Foglio 6

$$c_{ij} = (-1)^{i+j} \cdot \det[\tilde{A}]$$

$[\tilde{A}]$ = minore algebrico di ordine n-1 ottenuto da $[A]$ eliminando riga i-esima e colonna j-esima

Esercizio 1

le colonne sono l.i. => é invertibile

b) $$B = \begin{pmatrix} 1 & -3 \\ -1/2 & 2 \end{pmatrix}$$

ricordo $$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-cb} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\Rightarrow B^{-1} = \frac{1}{2-\frac{3}{2}} \begin{pmatrix} 2 & 3 \\ \frac{1}{2} & 1 \end{pmatrix} = 2 \begin{pmatrix} 2 & 3 \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}$$

d) $$D = \begin{pmatrix} 0 & -2 & 1 \\ -1 & 1 & 1/2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\det[A] = \sum_{j=1}^{n} a_{ij} \cdot c_{ij} \quad i \text{ fisso}$$

a_{ij} = generico elemento di $[A]$
c_{ij} = complemento algebrico o cofattore

$$\det D = +0 \cdot \det \begin{pmatrix} 1 & 1/2 \\ 2 & 0 \end{pmatrix} - (-2) \cdot \det \begin{pmatrix} -1 & 1/2 \\ 1 & 0 \end{pmatrix} + 1 \det \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= 0 + 2 \left(-1 \cdot 0 - \frac{1}{2}\right) + 1(-2-1) = -4$$ é invertibile!

$$\Rightarrow D^{-1} = \frac{1}{\det D} \cdot S^t$$

$S^t \rightarrow$ matrice trasposta della matrice che ha come elementi i complementi algebrici c_{ij} riferiti ad a_{ij}

$$S = \begin{pmatrix} +1 \cdot 0 - \frac{1}{2} \cdot 2 & -(-1 \cdot 0 - \frac{1}{2} \cdot 1) & +(-1 \cdot 2 - 1) \\ -(-2 \cdot 0 - 1 \cdot 2) & +(0 \cdot 0 - 1) & -(0 \cdot 2 - (-2) \cdot 1) \\ +(-2 \cdot \frac{1}{2} - 1 \cdot 1) & -(0 \cdot \frac{1}{2} - (-1) \cdot 1) & +(0 \cdot 1 - (-2)(-1)) \end{pmatrix} = \begin{pmatrix} -1 & +\frac{1}{2} & -3 \\ +2 & -1 & -2 \\ -2 & -1 & -2 \end{pmatrix}$$

$$D^{-1} = \frac{1}{-4} \begin{pmatrix} -1 & 2 & -2 \\ +1/2 & -1 & -1 \\ -3 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 & 1/2 \\ -1/8 & 1/4 & 1/4 \\ 3/4 & 1/2 & 1/2 \end{pmatrix}$$

Sia $\mathcal{V} = \{ v_1, \dots, v_n \}$ base di $\bar{V}$

$\mathcal{W} = \{ w_1, \dots, w_m \}$ base di $\bar{W}$

$\mathcal{Z} = \{ z_1, \dots, z_p \}$ base di $\bar{Z}$

$f : \bar{V} \to \bar{W} \rightsquigarrow A_{v,w,f} \in \mathcal{M}_{m \times n}(\mathbb{R})$

$g : \bar{W} \to \bar{Z} \rightsquigarrow A_{w,z,g} \in \mathcal{M}_{p \times m}(\mathbb{R})$

$g \circ f : \bar{V} \to \bar{Z} \rightsquigarrow A_{v,z,g\circ f} \in \mathcal{M}_{p \times n}(\mathbb{R})$

$$\begin{tikzcd}
\bar{V} \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & \bar{W} \arrow[d, "g"] \\
& \bar{Z}
\end{tikzcd}$$

$$\boxed{A_{v,z,g\circ f} = A_{w,z\,g} \cdot A_{v,w\,f}}$$

Foglio 4 es 12

b) $f: \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad$ $g: \mathbb{R}^2 \to \mathbb{R}^3$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-3y+2z \\ y+3x \end{pmatrix} \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y-x \\ x \end{pmatrix}$$

1ª modo:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & \mathbb{R}^2 \arrow[d, "g"] \\ & \mathbb{R}^3 \end{tikzcd}$$

$$g\circ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = g\begin{pmatrix} x-3y+2z \\ y+3x \end{pmatrix} = \begin{pmatrix} (x-3y+2z)+(y+3x) \\ (y+3x)-(x-3y+2z) \\ x-3y+2z \end{pmatrix} = \begin{pmatrix} 4x-2y+2z \\ 2x+4y-2z \\ x-3y+2z \end{pmatrix}$$

