Matrici di cambio di base

$V \qquad B_1$ e B_2 due basi di V

$$\boxed{T_{B_1}^{B_2} = A_{B_1, B_2, id_V}}$$

1) $V = \mathbb{R}^n \qquad B = \{v_1, \dots, v_n\} \qquad \mathcal{E} = \{e_1, \dots, e_n\}$

$$T_B^{\mathcal{E}} = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \in GL_n(\mathbb{R})$$

$$T_{\mathcal{E}}^B = \left(T_B^{\mathcal{E}}\right)^{-1}$$

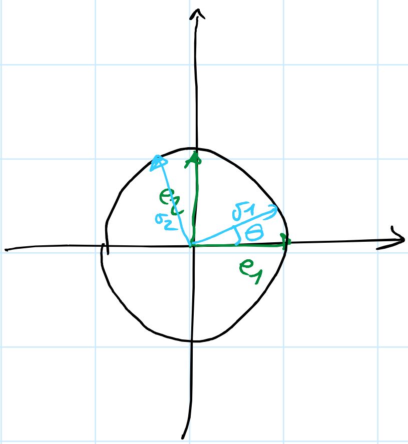
2) $\mathbb{R}^n \quad B_1, B_2$

$$\boxed{T_{B_1}^{B_2} = \underset{\substack{\uparrow \\ \text{dopo}}}{T_{\mathcal{E}}^{B_2}} \cdot T_{B_1}^{\mathcal{E}} = \left(T_{B_2}^{\mathcal{E}}\right)^{-1} T_{B_1}^{\mathcal{E}}}$$

$$\begin{tikzcd}
(\mathbb{R}^n, B_1) \arrow[rr, "T_{B_1}^{B_2}"] \arrow[dr, "T_{B_1}^{\mathcal{E}}"'] & & (\mathbb{R}^n, B_2) \\
& (\mathbb{R}^n, \mathcal{E}) \arrow[ur, "T_{\mathcal{E}}^{B_2}"'] &
\end{tikzcd}$$

Osservazione

$$R_\theta = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix}$$



$$\det(R_\theta) = \cos^2\theta + \operatorname{sen}^2\theta = 1$$

$$R_\theta = A_{\mathcal{E}, \mathcal{E}, r_\theta}$$

$$R_\theta = T_{B}^{\mathcal{E}} \qquad B = \left\{ \overset{v_1}{\begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix}} \right\}$$

$$= A_{B, \mathcal{E}, id_{\mathbb{R}^2}}$$

Uso delle matrici di cambio di base per il calcolo delle coordinate.

V, B_1 e B_2 basi di V

Fissiamo B_1 $\qquad f: V \longrightarrow \mathbb{R}^n$

$B_1 = \{v_1, \dots, v_n\}$ $\qquad v = \sum_{i=1}^{n} a_i v_i \longrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (v)_{B_1}$ coordinate del vettore v nella base B_1.

$$\begin{tikzcd} V \arrow[r, "id_V"] \arrow[d, "B_1"'] & V \arrow[d, "B_2"] \\ \mathbb{R}^n \arrow[r, "T_{B_1}^{B_2}"] & \mathbb{R}^n \end{tikzcd}$$

Se $(v)_{B_1} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\qquad \boxed{(v)_{B_2} = T_{B_1}^{B_2} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = T_{B_1}^{B_2} (v)_{B_1}}$

Esempio: calcolare le coordinate del vettore $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ nella base $B = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

$V = \mathbb{R}^2$. $\qquad (v)_{\mathcal{E}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad (v)_B$

Con $B_1 = \mathcal{E}$ base $B_2 = B$ $\qquad (v)_{B_2} = T_{B_1}^{B_2} (v)_{B_1}$

$$(v)_B = T_{\mathcal{E}}^{B} (v)_{\mathcal{E}} = T_{\mathcal{E}}^{B} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$T_{\mathcal{E}}^{B} = \left(T_B^{\mathcal{E}} \right)^{-1}$$

