μ Borel σ-finite measure on $\mathbb{R}$

$\updownarrow$

F right continuous, non decreasing $: \mathbb{R} \to \mathbb{R}$
(cumulative distr. function)

If μ is finite $\quad F(x) = \underline{\mu(-\infty, x]}$

If μ is σ-finite $\quad \begin{cases} G(x) = \mu(0, x] & x > 0 \\ G(0) = 0 \\ G(x) = -\mu(x, 0] & x < 0 \end{cases}$

$$G(x) = F(x) - \mu(-\infty, 0]$$

Lebesgue measure $\to F(x) = x$

Obs $\quad \mu(\{a\}) = F(a) - F(a^-) = F(a) - \lim_{x \to a^-} F(x).$

def MEASURABLE FUNCTION

(A, Σ, μ) measure space

A set
Σ is a σ-algebra on A
$\mu : \Sigma \to [0, +\infty]$ is a meas.

$$f : (A, \Sigma, \mu) \longrightarrow \underline{\underline{\mathbb{R}}} \qquad (\mathbb{R}, \underline{\underline{\mathcal{B}}},$$
$$a \longmapsto f(a) \in \mathbb{R}$$

f is measurable if the pre-image of every Borel set in $\mathbb{R}$ is an element of Σ.

$$\underbrace{\forall B \in \mathcal{B}}_{f^{-1}(B) \in \Sigma.} \qquad f^{-1}(B) = \{a \in A,\ f(a) \in B\} = \text{preimage of } B$$

f is measurable $\iff$ $\forall\, t \in \mathbb{R}$ $\underbrace{\{a \in A \mid f(a) \le t\}}_{f^{-1}(-\infty, t]} \in \Sigma$

<u>Observation</u>

μ Borel measure on $\mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ CONTINUOUS $\implies$ $f : (\mathbb{R}, \mathcal{B}, \mu) \to \mathbb{R}$ is measurable

$\forall x \in \mathbb{R}$ $\lim_{y \to x} f(y) = f(x)$

The preimage of an interval is always an interval for continuous functions

So f continuous is measurable with respect to every μ Borel on $\mathbb{R}$

$f : (A, \mathcal{E}, \underline{\underline{\mu}}) \to \underline{\mathbb{R}}$ measurable function

<u>Def</u> <u>PUSH-FORWARD</u> of μ by f $\quad (f_\# \mu)$

$f_\# \mu$ is a Borel measure on $\mathbb{R}$

$\forall B \in \mathcal{B}$

$$\underbrace{f_\# \mu(B)}_{\substack{\| \\ \nu(B)}} = \underline{\underline{\mu}}(\{\underline{\underline{a \in A}}, \underbrace{f(a) \in B}\})$$

$\mu \longrightarrow (f_\# \mu) = \nu$ $\qquad$ ν is a Borel measure measure on $\mathbb{R}$

$f : A \longrightarrow \mathbb{R}$

$a \xrightarrow{f} B \in \mathcal{B}$

Since f is meas. $\quad \{a \in A \mid f(a) \in B\} \in \mathcal{E}$

$\nu(B) = f_\# \mu(B) \qquad B \in \mathcal{B} \longrightarrow \{a \in A \;\; f(a) \in B\}$

$$(f_{\#}\mu)(B) = \mu(\underbrace{f^{-1}(B)}_{\sim \cdot \sim})$$

EX. RANDOM VARIABLE

$$(\Omega, \overset{\text{set filtrations}}{\mathcal{F}}, \mathbb{P})$$

$\mathcal{F}$ is a σ-algebra on Ω.

$X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ measurable function (RANDOM VARIABLE

$\Updownarrow$

X is a random variable $\Leftrightarrow$ $\{\omega \mid X(\omega) \leq t\} \in \mathcal{F}$ $\forall t \in \mathbb{R}$

law of the random variable X (distribution)

$\mathcal{L}_X$ is the Borel measure on $\mathbb{R}$ given by the push forward of $\mathbb{P}$ by X

$$\mathcal{L}_X = X_{\#}\mathbb{P}$$

$$X : \Omega \xrightarrow{X} \mathbb{R}$$

$$\mathbb{P} \longrightarrow \mathcal{L}_X$$

$\forall B \in \mathcal{B}$ $\quad B \subseteq \mathbb{R}$

$$\mathcal{L}_X(B) = (X_{\#}\mathbb{P})(B) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in B\})$$

