DENSITY RESULTS

Observ. 1 $\quad f \in W^{k,p}(U) \qquad \phi \in C_c^\infty(U)$

$f \cdot \phi \in W^{kp}(U)$ and $\underbrace{D^\alpha(f\phi)}_{\text{weak derivative}} = \underbrace{D^\alpha f}_{\text{weak der.}} \cdot \phi + f \underbrace{D^\alpha \phi}_{\text{classical derivative}}$

$\psi \in C_c^\infty(U)$

$$\int_U \frac{\partial}{\partial x_i}(f\phi)\,\psi\,dx = -\int_U f\phi \frac{\partial}{\partial x_i}\psi\,dx =$$

$$\phi \frac{\partial}{\partial x_i}\psi = \frac{\partial}{\partial x_i}\phi\psi - \frac{\partial}{\partial x_i}\phi\,\psi$$

$$= -\int_U f \frac{\partial}{\partial x_i}(\varphi\psi) + \int_U f \frac{\partial \phi}{\partial x_i}\psi\,dx = \int \Big(\underbrace{\frac{\partial}{\partial x_i} f}_{\text{weak derivative of } f}\Big)\phi\,\psi + \int f \frac{\partial}{\partial x_i}\phi\,\psi\,dx$$

<u>Lemma</u> $\quad f \in W^{r,p}(U) \qquad p \in [1, +\infty)$

$\rho_k(x) = k^n \rho(kn) \qquad \forall U' \subset\subset U \quad \exists k_0$ suff. large $\quad f * \rho_k \in C^\infty(U')$

$f * \rho_k \longrightarrow f$ in $W^{r,p}(U')$ $\quad$ (since $D^\alpha(f * \rho_k) = D^\alpha f * \rho_k$)

Theorem (MEYERS-SERRIN 1964)

1) $U \subseteq \mathbb{R}^n$ open $\qquad \overline{\mathcal{C}^\infty(U) \cap W^{r,p}(U)}^{\|\cdot\|_{r,p}} = W^{r,p}(U) \qquad p<+\infty$

(NB $\quad \overline{\mathcal{C}^\infty(U) \cap W^{r,\infty}(U)}^{\|\cdot\|_{r,\infty}} = \mathcal{C}^k(U) \cap W^{r,\infty}(U) \subsetneq W^{r,\infty}(U)$)

2) $\overline{\mathcal{C}_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{r,p}} = W^{r,p}(\mathbb{R}^n) \quad (= W_0^{r,p}(\mathbb{R}^n))$

whereas $U \subsetneq \mathbb{R}^n$ in general $\underbrace{W_0^{r,p}(U) \subsetneq W^{r,p}(U)}$

1) $\Rightarrow$ 2) $\quad \mathcal{C}_c^\infty(\mathbb{R}^n) \subseteq \mathcal{C}^\infty(\mathbb{R}^n) \cap W^{r,p}(\mathbb{R}^n)$ and it is dense there

$\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \qquad \psi(x) = \begin{cases} 0 & |x|>2 \\ 1 & |x|<1 \end{cases} \quad 0 \le \psi \le 1$

$\phi \in \mathcal{C}^\infty(\mathbb{R}^n) \cap W^{r,p}(\mathbb{R}^n) \qquad \phi_k(x) = \phi(x)\,\psi\left(\frac{x}{k}\right) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$\boxed{\phi_k \to \phi \text{ in } W^{r,p}(\mathbb{R}^n)}$ since

$\int_{\mathbb{R}^n} |\phi_k - \phi|^p = \int_{\mathbb{R}^n} |\phi|^p \left|\psi\left(\frac{x}{k}\right) - 1\right|^p \to 0$ lebesgue $+$

$$D\varphi_k = D\varphi \cdot \psi\left(\frac{x}{k}\right) + \frac{1}{k}\varphi\, D\psi\left(\frac{x}{k}\right)$$

$$\left[\int_{\mathbb{R}^n} |D\varphi_k - D\varphi|^p\right]^{1/p} \leq \left[\int_{\mathbb{R}^n} \underbrace{|D\varphi|^p \left|1-\psi\left(\frac{x}{k}\right)\right|^p}_{\to 0 \text{ Lebesgue}}\right]^{1/p} + \left[\underbrace{\int_{\mathbb{R}^n} \frac{1}{k^p}|D\psi|^p|\varphi|^p}_{\to 0 \text{ Lebesgue}}\right]^{1/p}.$$

So $\forall \phi \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ $\exists$ $\phi_k \in C_c^\infty(\mathbb{R}^n)$ $\phi_k \to \phi$ in $W^{n,p}(\mathbb{R}^n)$.

