COMPUTABILITY (17/11/2025)

OBSERVATION: A function which is total and not computable

$$f(x) = \begin{cases} \boxed{\varphi_x(x) + 1} & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \varphi_U(x,x) + 1 & \boxed{\text{if } \varphi_x(x)\downarrow} \\ 0 & \boxed{\text{otherwise}} \end{cases}$$

HALTING PROBLEM: The predicate below is UNDECIDABLE

$$Halt(x) = \begin{cases} true & \text{if } \varphi_x(x)\downarrow \quad (\text{i.e. } x \in W_x) \\ false & \text{if } \varphi_x(x)\uparrow \quad (\text{i.e. } x \notin W_x) \end{cases}$$

idea: by contradiction: we show that assuming $Halt(x)$ decidable we can prove f computable

$$f(x) = \begin{cases} \boxed{\varphi_x(x) + 1} & \text{if } \overbrace{\varphi_x(x)\downarrow}^{Halt(x)} \\ 0 & \text{otherwise} \end{cases}$$

$$\cancel{=} \underbrace{(\varphi_U(x,x) + 1) \cdot \underbrace{\chi_{Halt}(x)}_{\substack{0 \text{ if } \varphi_x(x)\uparrow \\ 1 \text{ if } \varphi_x(x)\downarrow}}}_{\substack{\varphi_x(x)+1 \quad \text{if } \varphi_x(x)\downarrow \\ \uparrow \quad \text{if } \varphi_x(x)\uparrow}}$$

instead

$$f(x) = \text{“}(\mu(t,y,z).\big((S(x,x,y,t) \wedge z = y+1 \wedge Halt(x)) \vee (z = 0 \wedge \neg Halt(x))\big)_z\text{”} \quad \begin{matrix} \rightsquigarrow y+1 \\ \rightsquigarrow 0 \end{matrix}$$

$$= \big(\mu w. \big((S(x,x,(w)_2,(w)_1) \wedge (w)_3 = (w)_2 + 1 \wedge Halt(x)) \vee ((w)_3 = 0 \wedge \neg Halt(x)) \big)\big)_3$$

$(w)_1 = t$
$(w)_2 = y$
$(w)_3 = z$

if you call

$$Q(x,\omega) = \left(S(x, x, (\omega)_2, (\omega)_1) \wedge (\omega)_3 = (\omega)_2 + 1 \wedge Halt(x) \right) \vee$$
$$(\omega)_3 = 0 \wedge \neg Halt(x))$$

decidable

$$f(x) = \left(\mu\omega.\ |\chi_Q(x,\omega) - 1|\right)_3$$

computable (minimalisation / composition of computable functions)

$\Rightarrow$ contradiction

$\Downarrow$

Halt(x) not decidable. $\square$

<u>EXERCISE</u> : Let $Q(x)$ decidable predicate

$f_1, f_2 : \mathbb{N} \to \mathbb{N}$ computable

and define

$$f(x) = \begin{cases} f_1(x) & \text{if } Q(x) \\ f_2(x) & \text{if } \neg Q(x) \end{cases}$$ is computable

<u>proof</u>

If f_1, f_2 are <u>total</u>

$$f(x) = \underbrace{f_1(x) \cdot \underset{\substack{1 \ \text{if } Q(x) \text{ true} \\ 0 \ \text{otherwise}}}{\underset{\parallel}{\chi_Q(x)}}}_{\substack{f_1(x) \ \text{if } Q(x) \text{ true} \\ 0 \ \text{otherwise}}} + \underbrace{f_2(x) \cdot \underset{\substack{1 \ \text{if } Q(x) \text{ false} \\ 0 \ \text{otherwise}}}{\underset{\parallel}{\chi_{\neg Q}(x)}}}_{\substack{f_2(x) \ \text{if } Q(x) \text{ false} \\ 0 \ \text{otherwise}}}$$

$\rightsquigarrow$ f is a composition of computable functions hence computable

In general, let $e_1, e_2 \in \mathbb{N}$ s.t. $\varphi_{e_1} = f_1$ and $\varphi_{e_2} = f_2$

$$f(x) = \Big(\mu(t,y).\ \big(S(e_1, x, y, t) \wedge Q(x) \big) \vee \big(S(e_2, x, y, t) \wedge \neg Q(x) \big) \Big)_y$$

$$= \Big(\mu w. \underbrace{\big(S(e_1, x, (w)_1, (w)_2) \wedge Q(x)\big) \vee \big(S(e_2, x, (w)_1, (w)_2) \wedge \neg Q(x)\big)}_{\text{decidable}} \Big)_1$$

$(w)_1 = y$
$(w)_2 = t$

$\rightsquigarrow$ f is computable $\square$

EXERCISE : TOTALITY

$Tot(x) \equiv$ "φ_x is total" $\equiv$ "P_x is terminating on every input"

is undecidable.