$g\circ f: \mathbb{R}^3 \to \mathbb{R}^3$

$$A_{\mathcal{E}_3\mathcal{E}_3\, g\circ f} = \begin{pmatrix} 4 & -2 & 2 \\ 2 & 4 & -2 \\ 1 & -3 & 2 \end{pmatrix}$$

2ª modo:

$$A_{\mathcal{E}_3\mathcal{E}_2\, f} = \begin{pmatrix} 1 & -3 & 2 \\ 3 & 1 & 0 \end{pmatrix} = A$$

$$A_{\mathcal{E}_2\mathcal{E}_3\, g} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} = B$$

$$\Rightarrow A_{\mathcal{E}_3\mathcal{E}_3\, g\circ f} = A_{\mathcal{E}_2\mathcal{E}_3\, g} \cdot A_{\mathcal{E}_3\mathcal{E}_2\, f} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 2 \\ 2 & 4 & -2 \\ 1 & -3 & 2 \end{pmatrix}$$

ESERCIZIO

1) data $A = A_{\mathcal{E}B\,id} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ calcolare B

$$\Rightarrow A_{B\mathcal{E}\,id} = \left(A_{\mathcal{E}B\,id}\right)^{-1} = \frac{1}{\det A} S^T$$

$$\det A = 1 \cdot (-1) - 1(-2) = 1$$

$$S = \begin{pmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad A_{B\mathcal{E}\,id} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow B = \left\{ \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

2) data $A = A_{CB\,id} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

note $C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}$ calcolare B

$$A_{B\mathcal{E}\,id} = A_{C\mathcal{E}\,id} \cdot A_{BC\,id} = A_{C\mathcal{E}\,id} \cdot \left(A_{CB\,id}\right)^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & -1 \\ 1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow C = \left\{ \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

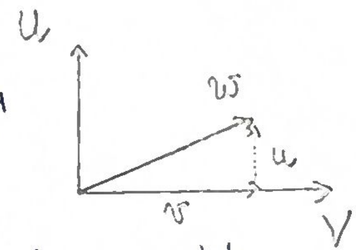
Foglio di esercizi 5

ES 4 : In $\mathbb{R}^4$ si considerano i sottospazi:

$$\mathcal{U}: \begin{cases} x + 2y - z - 2w = 0 \\ x + y + z + w = 0 \end{cases} \quad \text{e} \quad V = \left\langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle$$

a) scrivere la matrice rispetto alle basi canoniche di dominio e codominio ($\mathbb{R}^4$) della proiezione $\pi: \mathbb{R}^4 \to \mathbb{R}^4$ lungo $\mathcal{U}$ su V

b) scrivere la matrice rispetto alle basi canoniche di $\mathbb{R}^4$ della simmetria $\sigma: \mathbb{R}^4 \to \mathbb{R}^4$ di asse $\mathcal{U}$ e direzione V



a) proiezione lungo $\mathcal{U}$ su V : $\pi : \mathbb{R}^4 \to \mathbb{R}^4$

$p_V^{\mathcal{U}}(w) : \mathbb{R}^4 \to \mathbb{R}^4$

$p_V^{\mathcal{U}}(w) = v$

$\mathcal{U} \oplus V = \mathbb{R}^4$

$w = u + v$

$u \in \mathcal{U}$, $v \in V$, $w \in \mathbb{R}^4$

$$\begin{array}{ccc} & & \text{conoscere} \\ \mathbb{R}^4_{\mathcal{E}_4} & \xrightarrow[\pi]{M} & \mathbb{R}^4_{\mathcal{E}_4} \\ {\scriptstyle T_{\mathcal{E}_4}^{B_1}}\downarrow {\scriptstyle id} & & {\scriptstyle id}\downarrow {\scriptstyle T_{B_1}^{\mathcal{E}_4}} \\ \mathbb{R}^4_{B_1} & \xrightarrow[\pi]{A} & \mathbb{R}^4_{B_1} \\ & & \text{conosco} \end{array}$$

$$M = T_{B_1}^{\mathcal{E}_4} \, A \, T_{\mathcal{E}_4}^{B_1} = T_{B_1}^{\mathcal{E}_4} \, A \left(T_{B_1}^{\mathcal{E}_4}\right)^{-1}$$

$B_1 = B_u + B_v$

cerchiamo $A = A_{B_1 B_1 \pi}$:

→ verifichiamo che V e $\mathcal{U}$ siano in somma diretta:

$$\begin{pmatrix}\alpha\\\alpha\\\beta\\\beta\end{pmatrix} \overset{?}{\in} \mathcal{U} \to \begin{cases} \alpha + 2\alpha - \beta - 2\beta = 0 \\ \alpha + \alpha + \beta + \beta = 0 \end{cases} \quad \begin{cases} 3\alpha - 3\beta = 0 \\ 2\alpha + 2\beta = 0 \end{cases} \iff \alpha = \beta = 0$$