$\mathcal{L}_X$ is a Borel measure on $\mathbb{R}$, which is finite

$$\mathcal{L}_X(\mathbb{R}) = \mathbb{P}(\omega \mid X(\omega) \in \mathbb{R}) = \mathbb{P}(\Omega) = 1$$

$\mathcal{P}(\mathbb{R})$ = "probability measures on $\mathbb{R}$" =

$= \{ \mu$ Borel measure on $\mathbb{R}$ $\quad \mu : \mathcal{B} \to [0, +\infty]$

such that $\mu(\mathbb{R}) = 1 \}$

Theorem If $\mu \in \mathcal{P}(\mathbb{R})$ (μ is a Borel measure on $\mathbb{R}$ $\mu(\mathbb{R}) = 1$)

$\exists (\Omega, \mathcal{F}, \mathbb{P})$, $\exists X : \Omega \to \mathbb{R}$ random variable

such that $\mu = \mathcal{L}_X = X \# \mathbb{P}$

If $X : \Omega \to \mathbb{R}$ is a random variable $\mathcal{L}_X \in \mathcal{P}(\mathbb{R})$.

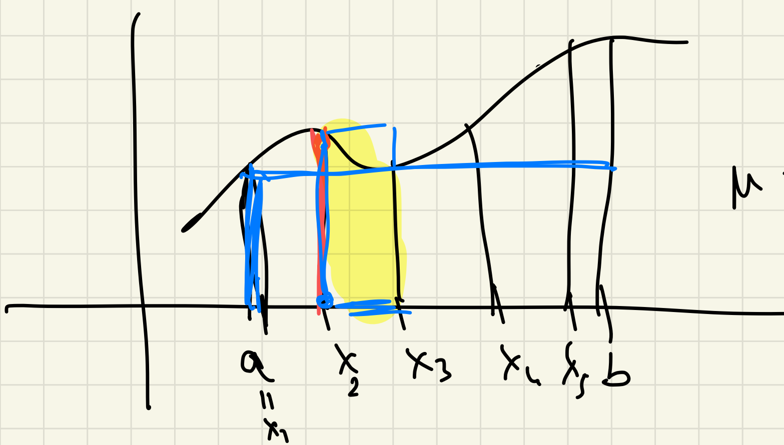
Integration in $\mathbb{R}$ with respect to a Borel measure

$\underbrace{\mu \text{ Borel measure on } \mathbb{R}}$ $\longrightarrow$ F cumulative distribut. function

$f : \mathbb{R} \to \mathbb{R}$ continuous

$a < b$ $\quad \displaystyle\int_a^b f(x)\, d\mu = \sup \sum_{i=1}^{k-1} \left(\min_{[x_i, x_{i+1}]} f \right) \cdot \left[F(x_{i+1}) - F(x_i) \right]$

$a \leq x_1 \leq x_2 < x_3 \ldots \leq x_k \leq b$



$F(x) = x \qquad F(x_{i+1}) - F(x_i) =$

$x_{i+1} - x_i$

μ finite

$F(x_{i+1}) - F(x_i) =$

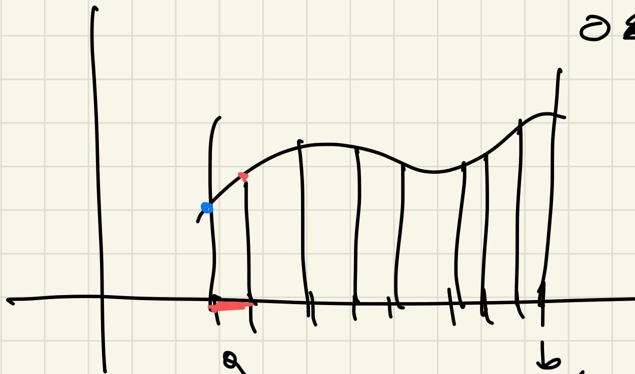
$= \mu(-\infty, x_{i+1}] - \mu(-\infty, x_i] =$