In fact

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } Tot(x) \\ 0 & \text{otherwise} \end{cases}$$

$\to$ f is total $\begin{cases} \text{if } Tot(x) & f(x) = \varphi_x(x) + 1 \downarrow \\ \text{if } \neg Tot(x) & f(x) = 0 \end{cases}$ always defined

$\to$ f is different from all total computable functions

(in fact $\forall x \in \mathbb{N}$ if φ_x is total then $f \neq \varphi_x$ on x

$$f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$$)

$\hookrightarrow$ f is not computable

Assume, by contradiction, that $Tot(x)$ is decidable. Then

$$f(x) = \begin{cases} f_1(x) & \text{if } Tot(x) \\ f_2(x) & \text{otherwise} \end{cases}$$

where

$f_1(x) = \varphi_x(x) + 1 = \Psi_U(x, x) + 1$
$f_2(x) = 0$

computable functions

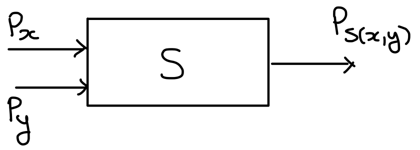
the f is computable (by previous exercise), contradiction!

$\Rightarrow$ $Tot(x)$ not decidable. $\square$

EFFECTIVE OPERATIONS ON COMPUTABLE FUNCTIONS

① there is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\forall x, y \qquad \varphi_{s(x,y)}(z) = \varphi_x(z) * \varphi_y(z) \qquad \forall z \in \mathbb{N}$$



$P_x, P_y \rightsquigarrow$

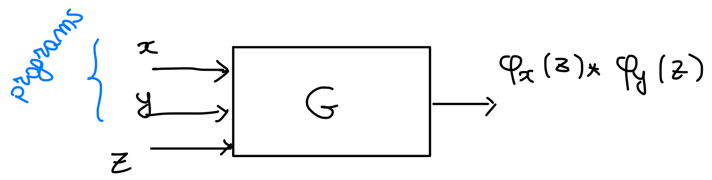
def $P_{s(x,y)}(z)$:
$v_x = P_x(z)$
$v_y = P_y(z)$
return $v_x * v_y$

define a function $g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x, y, z) = \varphi_x(z) * \varphi_y(z)$$

$$= \Psi_U(x, z) * \Psi_U(y, z)$$

computable (composition of computable functions)



function s takes G and hard code x, y into G

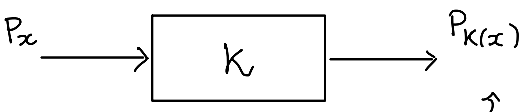
by smn theorem (corollary of) there is $s : \mathbb{N}^2 \to \mathbb{N}$ total computable such that $\forall x, y, z$

$$\varphi_{s(x,y)}(z) = g(x, y, z) = \varphi_x(z) * \varphi_y(z)$$

<u>EXERCISE</u> : Effectiveness of inverting a function

There is a total computable function $K : \mathbb{N} \to \mathbb{N}$ such that

$\forall x \in \mathbb{N}$ if φ_x is injective then $\varphi_{K(x)}(z) = \varphi_x^{-1}(z) \quad \forall z$



computes the inverse of the function computed by P_x

idea: in order to compute $\varphi_x^{-1}(z)$

$y=0$? $\varphi_x(0)=z$?
$y=1$? $\varphi_x(1)=z$?

$\vdots$

define

$g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x,z) = \varphi_x^{-1}(z) = \begin{cases} y & \text{s.t. } \varphi_x(y)=z \text{ when it exists} \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \left(\mu(y,t) \quad S(x,y,z,t)\right)_y$$

$$= \left(\mu\omega. \quad S(x, (\omega)_1, z, (\omega)_2)\right)_1$$

$$= \left(\mu\omega \mid \chi_S(x, (\omega)_1, z, (\omega)_2) - 1 \mid\right)_1$$

computable

Hence, by smn theorem, there is $k : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$$\varphi_{k(x)}(z) = g(x,z) = (\varphi_x)^{-1}(z) \qquad \text{if } \varphi_x \text{ is injective}$$

What do we get when φ_x is not injective?

$\varphi_{k(x)}(z)$ is one of the (possibly many) y s.t. $\varphi_x(y)=z$

$\boxed{\text{QUESTION}}$: Given $f : \mathbb{N} \to \mathbb{N}$ computable, define for all $z \in \mathbb{N}$

$$f^{-1}[z] = \{ y \in \mathbb{N} \mid f(y) = z \}$$

and then let

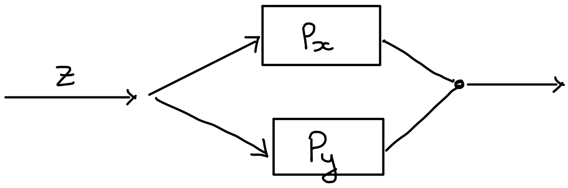
$$g(z) = \begin{cases} \min(f^{-1}[z]) & \text{if } f^{-1}[z] \neq \emptyset \\ \uparrow & \text{otherwise} \end{cases}$$

Is g computable?