$\Rightarrow \mathcal{U} \cap V = \{0_{\mathbb{R}^4}\}$ e $\mathbb{R}^4 = \mathcal{U} \oplus V$

→ cerchiamo una base di $\mathcal{U}$: $\begin{cases} x + 2y - z - 2w = 0 \\ y - 2z - 3w = 0 \end{cases} \quad \begin{cases} x = -3z - 4w \\ y = 2z + 3w \end{cases}$

$$\begin{pmatrix} -3z - 4w \\ 2z + 3w \\ z \\ w \end{pmatrix} = z \begin{pmatrix}-3\\2\\1\\0\end{pmatrix} + w \begin{pmatrix}-4\\+3\\0\\1\end{pmatrix}$$

$$\Rightarrow B_1 = \left[\underset{v_1}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} \underset{v_2}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}} \underset{v_3}{\begin{pmatrix}-3\\2\\1\\0\end{pmatrix}} \underset{v_4}{\begin{pmatrix}-4\\3\\0\\1\end{pmatrix}} \right]$$

$A = A_{B_1 B_1 \pi}$: $\quad \pi\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = 1\cdot v_1 + 0v_2 + 0v_3 + 0v_4$

$$\pi\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} = 0v_1 + 1v_2 + 0v_3 + 0v_4$$

$$\pi\begin{pmatrix}-3\\2\\1\\0\end{pmatrix} = \pi\begin{pmatrix}-4\\3\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = 0v_1 + 0v_2 + 0v_3 + 0v_4$$

$$\Rightarrow A_{B_1 B_1 \pi} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$T_{B_1}^{\mathcal{E}_4} = A_{B_1 \mathcal{E}_4\, id} = \begin{pmatrix}{}^{+}1&0&-3&-4\\{}^{-}1&0&2&3\\{}^{+}0&1&1&0\\{}^{-}0&1&0&1\end{pmatrix}$$

infatti: $\quad id\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = 1e_1 + 1e_2 + 0e_3 + 0e_4$

$$id\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} = 0e_1 + 0e_2 + 1e_3 + 1e_4$$

$$id\begin{pmatrix}-3\\2\\1\\0\end{pmatrix} = \begin{pmatrix}-3\\2\\1\\0\end{pmatrix} = -3e_1 + 2e_2 + 1e_3 + 0e_4$$

$$id\begin{pmatrix}-4\\3\\0\\1\end{pmatrix} = \begin{pmatrix}-4\\3\\0\\1\end{pmatrix} = -4e_1 + 3e_2 + 0e_3 + 1e_4$$

$$T_{\mathcal{E}_4}^{B_1} = \left(T_{B_1}^{\mathcal{E}_4}\right)^{-1} = \frac{1}{\det\left(T_{\mathcal{E}_4}^{B_1}\right)} \cdot S^T$$

$$\det\left(T_{\mathcal{E}_4}^{B_1}\right) = +1\cdot\det\begin{pmatrix}0&2&3\\1&1&0\\1&0&1\end{pmatrix} - 1\cdot\det\begin{pmatrix}0&-3&-4\\1&1&0\\1&0&1\end{pmatrix} + 0\det\begin{pmatrix}0&-3&-4\\0&2&3\\1&0&1\end{pmatrix} - 0\det\begin{pmatrix}0&-3&-4\\0&2&3\\1&1&0\end{pmatrix}$$

$$\det\begin{pmatrix}{}^{+}0&2&3\\{}^{-}1&1&0\\{}^{+}1&0&1\end{pmatrix} = 0\cdot\det\begin{pmatrix}1&0\\0&1\end{pmatrix} - 1\det\begin{pmatrix}2&3\\0&1\end{pmatrix} + 1\det\begin{pmatrix}2&3\\1&0\end{pmatrix} = -5$$

$$\det\begin{pmatrix}{}^{+}0&-3&-4\\{}^{-}1&1&0\\{}^{+}1&0&1\end{pmatrix} = 0\det\begin{pmatrix}1&0\\0&1\end{pmatrix} - 1\det\begin{pmatrix}-3&-4\\0&1\end{pmatrix} + 1\det\begin{pmatrix}-3&-4\\1&0\end{pmatrix} = 7$$

$$\Rightarrow \det\left(T_{\mathcal{E}_4}^{B_1}\right) = -12$$

$$S^T = \begin{bmatrix}{}^{+}S_{11}&{}^{-}S_{12}&{}^{+}S_{13}&{}^{-}S_{14}\\{}^{-}S_{21}&{}^{+}S_{22}&{}^{-}S_{23}&{}^{+}S_{24}\\{}^{+}S_{31}&{}^{-}S_{32}&{}^{+}S_{33}&{}^{-}S_{34}\\{}^{-}S_{41}&{}^{+}S_{42}&{}^{-}S_{43}&{}^{+}S_{44}\end{bmatrix} = \begin{pmatrix}-5&-7&-1&1\\-1&1&-5&-7\\1&-1&-7&7\\1&-1&5&-5\end{pmatrix}$$