$= \mu(x_i, x_{i+1}]$

Equivalently (f continuous)

$$\int_a^b f(x)\,d\mu(x) = \lim_{\varepsilon \to 0^+} \left[\sum_{i=1}^{k-1} \underline{\underline{f(x_i)}} \left[F(x_{i+1}) - F(x_i)\right], \quad a \le x_1 < x_2 < \dots \le x_n \le b, \quad 0 \le x_i - x_{i-1} \le \varepsilon \right]$$

$$\downarrow$$



$$= \inf_{a \le x_1 < x_2 \dots x_n \le b} \sum_{i=1}^{k-1} \left(\max_{[x_i, x_{i+1}]} f \right) \left(F(x_{i+1}) - F(x_i) \right)$$

Ex X random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$
measurable

$\downarrow$

$\mathcal{L}_X$ law of X (Borel measure on $\mathbb{R}$ $\mathcal{L}_X(\mathbb{R}) = 1$
$\mathcal{L}_X \in \mathcal{P}(\mathbb{R})$)

$f : \mathbb{R} \to \mathbb{R}$ continuous function

$$\int_{\mathbb{R}} f(x)\, d\mathcal{L}_X = \mathbb{E}(f(X))$$

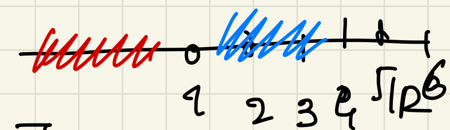
$\mathbb{E}(X) = \text{mean of } X = \int_{\mathbb{R}} x \, d\mathcal{L}_X$

$\mathbb{E}(|X|^n) = n$-moment of X

$$= \int_{\mathbb{R}} |x|^n \, d\mathcal{L}_X$$

Ex $\quad X : \left(\underbrace{\{\text{outcome of the dice}\}}_{\Omega},\ \underset{\downarrow}{\mathbb{P}} \right) \longrightarrow X(\omega)$