(1) $U_k = \{ x \in U \quad |x| < k, \quad \text{dist}(x, \mathbb{R}^n \setminus U) > \frac{1}{k} \}$, $U_0 = \emptyset$

$U = \bigcup_k U_k \qquad U_k \subset\subset U$

$V_0 := U_1 \quad V_1 := U_2 \setminus \overline{U}_1 \quad V_k := U_{k+1} \setminus \overline{U_{k-1}} = \{ x \in U_{k+1}$ such that either $|x| > k-1$ or $d(x, \mathbb{R}^n \setminus U) < \frac{1}{k-1} \}$

$\xi_k \in C_c^\infty(U) \quad 0 \leq \xi_k \leq 1 \quad \text{supp}\, \xi_k \subseteq V_k \quad \forall x \in U \quad \underbrace{\sum_k \xi_k(x)}_{\text{finite sum.}} = 1$

$f = \sum_k f \xi_k$ (finite sum for every x)

by Lemma $(f\xi_k) * \rho_h \longrightarrow f\xi_k$ in $W^{r,p}(U_{k+2} \setminus \overline{U}_{k-2})$

Fix $\delta > 0$ and find $h(\delta, k) > (k+1)(k+2)$ such that

$$\| f\xi_k * \rho_{h(\delta,k)} - f\xi_k \|_{W^{r,p}(V_k)} \leq \frac{\delta}{2^{k+1}}$$

$f = \sum f\xi_k$ $\qquad \phi = \sum_k [f\xi_k * \rho_{h(\delta,k)}] \in C^\infty(U)$

the sum is finite $\forall x$ $\qquad$ that sum is finite $\forall x$!

Let $A \subset V_{k_0} \subset\subset U$ $\qquad \|\phi - f\|_{W^{r,k}(A)} \leq \sum_{k \leq k_0+2} \|f\xi_k * \rho_{h(\delta,k)} - f\xi_k\|$

$$\leq \sum_{k \leq k_0+2} \frac{\delta}{2^{k+1}} \leq \delta$$

$\Longrightarrow \forall \delta > 0 \; \exists \; \phi = \phi_\delta = \sum_k f\xi_k * \rho_{h(\delta,k)}$ such that

$\|f - \phi_\delta\|_{W^{r,p}(A)} \leq \delta \quad \forall A \subset\subset U \rightarrow$

$\Rightarrow \|f - \phi_\delta\|_{W^{k,p}(U)} \leq \delta$ by monotone convergence
($\chi_{U_k} \to \chi_U$ monotonically)

~ . ~

CONTINUOUS EMBEDDINGS of $W^{1,p}(\mathbb{R}^n)$.

$p < n \to$ GNS
$p = n$ "GNS"
$p > n \to$ MORREY

GAGLIARDO-NIRENBERG-SOBOLEV INEQUALITY

$p < n$. Let p^* = SOBOLEV CONJUGATE $p^* = \dfrac{np}{n-p}$ $\left(\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}\right)$

($p^* \to +\infty$ as $p \nearrow n$) $p^* > p$

$\forall p < n$ $\exists C = C(n,p)$ (explicit) ≤ 1 for $p = 1$
$\leq \dfrac{n-1}{n-p} \cdot p$ $p > 1$

such that $\forall f \in W^{1,p}(\mathbb{R}^n)$ it holds $f \in L^{p^*}(\mathbb{R}^n)$

and $\|f\|_{L^{p^*}} \leq C(n,p) \underbrace{\|\,|Df|\,\|_{L^p}}_{\| \sum_i |\frac{\partial f}{\partial x_i}| \|_{L^p}}$ $\left(\leq \tilde{C}(n,p) \|f\|_{W^{1,p}_k(\mathbb{R}^n)}\right)$

Obs 1 $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ continuously

Obs 2 since $f \in W^{1,p}(\mathbb{R}^n) \to f \in L^p(\mathbb{R}^n)$ by GNS $f \in L^q(\mathbb{R}^n)$ $\forall q \in [p, p^*]$

$$\Rightarrow \|f\|_{L^q} \le \|f\|_p^{\theta} \|f\|_{p^*}^{1-\theta} \le C(n,p)^{1-\theta} \|f\|_p^{\theta} \cdot \| |Df| \|_{L^p}^{1-\theta} \le$$

$$\frac{1}{q} = \frac{\theta}{p} + \frac{(1-\theta)}{p^*} \qquad \le \tilde{C}(n,p) \|f\|_{W^{1,p}}$$

Obs 3 why p^*? by rescaling $f_\lambda(x) = f(\lambda x)$

$$\|f_\lambda\|_q = \lambda^{-n/q} \|f\|_q \qquad \| |Df_\lambda| \|_p = \lambda \lambda^{-n/p} \| |Df| \|_p$$

$$\Rightarrow \lambda^{-n/q} = \lambda^{1-n/p} \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n} \qquad q = p^* !$$

Obs 4 $C(n,1) \le 1$ $\qquad C(n,p) \le \frac{n-1}{n-p} \cdot p$

$C(n,p)$ is explicit (optimal constant $\to$ $C(n,1)$ related to isoperimetric ratio). Family of functions which satisfies equality are AUBIN-TALENTI functions $p \neq 1$

$$u(x) = \left(a + b\,|x|^{p/p-1}\right)^{1-\frac{n}{p}}.$$