$$S_{11} = +\det\begin{pmatrix}0&2&3\\1&1&0\\1&0&1\end{pmatrix} = -5$$

$$S_{21} = -\det\begin{pmatrix}1&2&3\\0&1&0\\0&0&1\end{pmatrix} = -1$$

$$S_{31} = +\det\begin{pmatrix}1&0&3\\0&1&0\\0&1&1\end{pmatrix} = +1$$

$$S_{41} = -\det\begin{pmatrix}1&0&2\\0&1&1\\0&1&0\end{pmatrix} = +1 \quad \dots$$

$$S_{12} = -\det\begin{pmatrix} 0 & -3 & -4 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = -\left(+1(+4) + 1(+3)\right) = -7$$

$$S_{22} = +\det\begin{pmatrix} 1 & -3 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = +1$$

$$S_{32} = -\det\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = -1$$

$$S_{42} = +\det\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = +1(-1) = -1$$

$$S_{13} = +\det\begin{pmatrix} 0 & -3 & -4 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix} = -1$$

$$S_{23} = -\det\begin{pmatrix} 1 & -3 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} = -1(5) = -5$$

$$S_{33} = +\det\begin{pmatrix} 1 & 0 & -4 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} = -1(3+4) = -7$$

$$S_{43} = -\det\begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = -(-1)(2-(-3)) = 5$$

$$S_{14} = -\det\begin{pmatrix} 0 & -3 & -4 \\ 0 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} = -(-1) = 1$$

$$S_{24} = +\det\begin{pmatrix} 1 & -3 & -4 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} = -1(3-(-4)) = 7$$

$$S_{34} = -\det\begin{pmatrix} 1 & 0 & -4 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} = -(-1)(3-(-4)) = 7$$

$$S_{44} = +\det\begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} = -1(2-(-3)) = -5$$

$$M = A_{\mathcal{E}_4\mathcal{E}_4\pi} = T_{B1}^{\mathcal{E}_4}\, A \left(T_{B1}^{\mathcal{E}_4}\right)^{-1}$$

$$= \begin{pmatrix} 1 & 0 & -3 & -4 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{-12} \begin{pmatrix} -5 & -7 & -1 & 1 \\ -1 & 1 & -5 & -7 \\ 1 & -1 & -7 & 7 \\ 1 & -1 & 5 & -5 \end{pmatrix}$$

$$= \frac{1}{+12} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} +5 & +7 & +1 & -1 \\ +1 & -1 & +5 & +7 \\ -1 & +1 & +7 & -7 \\ -1 & +1 & -5 & +5 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 5 & 7 & 1 & -1 \\ 5 & 7 & 1 & -1 \\ 1 & -1 & 5 & +7 \\ 1 & -1 & 5 & +7 \end{pmatrix}$$

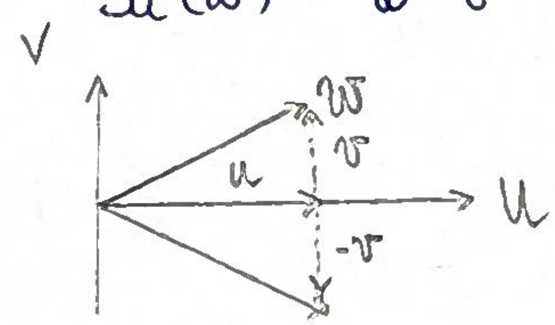
b) simmetria di asse U e direzione V: $\sigma : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ $U \oplus V = \mathbb{R}^4$

$s_u^v(w) = u - v$

$u \in U$

$v \in V$

$w = u + w$



$$\sigma_u^v = id_{\mathbb{R}^4} - 2\pi_v^u$$

$$\Rightarrow \quad A_{\mathcal{E}_4\mathcal{E}_4\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 5 & 7 & 1 & -1 \\ 5 & 7 & 1 & -1 \\ 1 & -1 & 5 & 7 \\ 1 & -1 & 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1/6 & -7/6 & -1/6 & +1/6 \\ -5/6 & -1/6 & -1/6 & 1/6 \\ -1/6 & 1/6 & 1/6 & -7/6 \\ -1/6 & 1/6 & -5/6 & -1/6 \end{pmatrix}$$