EXERCISE : There is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ s.t.

$$\forall x, y \in \mathbb{N} \qquad W_{s(x,y)} = W_x \cup W_y$$

$$\varphi_{s(x,y)}(z)\downarrow \quad \text{iff} \quad \varphi_x(z)\downarrow \ \text{or} \ \varphi_y(z)\downarrow$$



$g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x, y, z) = \begin{cases} \overset{1}{\downarrow} & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$\mathbb{1}\left(\mu t.\ H(x, z, t) \vee H(y, z, t)\right)$$

$$\text{where} \quad \mathbb{1} : \mathbb{N} \to \mathbb{N} \qquad \mathbb{1}(x) = 1 \quad \forall x$$

g is computable, hence by smn theorem there is $s : \mathbb{N}^2 \to \mathbb{N}$ total and computable s.t.

$$\underline{\varphi_{s(x,y)}}(z) = g(x, y, z) = \begin{cases} 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Hence s is the desired function, i.e. $W_{s(x,y)} = W_x \cup W_y$

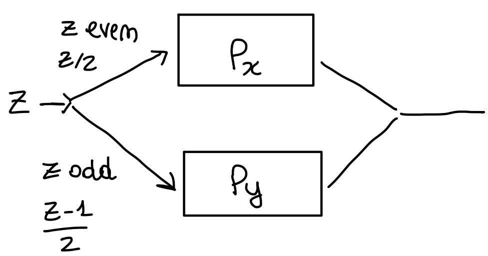
$$z \in W_{s(x,y)} \quad \text{iff} \quad \underset{\substack{\| \\ g(x,y,z)}}{\varphi_{s(x,y)}(z)}\downarrow \quad \text{iff} \quad \varphi_x(z)\downarrow \ \text{or} \ \varphi_y(z)\downarrow$$

$$\text{iff} \quad z \in W_x \quad \text{or} \quad z \in W_y$$

EXERCISE There is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that

$$E_{s(x,y)} = E_x \cup E_y$$

($P_{s(x,y)}$ produces as outputs all and only the outputs of P_x and P_y)



	0	1	2	3	4
P_x	1	2	$\cancel{\uparrow}$	4	0
P_y	$\uparrow$	0	3	4	3

$$g(x,y,z) = \begin{cases} \varphi_x(z/2) & \text{if } z \text{ even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{if } z \text{ odd} \end{cases}$$

$$\cancel{=} \quad \cancel{\varphi_U(x, qt(2,z)) \cdot \overline{sg}(rm(2,z)) + \varphi_U(y, qt(2,z)) \cdot sg(rm(2,z))}$$

$$= \Big(\mu(v,t). \quad \big(S(x, z/2, v, t) \wedge z \text{ even}\big) \vee \big(S(y, \tfrac{z-1}{2}, v, t) \wedge z \text{ odd}\big)\Big)_v$$

$$= \underbrace{\Big(\mu w. \quad \big(S(x, qt(2,z), (w)_1, (w)_2) \wedge z \text{ even}\big) \vee \big(S(y, qt(2,z), (w)_1, (w)_2) \wedge z \text{ odd}\big)\Big)_1}_{\text{computable}}$$

By smn theorem there is $s : \mathbb{N}^2 \to \mathbb{N}$ total and computable s.t.

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} \varphi_x(z/2) & \text{if } z \text{ is even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{otherwise } (z \text{ is odd}) \end{cases}$$

I claim that s is the desired function. $E_{s(x,y)} = E_x \cup E_y$

($\subseteq$) $E_{s(x,y)} \subseteq E_x \cup E_y$

Let $v \in E_{s(x,y)}$ i.e. $\exists z$ s.t. $\varphi_{s(x,y)}(z) = v$

then either

$$\left.\begin{array}{lll} v = \varphi_x(z/2) & \leadsto & v \in E_x \\ \text{or} \quad v = \varphi_y\left(\frac{z-1}{2}\right) & \leadsto & v \in E_y \end{array}\right\} \to v \in E_x \cup E_y$$

($\supseteq$) $E_{s(x,y)} \supseteq E_x \cup E_y$

let $v \in E_x \cup E_y$ then either

(a) $v \in E_x \leadsto \exists z$ s.t. $\varphi_x(z) = v$

$\leadsto \varphi_{s(x,y)}(2z) = \varphi_x(\cancel{2}z/\cancel{2}) = \varphi_x(z) = v$

$\leadsto v \in E_{s(x,y)}$

or (b) $v \in E_y$

same

Hence we are done. $\square$

<u>EXERCISE</u>: variant of URM

URM^P

$Z(n)$

$\cancel{S(n)}$ $P(n)$ $r_n \leftarrow r_n \dot{-} 1$

$T(m,n)$

$J(m,n,t)$

$\mathcal{C}^P$ URM^P computable functions vs $\mathcal{C}$

<u>EXERCISE</u>: Are there $f, g : \mathbb{N} \to \mathbb{N}$ functions s.t.

1. f computable, g not computable $f \circ g$ computable?
2. f not computable, g not computable $f \circ g$ computable?