$\mathcal{F} = \mathcal{P}(\Omega) \qquad \mathbb{P}\{1\} = \frac{1}{6}$



$$\mathcal{L}_X = X_\# \mathbb{P} \ : \mathcal{B} \longrightarrow [0,+\infty]$$

$\forall \omega \in \Omega \qquad X(\omega) \in \{1,2,3,4,5,6\}$

$\mathcal{L}_X\{i\} = \frac{1}{6}$

$i = 1,2,3,4,5,6$

$B \in \mathcal{B}$

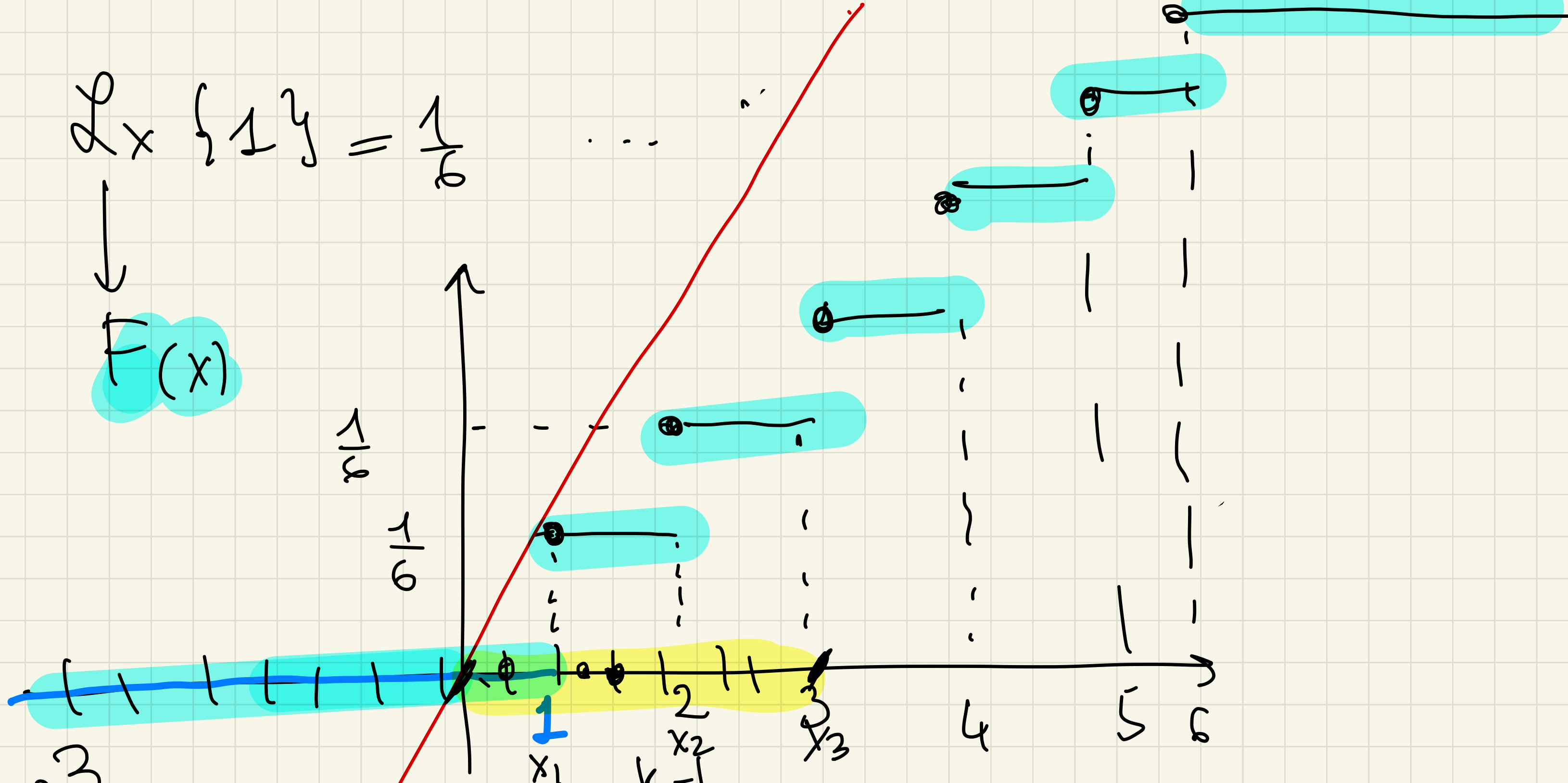
$$\mathcal{L}_X(B) = \mathbb{P}\{\omega \mid X(\omega) \in B\} =$$

$$= \frac{\text{number of } \{1,2,3,4,5,6\} \text{ in } B}{6}$$

$$\mathcal{L}_X\{1\} = \frac{1}{6}$$

$\downarrow$

$F(x)$



$$\int_{-\infty}^{3} x \, d\mathcal{L}_X = \sup_{-\infty < x_1 < x_2 \dots < x_k \le 3} \sum_{i=1}^{k-1} x_i \left[F(x_{i+1}) - F(x_i)\right] =$$

$$= 1 \cdot \left[F(2) - F(1)\right] + 2 \cdot \left[F(3) - F(2)\right] = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$$

In particular we may define the

$\int_a^b f(x)\,dx$ (Riemann integral) for $f:\mathbb{R}\to\mathbb{R}$ continuous

Lebesgue measure $F(x)=x$

Obs $f\geq 0$ continuous

$$B\in\mathcal{B} \longrightarrow \mu_f(B)=\int_B f(x)\,dx$$

$$B=[a,b] \longrightarrow \mu_f[a,b]=\int_a^b f(x)\,dx$$

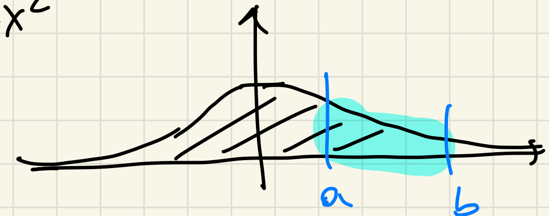
$f(x)\equiv 1\ \forall x$

$\mu_f[a,b]=\int_a^b dx = b-a$

μ_f is actually a Borel measure on $\mathbb{R}$

We fix $f:\mathbb{R}\to\mathbb{R}$ positive and we "move" the DOMAIN where we integrate f

Ex $f(x) = e^{-x^2}$



$$\mu_f([a,b]) = \int_a^b e^{-x^2}\,dx \quad = \text{area}$$

It is possible to prove that $f : \mathbb{R} \to \mathbb{R}$ $f \geq 0$ and continuous then $\mu_f : B \longrightarrow \int_B f(x)\,dx$ is actually a Borel measure on $\mathbb{R}$ (which is σ-finite)