METODO ALTERNATIVO:

a) troviamo una decomposizione dei vettori della base canonica di $\mathbb{R}^4$ in somma di un vettore in V e uno in U

$$e_1 = u + v \implies u = e_1 - v = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}\alpha\\\alpha\\\beta\\\beta\end{pmatrix} = \begin{pmatrix}1-\alpha\\-\alpha\\-\beta\\-\beta\end{pmatrix} \overset{?}{\in} U$$

$$\begin{cases} 1-\alpha-2\alpha+\beta+2\beta = 0 \\ 1-\alpha-\alpha-\beta-\beta = 0 \end{cases} \qquad \begin{cases} 3\alpha - 3\beta = 1 \\ 2\alpha + 2\beta = 1 \end{cases} \qquad \begin{pmatrix}3 & -3\\2 & 2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}3 & -3\\2 & 2\end{pmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{12}\begin{pmatrix}2 & 3\\-2 & 3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{12}\begin{pmatrix}5\\1\end{pmatrix} \implies u = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}5/12\\5/12\\1/12\\1/12\end{pmatrix} \in U$$

$$P_V^U(w) = v \implies P_V^U(e_1) = v = \frac{1}{12}\begin{pmatrix}5\\5\\1\\1\end{pmatrix} = \frac{5}{12}e_1 + \frac{5}{12}e_2 + \frac{1}{12}e_3 + \frac{1}{12}e_4$$

$$e_2 = u + v \implies u = e_2 - v = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}\alpha\\\alpha\\\beta\\\beta\end{pmatrix} = \begin{pmatrix}-\alpha\\1-\alpha\\-\beta\\-\beta\end{pmatrix} \overset{?}{\in} U$$

$$\begin{cases} -\alpha+2-2\alpha+\beta+2\beta = 0 \\ -\alpha+1-\alpha-\beta-\beta = 0 \end{cases} \qquad \begin{cases} -3\alpha+3\beta+2 = 0 \\ -2\alpha-2\beta+1 = 0 \end{cases} \qquad \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}3 & -3\\2 & 2\end{pmatrix}^{-1}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \frac{1}{12}\begin{pmatrix}2 & 3\\-2 & 3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{12}\begin{pmatrix}7\\-1\end{pmatrix} \implies u = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}7/12\\7/12\\-1/12\\-1/12\end{pmatrix} \in U$$

$$P_V^U(w) = v \implies P_V^U(e_2) = \frac{1}{12}\begin{pmatrix}7\\7\\-1\\-1\end{pmatrix} = \frac{7}{12}e_1 + \frac{7}{12}e_2 - \frac{1}{12}e_3 - \frac{1}{12}e_4$$

$$e_3 = v + u \implies u = \begin{pmatrix} -\alpha \\ -\alpha \\ 1-\beta \\ -\beta \end{pmatrix} \overset{?}{\in} U \quad \begin{cases} -\alpha - 2\alpha - 1 + \beta + 2\beta = 0 \\ -\alpha - \alpha + 1 - \beta - \beta = 0 \end{cases} \quad \begin{cases} -3\alpha + 3\beta - 1 = 0 \\ -2\alpha - 2\beta + 1 = 0 \end{cases}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \implies \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{12} \begin{pmatrix} 1 \\ 1 \\ 5 \\ 5 \end{pmatrix} \in U$$

$$p(e_3) = \frac{1}{12} \begin{pmatrix} 1 \\ 1 \\ 5 \\ 5 \end{pmatrix} = \frac{1}{12} e_1 + \frac{1}{12} e_2 + \frac{5}{12} e_3 + \frac{5}{12} e_4$$

$$e_4 = u + v \implies u = \begin{pmatrix} -\alpha \\ -\alpha \\ -\beta \\ 1-\beta \end{pmatrix} \overset{?}{\in} U \quad \begin{cases} -\alpha - 2\alpha + \beta - 2 + 2\beta = 0 \\ -\alpha - \alpha - \beta + 1 - \beta = 0 \end{cases} \quad \begin{cases} -3\alpha + 3\beta - 2 = 0 \\ -2\alpha - 2\beta + 1 = 0 \end{cases}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

$$p(e_4) = \frac{1}{12} \begin{pmatrix} -1 \\ -1 \\ 7 \\ 7 \end{pmatrix} = -\frac{1}{12} e_1 - \frac{1}{12} e_2 + \frac{7}{12} e_3 + \frac{7}{12} e_4$$

$$\implies A_{\varepsilon_4 \varepsilon_4}{}^{\pi} = \frac{1}{12} \begin{pmatrix} 5 & 7 & 1 & -1 \\ 5 & 7 & 1 & -1 \\ 1 & -1 & 5 & 7 \\ 1 & -1 & 5 & 7 \end{pmatrix}$$

ESERCIZIO $\rightarrow$ sia $\phi: \mathbb{R}^4 \to \mathbb{R}^3$

$$\phi\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix}$$

a) verificare che ϕ é lineare

b) trovare una base di $\operatorname{Ker}\phi$ e una base di $\operatorname{Im}\phi$

c) trovare la matrice associata a ϕ rispetto alle basi

$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ di dominio e $C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ di codominio

a) ϕ é lineare se $\cdot\ \forall\, u_1, u_2 \in \mathbb{R}^4,\ \phi(u_1+u_2) = \phi(u_1) + \phi(u_2)$ ①

