int f\cdot\phi - \varepsilon^{\cancel{2}} \int_U |Df|^2 \int |\phi|^2\Big]$$

$$= \frac{2}{\int_U |f|^2 \int_U |f+\varepsilon\phi|^2} \left[\int_U |f|^2 \int_U Df \cdot D\phi \, dx - \underbrace{\int_U |Df|^2}_{= \frac{1}{C_U} \int_U |f|^2} \int_U f \cdot \phi \, dx \right] + \varepsilon (\dots)$$

$$= \frac{2 \cancel{\left(\int_U |f|^2\right)}}{\cancel{\int_U |f|^2} \int_U |f+\varepsilon\phi|^2} \left[\int_U Df \cdot D\phi \, dx - \frac{1}{C_U} \int_U f\phi \, dx \right] + \varepsilon \frac{\cancel{\|f\|_2^2} \|D\phi\|_2^2}{\cancel{\|f\|_2^2} \|f+\varepsilon\phi\|^2} - \varepsilon \frac{\|Df\|_2^2 \|\phi\|^2}{\|f\|_2^2 \|f+\varepsilon\phi\|_2^2}$$

$$\lim_{\varepsilon \to 0} \frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} = \frac{2}{\|f\|_2^2} \left[\int_U Df \cdot D\phi - \frac{1}{C_U} \int f\phi \, dx \right]$$

$$= 0 \quad \forall \phi \in C_c^\infty(U)$$

f satisfies the (MIN) $\Rightarrow$ $\int_U Df \cdot D\phi \, dx = \frac{1}{C_U} \int_U f \phi \, dx \quad \forall \phi \in C_c^\infty(U)$

$$\Rightarrow \quad \underbrace{-\int_U \Delta\phi \cdot f \, dx}_{} = \frac{1}{C_U}\int f\cdot\phi \, dx \quad \forall \phi \in C_c^\infty(U)$$

$$-\Delta T_f(\phi) = \frac{1}{C_U}\int_U f\,\phi\, dx$$

$\Rightarrow$ f satisfies in the sense of distributions

$$-\Delta f = \frac{1}{C_U} f \qquad f \in W_0^{1,2}(U)$$

in some weak sense "$f=0$ on ∂U."

$\frac{1}{C_U}$ is an eigenvalue of the laplace operator and f is the associated eigenfunctions.

2) $\frac{1}{C_U}$ is the minimal among possible eigenvalues:

if $\exists\, g \in W_0^{1,2}(U)$ with $\underset{\text{in } U}{-\Delta g = \lambda g}$ in the sense of distn, $\exists\, \lambda \in \mathbb{R}$ $\Big] \Rightarrow \lambda \geq \frac{1}{C_U}$

Let λ such that $\exists\, g \in W_0^{1,2}(U)$ $-\Delta g = \lambda g$ in the sense of distributions.

$$\Rightarrow \quad -\int_U \Delta\phi\, g\, dx = \lambda \int_U \phi g \, dx \quad \forall\, \phi \in C_c^\infty(U)$$

$$+\int_U D\phi \cdot \underbrace{Dg}_{\text{weak derivative of } g} dx = \lambda \int_U \phi g \, dx \quad \forall\, \phi \in C_c^\infty(U)$$

by density

$\overline{C_c^\infty(U)}^{\|\cdot\|_{1,2}} = W_0^{1,2}(U)$

$$\int_U Dh \cdot Dg \, dx = \lambda \int h \cdot g \, dx \quad \forall\, h \in W_0^{1,2}(U)$$

taking $h = g \Rightarrow$

$$\int_U |Dg|^2 dx = \lambda \int_U |g|^2 dx \qquad \lambda =: \frac{\int_U |Dg|^2 dx}{\int_U |g|^2 dx} \geq \frac{1}{C_U}$$

the best constant in the "Poincaré" inequality in U is the reciprocal of the minimal eigenvalue of the laplacian

$$\inf \{ c,\ \exists f \in W_0^{1,2}(U)\ -\Delta f = cf \text{ in } U \text{ in the sense of distributions} \}$$

$$= \inf_{f \in W_0^{1,2}(U)} \frac{\int_U |Df|^2 dx}{\int |f|^2 dx} = \left(\frac{1}{\tilde{C}^2}\right)$$

$\tilde{C}$ = best constant in the Poincaré inequality.

Theorem (MORREY)

Let $n < p < +\infty$ $\exists\, C(n,p)$

$\forall f \in W^{1,p}(\mathbb{R}^n) \to f \in L^\infty(\mathbb{R}^n)$ $\quad \|f\|_\infty \leq C(n,p)\, \|f\|_{W^{1,p}}$

and for a.e. x, y $\quad |f(x) - f(y)| \leq C(n,p)\, \| |Df| \|_p\, |x-y|^{1-n/p}$.

So up to choosing a representative we have that

$$f \in W^{1,p}(\mathbb{R}^n) \to f \in L^\infty(\mathbb{R}^n) \cap C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$$

<u>Obs</u>

$$W^{1,p}(\mathbb{R}^n) \subsetneq L^\infty(\mathbb{R}^n) \cap C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$$