Definition μ Borel measure on $\mathbb{R}$

μ is ABSOLUTELY CONTINUOUS with respect to Lebesgue ($\mu \ll \mathcal{L}$) if $\forall B \in \mathcal{B}$ $\underline{|B| = 0}$ (Lebesgue measure of B, "length of B") then $\underline{\mu(B) = 0}$

μ is SINGULAR with respect to Lebesgue ($\mu \perp \mathcal{L}$) if $\mathbb{R} = B_1 \cup B_2$ $B_1, B_2 \in \mathcal{B}$

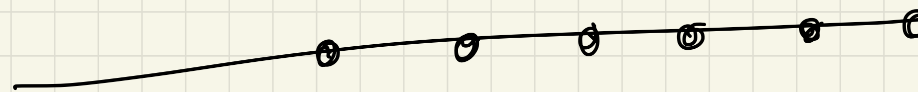
$B_1 \cap B_2 = \emptyset$, $|B_1| = 0$ $\mu(B_2) = 0$

Ex $\quad \mathcal{L}_X \qquad X$ "launching of a dice"

$$\mathcal{L}_X\{1\} = \frac{1}{6} = \mathcal{L}_X\{2\} = \dots = \mathcal{L}_X\{6\}$$

$$\mathbb{R} = \left(\mathbb{R} \setminus \{1,2,3,4,5,6\}\right) \cup \{1,2,3,4,5,6\})$$

Lebesgue measure of $\{1,2,3,4,5,6\} = 0$



$$|\{1,2,3,4,5,6\}| = |\{1\}| + |\{2\}| + |\{3\}| + \dots + |\{6\}| = 0$$

$$\mathcal{L}_X\left(\mathbb{R} \setminus \{1,2,3,4,5,6\}\right) = 0$$

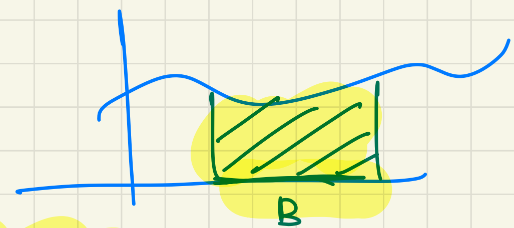
<u>Obs</u> $f \geq 0$ continuous $\mu_f([a,b]) = \int_a^b f(x)\,dx$

$$\mu_f \ll \mathcal{L}$$

since $\forall B \in \mathcal{B}$ $\underbrace{|B| = 0}$ then $\int_B f(x)\,dx = 0$

$\mu_f(B) = 0$

$$\mu_f\{a\} = \int_a^a f(x)\,dx = 0$$



$f \geq 0$ $\underbrace{\int_B f(x)\,dx}$ = area of the region with base $\underline{B}$ and under the graph of f

Theorem of (Lebesgue - Radon - Nikodym)

μ σ-finite Borel measure on $\mathbb{R}$.

then there exists a UNIQUE ABSOLUTELY CONTINUOUS MEASURE $\mu_a \ll \mathcal{L}$ and a UNIQUE SINGULAR MEASURE $\mu_s \perp \mathcal{L}$ such that

$$\mu = \underline{\underline{\mu_a}} + \underline{\underline{\mu_s}} \qquad (\mu(B) = \mu_a(B) + \mu_s(B))$$

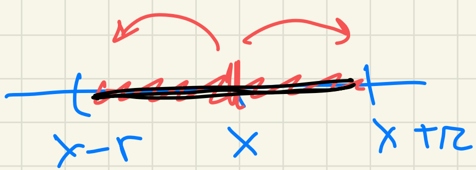
$$\left(\begin{array}{lll} \text{if } \mu \ll \mathcal{L} & \mu = \mu_a & \mu_s = 0 \\ \text{if } \mu \perp \mathcal{L} & \mu = \mu_s & \mu_a = 0 \end{array} \right)$$

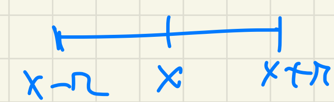
Moreover μ_a can be written as the integral with respect to Lebesgue of a function f_μ (called the DENSITY of μ).