$\cdot\ \forall\, \lambda \in \mathbb{R},\ \forall u \in \mathbb{R}^4\ \ \phi(\lambda u) = \lambda\phi(u)$ ②

① $$\phi\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ w_1+w_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 + y_1+y_2 + 2(z_1+z_2) \\ y_1+y_2 - (z_1+z_2) + w_1+w_2 \\ 2(x_1+x_2) + y_1+y_2 - (w_1+w_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1+y_1+2z_1 \\ y_1-z_1+w_1 \\ 2x_1+y_1-w_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2+2z_2 \\ y_2-z_2+w_2 \\ 2x_2+y_2-w_2 \end{pmatrix} = \phi\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} + \phi\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} \quad \checkmark$$

② $$\phi\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{pmatrix} = \begin{pmatrix} \lambda x + \lambda y + 2\lambda z \\ \lambda y - \lambda z + \lambda w \\ 2\lambda x + \lambda y - \lambda w \end{pmatrix} = \lambda\begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix} = \lambda\phi\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad \checkmark$$

$\Rightarrow$ ϕ é lineare!

b) la matrice di ϕ rispetto alle basi canoniche di dominio e codominio é:

$$A_{\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^3}\phi} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix}$$

$\operatorname{Ker}\phi = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \mid \phi\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\Rightarrow$ devo risolvere il sistema:

$$\begin{cases} x+y+2z = 0 \\ y-z+w = 0 \\ 2x+y-w = 0 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x+y+2z=0 \\ y-z+w=0 \\ z=0 \end{cases} \quad \begin{cases} x=+w \\ y=-w \\ z=0 \end{cases} \rightarrow \begin{pmatrix} +w \\ -w \\ 0 \\ w \end{pmatrix} = w \begin{pmatrix} +1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \mathcal{B}_{\ker\phi} = \left\{ \begin{pmatrix} +1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \rightarrow \dim \ker\phi = 1 \ (\neq 0)$ ϕ no iniettiva

inoltre $A_{\mathcal{E}_4\mathcal{E}_3\phi}$ ha rango 3 $\Rightarrow \dim(\operatorname{im}\phi) = \dim(\mathbb{R}^3)$

$\Rightarrow \phi$ suriettiva $\Rightarrow \operatorname{im}\phi = \mathbb{R}^3$

$\Rightarrow$ una base di $\operatorname{Im}\phi$ é $\mathcal{B}_{\operatorname{Im}\phi} = \mathcal{E}_{\mathbb{R}^3} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

oppure so che $\operatorname{Im}\phi = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \exists \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \text{ t.c. } \phi\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix} \right\}$

considero $\mathcal{E}_{\mathbb{R}^4} \Rightarrow \operatorname{Im}\phi = \left\langle \phi\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \phi\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \phi\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \phi\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$ (*)

so che $\dim\mathbb{R}^4 = \dim\operatorname{Im}\phi + \dim\ker\phi \Rightarrow \dim\operatorname{Im}\phi = 3$

$\Rightarrow$ estraggo una base da (*)

$\mathcal{B}_{\operatorname{Im}\phi} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$

c) $\phi\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \begin{cases} \alpha = 2 \\ \beta = 1 \\ \beta+\gamma = 3 \end{cases} \begin{cases} \alpha = 2 \\ \beta = 1 \\ \gamma = 2 \end{cases}$ 1ª colonna

$\phi\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \begin{cases} \alpha = 3 \\ \beta = 0 \\ \gamma = 1 \end{cases}$ 2ª colonna

$\phi\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \begin{cases} \alpha = 2 \\ \beta = 0 \\ \gamma = -1 \end{cases}$ 3ª colonna

$\phi\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \begin{cases} \alpha = 1 \\ \beta = 1 \\ \gamma = 0 \end{cases}$ 4ª colonna

$$A_{\mathcal{B}\mathcal{C}\phi} = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 0 \end{pmatrix}$$

d) Esiste un sottospazio $W \subset \mathbb{R}^4$ tale che $\dim(\Phi(W)) = \dim(W)$? In caso affermativo esibirne uno di dimensione massima.

