Lezione 12

martedì 18 novembre 2025 14:30

Matrice $n \times n$ $$A = \begin{pmatrix} \cdots & \cdots & \cdots \\ & a_{ij} & \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Indico con A_{ij} la matrice ottenuta cancellando la riga i e la colonna j.

Sia $$\underbrace{b_{ij} = (-1)^{i+j} \det A_{ij}}_{}$$

La matrice inversa di A è data da:

$$\underbrace{A^{-1} = \frac{1}{\det A} \begin{pmatrix} \cdots & \cdots & \cdots \\ \vdots & b_{ij} & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix}^T}_{}$$

$(\det A \neq 0)$

Esempio. $$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

Per calcolare A^{-1} bisogna:

(1) calcolare $\det A$

(2) calcolare i vari b_{ij}

$$b_{11} = (-1)^{1+1} \det \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix} = 6$$

$$b_{12} = (-1)^{1+2} \det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = -2$$

$$b_{12} = (-1) \det \begin{pmatrix} & \\ 1 & 2 \end{pmatrix} = -2$$

$$b_{13} = (-1)^{1+3} \det \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} = -1$$

$$b_{21} \ ; \ b_{22} \ ; \ b_{23} \ ; \ b_{31} \ ; \ b_{32} \ ; \ b_{33}$$

Alla fine si trova:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 6 & -2 & -1 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}^{T}$$

$$= \frac{1}{\det A} \begin{pmatrix} 6 & * & * \\ -2 & * & * \\ -1 & * & * \end{pmatrix}$$

Calcolo di A^{-1} con l'eliminazione di Gauss

Osservazione.

$$A = \begin{pmatrix} 2 & 3 & -1 & 2 \\ 4 & 5 & -1 & 6 \\ -2 & -4 & 5 & 1 \end{pmatrix}$$

riduciamo A in forma a scala:

$$\left(\underbrace{\begin{matrix} 2 & 3 & -1 & 2 \\ 4 & 5 & -1 & 6 \\ -2 & -4 & 5 & 1 \end{matrix}}_{A} \left| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \right.\right) \longrightarrow \left(\begin{matrix} 2 & 3 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & 4 & 3 \end{matrix} \left| \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \right.\right) \longrightarrow$$

2^{a} riga − 2 × 1^{a} riga
3^{a} riga + 1^{a} riga

A^{1}

$$\longrightarrow \left(\begin{array}{cccc|ccc} 2 & 3 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 3 & -1 & 1 \\ & & & & \multicolumn{3}{c}{\underbrace{\hphantom{3 \quad -1 \quad 1}}_{B}} \end{array}\right)$$

3^a riga − 2^a riga

A' e' la forma a scala di A

$$B \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 & 2 \\ 4 & 5 & -1 & 6 \\ -2 & -4 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 3 & 1 \end{pmatrix} = A'$$

Idea per calcolare A^{-1}: ridurre A in forma a scala, ma in modo che $A' = I$

Esempio: $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

$\det A = 1$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1 \quad 2 \quad 1}}^{A}} & & & \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ \multicolumn{3}{c|}{\underbrace{\hphantom{1 \quad 0 \quad 0}}_{A' = I}} & \multicolumn{3}{c}{\underbrace{\hphantom{6 \quad -2 \quad -3}}_{B}} \end{array}\right)$$

$B \cdot A = A' = I \quad \Rightarrow \quad B = A^{-1}$.

Esercizio: calcolare l'inversa della matrice $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

$$\left(\overbrace{\begin{matrix}2 & 0 & -1\\ 1 & 1 & 2\\ 2 & 2 & 3\end{matrix}}^{A}\middle|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right)\to\left(\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 2 & 0 & -1 & 1 & 0 & 0\\ 2 & 2 & 3 & 0 & 0 & 1\end{array}\right)\to$$

$$\to\left(\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & -2 & -5 & 1 & -2 & 0\\ 0 & 0 & -1 & 0 & -2 & 1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & -2 & -5 & 1 & -2 & 0\\ 0 & 0 & 1 & 0 & 2 & -1\end{array}\right)\to$$

$$\to\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 0 & -3 & 2\\ 0 & -2 & 0 & 1 & 8 & -5\\ 0 & 0 & 1 & 0 & 2 & -1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 0 & -3 & 2\\ 0 & 1 & 0 & -1/2 & -4 & 5/2\\ 0 & 0 & 1 & 0 & 2 & -1\end{array}\right)\to$$

$$\to\left(\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{A'=I}\middle|\underbrace{\begin{matrix}1/2 & 1 & -1/2\\ -1/2 & -4 & 5/2\\ 0 & 2 & -1\end{matrix}}_{A^{-1}}\right)$$