$$\mu_a(B) = \int_B \underline{\underline{f_\mu}}(x)\,dx \qquad \underline{f_\mu \geq 0}$$

$$\underline{\underline{f_\mu(x)}} = \text{Radon-Nikodym derivative of } \mu = \lim_{r \to 0^+} \frac{\mu((x-r, x+r))}{\underbrace{2r}_{\text{Lebesgue measure of } (x-r,\, x+r)}}.$$

for a.e x





μ Borel measure $\longrightarrow$ F cumulative distribution function

(μ finite $\qquad f_\mu(x) = \lim_{r \to 0^+} \dfrac{\mu(x-r, x+r)}{2r} =$

$$\mu(x-r, x+r) = \underbrace{\mu(-\infty, x+r)} - \mu(-\infty, x-r] =$$

$$= \underbrace{F((x+r)^-)}_{\lim_{y \to (x+r)^-} F(y)} - F(x-r)$$

$F(x) = \mu(-\infty, x]$

$F(x^-) = \mu(-\infty, x)$

$= \lim_{y \to x^-} F(y)$

$$f_\mu(x) = \lim_{r \to 0^+} \frac{F((x+r)^-) - F(x-r)}{2r} =$$

$$= \lim_{r \to 0^+} \frac{1}{2} \left[\frac{F(x+r) - F(x)}{r} + \frac{F(x) - F(x-r)}{r} \right]$$

$\downarrow$ F is differentiable at x.

$$= \frac{1}{2} \left(F'(x) + F'(x) \right) = F'(x)$$

$f_\mu(x) = F'(x)$ at the points where F is differentiable.

If μ is a Borel measure (μ finite)

$F(x) = \mu(-\infty, x]$ cumulative distribution function

by RADON NYKODYM

$\mu = \mu_a + \mu_s$ $\qquad \mu_a \ll \mathcal{L} \quad \mu_s \perp \mathcal{L}$

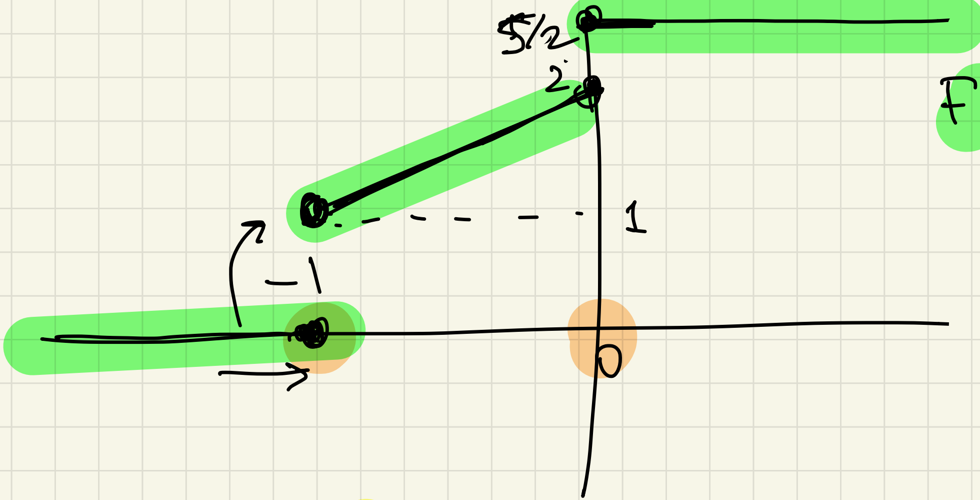
$$\mu_a(A) = \int_A f_\mu \, dx = \int_A \underline{\underline{F'(x)}} \, dx$$

what is μ_s?

J = jumps of $\mu = \{ a \in \mathbb{R} \quad F(a) = \lim_{y \to a^-} F(y) \}$

$$\mu\{a\} = F(a) - F(a^-) = F(a) - \lim_{y \to a^-} F(y)$$

$$\mu_s^J(B) = \sum_{a \in J} \mu(\{a\}) \cdot \delta_{\{a\}}(B) \qquad \delta_{\{a\}}(B) = \begin{cases} 0 & a \notin B \\ 1 & a \in B \end{cases}$$



$$F(x) = \begin{cases} 0 & x < -1 \\ x+2 & -1 \le x < 0 \\ \frac{5}{2} & x \ge 0 \end{cases}$$