W è tale che $\dim(\Phi(W)) = \dim W \iff W \cap \operatorname{Ker}\Phi = \{0_{\mathbb{R}^4}\}$

$\Rightarrow$ uno massimale con questa proprietà è tale che $\mathbb{R}^4 = W \oplus \operatorname{Ker}\Phi$

$\Rightarrow$ completiamo la base del $\operatorname{Ker}\Phi = \left\{\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}\right\}$ a una base di $\mathbb{R}^4$

$$\begin{array}{cccc|l}
1 & -1 & 0 & 1 & \}\ \operatorname{Ker}\Phi \\
0 & 1 & 0 & 0 & \\
0 & 0 & 1 & 0 & \Big\}\ W \\
0 & 0 & 0 & 1 &
\end{array}$$

$\Rightarrow \quad W = \left\langle \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\rangle$

e) Trovare la matrice associata a ϕ rispetto alle basi:

$\mathcal{V} = \left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$ di dominio e $\mathcal{W} = \left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ di codominio

$$\begin{tikzcd}
\mathcal{E}_4 \arrow[r, "M", "\phi"'] \arrow[d, "T_{\mathcal{E}_4}^{\mathcal{V}}"', "id"] & \mathcal{E}_3 \arrow[d, "id"', "T_{\mathcal{W}}^{\mathcal{E}_3}"] \\
\mathcal{V} \arrow[r, "A", "\phi"'] & \mathcal{W}
\end{tikzcd}$$

$M = T_{\mathcal{W}}^{\mathcal{E}_3} A\, T_{\mathcal{E}_4}^{\mathcal{V}}$

$T_{\mathcal{E}_3}^{\mathcal{W}} M\, T_{\mathcal{V}}^{\mathcal{E}_4} = A$

conoscere

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "A", "\phi"'] \arrow[d, "T_{\mathcal{V}}^{\mathcal{E}_4}"', "id"] & \mathcal{W} \arrow[d, "id"', "T_{\mathcal{E}_3}^{\mathcal{W}}"] \\
\mathcal{E}_4 \arrow[r, "M", "\phi"'] & \mathcal{E}_3
\end{tikzcd}$$

conosco

$A = \left(T_{\mathcal{W}}^{\mathcal{E}_3}\right)^{-1} M\, T_{\mathcal{V}}^{\mathcal{E}_4}$

$T_{\mathcal{W}}^{\mathcal{E}_3} = A_{\mathcal{W}\mathcal{E}_3 id} = \begin{pmatrix}1 & -1 & 0\\ 2 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$

$T_{\mathcal{V}}^{\mathcal{E}_4} = A_{\mathcal{V}\mathcal{E}_4 id} = \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$

calcoliamo l'inversa di $T_{\mathcal{W}}^{\mathcal{E}_3}$:

$\det\left(T_{\mathcal{W}}^{\mathcal{E}_3}\right) = +1\left(1-(-2)\right) = 3 \quad \Rightarrow \quad \left(T_{\mathcal{W}}^{\mathcal{E}_3}\right)^{-1} = \dfrac{1}{3}\begin{pmatrix}1 & 1 & 0\\ -2 & 1 & 0\\ 1 & -2 & 3\end{pmatrix}$

$$A_{\mathcal{V}\mathcal{W}\phi} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 2 & 1 & 1 \\ -2 & -1 & -5 & 1 \\ 7 & 2 & 4 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 3 & 2 & 2 \\ -3 & -6 & -4 & -1 \\ 9 & 6 & -1 & 2 \end{pmatrix}$$

ESERCIZIO → Sia $\Phi_k : \mathbb{R}^3 \to \mathbb{R}^3$ l'endomorfismo di $\mathbb{R}^3$ $(k \in \mathbb{R})$, la cui matrice rispetto alla base canonica $\mathcal{E}_3$ di $\mathbb{R}^3$ è :

$$A_{\mathcal{E}_3\mathcal{E}_3}\Phi_k = A_k = \begin{pmatrix} 1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

1) determinare, al variare di k, una base di $\ker \Phi_k$ e $\operatorname{Im} \Phi_k$
2) determinare, al variare di k, la controimmagine $\Phi_k^{-1}\begin{pmatrix}1\\2k\\k\end{pmatrix}$
3) determinare, al variare di k, la matrice $A_{\mathcal{E}_3 B}\Phi_k$ associata a Φ_k rispetto alla base canonica nel dominio e $B = \left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ nel codominio.

1) $\Phi_k : \mathbb{R}^3 \to \mathbb{R}^3$ è invertibile $\Leftrightarrow$ $\det A_k \neq 0$ ossia per $k \neq \pm 1$

per $k \neq \pm 1$ $\quad \operatorname{Im}\Phi_k = \mathbb{R}^3 = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$, $\ker \Phi_k = \{0_{\mathbb{R}^3}\}$

per $k = \pm 1$ $\quad \operatorname{Im}\Phi_k = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$, $\ker \Phi_k = \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$