$$\mu([a,b]) = F(b) - F(a)$$

$$F'(x) = \begin{cases} 0 & x < -1 \\ 1 & -1 < x < 0 \\ 0 & x > 0 \end{cases}$$

F' is not defined in $x = -1$, $x = 0$

$$J_\mu = \{ a \in \mathbb{R} \quad F(a) > F(a^-) \} = \{ -1, 0 \}$$

$$\mu_a(A) = \int_A F'(x)\,dx = \int_{A \cap [-1,0]} 1\,dx$$

$$\mu_a(0,3) = \int_{(0,3)\cap[-1,0]} 1\,dx = 0$$

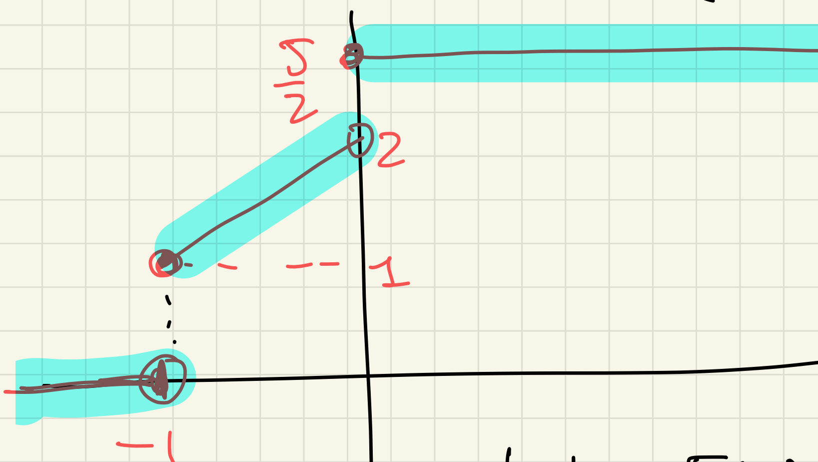
$$\mu_a\left(-\frac{1}{2}, 3\right) = \int_{-1/2}^{0} 1\,dx = \frac{1}{2}$$

μ_S^J

$J = \{-1, 0\}$

$F(-1) - \underbrace{F((-1)^-)} = 1 - 0 = 1$

$F(0) - F(0^-) = \frac{5}{2} - 2 = \frac{1}{2}$



$\mu\{-1\} = 1$

$\mu\{0\} = \frac{1}{2}$

$\mu\{a\} = F(a) - \lim_{y \to a^-} F(y) = F(a) - F(a^-)$

$\boxed{\mu_S^J(B)} = 1 \cdot \delta_{\{-1\}}(B) + \frac{1}{2}\delta_{\{0\}}(B)$

$\delta_{\{-1\}}(B) = \begin{cases} 0 & -1 \notin B \\ 1 & -1 \in B \end{cases}$

$\delta_{\{0\}}(B) = \begin{cases} 0 & 0 \notin B \\ 1 & 0 \in B \end{cases}$

$$\mu_S^J(B) = 1\,\delta_{\{-1\}}(B) + \frac{1}{2}\,\delta_{\{0\}}(B) =$$

$$= \begin{cases} 0 & \text{if } -1, 0 \notin B \\ 1 & \text{if } -1 \in B,\ 0 \notin B \\ \frac{1}{2} & \text{if } -1 \notin B\ \ 0 \in B \\ 1 + \frac{1}{2} = \frac{3}{2} & \text{if } -1, 0 \in B \end{cases}$$

$$\mu_S^J \perp \mathcal{L} \qquad \mathbb{R} = (\mathbb{R} \setminus \{-1, 0\}) \cup \{-1, 0\}$$

$$|\{-1, 0\}| = 0 \qquad \mu_S^J(\mathbb{R} \setminus \{-1, 0\}) = 0$$

$$F(x) = \begin{cases} 0 & x < -1 \\ x+2 & -1 \leq x < 0 \\ \frac{5}{2} & x \geq 0 \end{cases}$$

$$\downarrow$$

$$\mu(A) = \underbrace{\int_{A \cap [-1,0]} 1 \cdot dx}_{\mu_a(A)} + \underbrace{1 \cdot \delta_{\{-1\}}(A) + \frac{1}{2}\delta_{\{0\}}(A)}_{\mu_s^J(A)}$$

Actually the singular part of μ is not only the jump part but may contain some other very strange singular measure (CANTORIAN MEASURE)