2) $\Phi_k^{-1}\begin{pmatrix}1\\2k\\k\end{pmatrix}$ è l'insieme delle soluzioni del sistema

$$\left(\begin{array}{ccc|c} 1 & 0 & k^2-1 & 1 \\ 0 & 2 & 0 & 2k \\ 1 & 0 & 0 & k \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & k \\ 0 & 2 & 0 & 2k \\ 0 & 0 & k^2-1 & 1-k \end{array}\right)$$

per $k \neq \pm 1$ ammette unica soluzione $\Phi_k^{-1}\begin{pmatrix}1\\2k\\k\end{pmatrix} = \left\{\begin{pmatrix}k\\k\\-\frac{1}{k+1}\end{pmatrix}\right\}$

per $k = -1$ non ammette soluzioni $\Phi_k^{-1}\begin{pmatrix}1\\2k\\k\end{pmatrix} = \emptyset$

per $k = 1$ ammette infinite soluzioni $\Phi_k^{-1}\begin{pmatrix}1\\2k\\k\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$

3) $A_{\mathcal{E}_3 B}\,\phi_k = A_{\mathcal{E}_3 B}\, id \cdot A_{\mathcal{E}_3 \mathcal{E}_3}\,\phi_k$

nota !

$$A_{\mathcal{E}_3 B}\, id = \left(A_{B\mathcal{E}_3}\, id\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}^{-1}$$

$A_{\mathcal{E}_3 B}\, id = ?$

$$id\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix} + (-1)\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$id\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = 1\begin{pmatrix}0\\0\\-1\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$id\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = -1\begin{pmatrix}0\\0\\1\end{pmatrix} + 0\begin{pmatrix}0\\1\\1\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A_{\mathcal{E}_3 B}\, id_{\mathbb{R}^3} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow A_{\mathcal{E}_3 B}\,\phi_k = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} +1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 2 & 1-k^2 \\ 1 & 0 & k^2-1 \end{pmatrix}$$

$$A_{\mathcal{E}_3\mathcal{E}_3\Phi_k} = A_k = \begin{pmatrix} 1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

trovare $A_{\mathcal{E}_3 B \Phi_k}$ ($\to$ le colonne sono le coordinate delle immagini tramite Φ_k dei vettori della base $\mathcal{E}_3$, rispetto alla base B)

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = +2\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k^2-1 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - (k^2-1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k^2-1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A_{\mathcal{E}_3 B \Phi_k} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 2 & 1-k^2 \\ 1 & 0 & k^2-1 \end{pmatrix} \quad ✓$$

Foglio 5 Es 7

$f_a : \mathbb{R}^3 \to \mathbb{R}^3$

$$f_a\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay \\ x - y + 1 \\ x + az \end{pmatrix}$$

a) f_a iniettiva $\Leftrightarrow$ $\ker f_a = \{0_V\}$ $\Rightarrow$ $\det A_{E_3E_3} f_a \neq 0$

$$A_{E_3E_3} f_a = \begin{pmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 1 & 0 & a \end{pmatrix} = A_a$$

$$\det A_a = 0 \cdot (-1 \cdot a - 1 \cdot 0) - a(+1 \cdot a - 1) + 0(+1 \cdot 0 - 1(-1)) = -a(a-1)$$

$\Rightarrow$ f_a é iniettiva per $a \neq 0, 1$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \to \ker f_0 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \to \ker f_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

b) $f^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = ?$ $\quad A_a \underline{x} = \underline{b} \Rightarrow \underline{x} = A_a^{-1}\underline{b}$ con $\underline{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

devo trovare A_a^{-1} ma so che per $a = 0, 1$ $\det A_a = 0$

$\Rightarrow$ prima considero questi due casi:

$\boxed{a = 0}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

per $a=0$ $\text{rk}\, A \neq \text{rk}\, A|b$ $\Rightarrow$ $f^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \emptyset$

$\boxed{a=1}$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & +1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

soluz. partic.

$rkA = rkA|b \Rightarrow \exists S$

$$\begin{cases} x + z = 1 \\ y = 1 \end{cases} \xrightarrow{z=0} \begin{cases} x = 1 \\ y = 1 \\ z = 0 \end{cases} \; (*)$$

$$\Rightarrow f_1^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

soluzione partic. che risolve il sistema (*)

nucleo della matrice della funzione $f_{a=1}$

per $a \neq 0, 1$: ora posso calcolare A_a^{-1}

$\det A_a = a(1-a)$

$$\text{Cof. } A_a = \begin{pmatrix} -a & -(a-1) & 1 \\ -a^2 & 0 & -(-a) \\ a & 0 & -a \end{pmatrix} = \begin{pmatrix} -a & 1-a & 1 \\ -a^2 & 0 & a \\ a & 0 & -a \end{pmatrix}$$

$$f_a^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{a(1-a)} \cdot \begin{pmatrix} -a & -a^2 & a \\ 1-a & 0 & 0 \\ 1 & a & -a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{a(1-a)} \begin{pmatrix} -a + a \\ 1-a \\ 1-a \end{pmatrix} = \begin{pmatrix} 0 \\ 1/a \\ 1/a \end{pmatrix}$$