$B = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad T_B^{\mathcal{E}} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ $\quad \det T_B^{\mathcal{E}} = 2$

$$\left(\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \quad T_{\mathcal{E}}^{B} = \begin{pmatrix} 1/2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$(v)_B = \begin{pmatrix} 1/2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3+1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -2 \end{pmatrix}$$

$(w)_B \qquad w = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \qquad (w)_B = \begin{pmatrix} 1/2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -5 \end{pmatrix}$

Uso generale delle matrici di cambio di base

Sia $f: V \longrightarrow W$ lineare

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Fissiamo $\mathcal{E}_n = \{e_1, \dots, e_n\}$ b.c. $\mathbb{R}^n$

$\mathcal{E}_m = \{e_1, \dots, e_m\}$ b.c. $\mathbb{R}^m$

$A = A_{\mathcal{E}_n, \mathcal{E}_m, f}$

Sia $B_1 = \{v_1, \dots, v_n\}$ base di $\mathbb{R}^n$

e $B_2 = \{w_1, \dots, w_m\}$ base di $\mathbb{R}^m$

Calcolare $M = A_{B_1, B_2, f}$ con le matrici di cambio di base

$$\begin{tikzcd}
(\mathbb{R}^n, B_1) \arrow[r, "M"', "f"] \arrow[d, "T^{\mathcal{E}_n}_{B_1}"'] & (\mathbb{R}^m, B_2) \\
(\mathbb{R}^n, \mathcal{E}_n) \arrow[r, "A", "f"'] & (\mathbb{R}^m, \mathcal{E}_m) \arrow[u, "T^{B_2}_{\mathcal{E}_m}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, "\text{Cerco}"] \arrow[d] & \bullet \\
\bullet \arrow[r, "\text{Conosco}"'] & \bullet \arrow[u]
\end{tikzcd}$$

$$\boxed{M = T^{B_2}_{\mathcal{E}_m} \, A \, T^{\mathcal{E}_n}_{B_1}}$$

$m \times m \quad m \times n \quad n \times n$

$m \times \quad\quad n$

$T^{\mathcal{E}_n}_{B_1} \in M_{n,n}(\mathbb{R})$

$A \in M_{m,n}(\mathbb{R})$

$T^{B_2}_{\mathcal{E}_m} \in M_{m,m}(\mathbb{R})$

Esercizio:

$$f: \mathbb{R}^3 \longrightarrow M_{2,2}(\mathbb{R})$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z & y+z \\ y-2z & x+5z \end{pmatrix} = w \qquad \begin{pmatrix} x+2z \\ y+z \\ y-2z \\ x+5z \end{pmatrix} = w_{\mathcal{E}_{M_{2,2}(\mathbb{R})}}$$

1) Determinare la matrice $A = A_{\mathcal{E}_3, \mathcal{E}_{M_{2,2}(\mathbb{R})}, f}$

dove $\mathcal{E}_3 = \{e_1, e_2, e_3\}$

$$\mathcal{E}_{M_{2,2}(\mathbb{R})} = \left\{ e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Determinare la matrice $M = A_{B, \mathcal{C}, f}$ con

$$B = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \mathcal{C} = \{e_{12}, e_{11}, e_{22}, e_{21}\}.$$

Svolg.:

$$f: \mathbb{R}^3 \longrightarrow M_{22}(\mathbb{R})$$

$$A = A_{\mathcal{E}_3,\, \mathcal{E}_{M_{2,2}(\mathbb{R})},\, f} \in M_{4,3}(\mathbb{R})$$

$$\begin{pmatrix} x+2z \\ y+z \\ y-2z \\ x+5z \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$\begin{tikzcd}
(\mathbb{R}^3, \mathcal{B}) \arrow[r, "M"] \arrow[d, "T_{\mathcal{B}}^{\mathcal{E}_3}"'] & (M_{2,2}(\mathbb{R}), \mathcal{C}) \\
(\mathbb{R}^3, \mathcal{E}_3) \arrow[r, "A"] & (M_{2,2}(\mathbb{R}), \mathcal{E}_{M_{2,2}(\mathbb{R})}) \arrow[u, "T_{\mathcal{E}_{M_{2,2}(\mathbb{R})}}^{\mathcal{C}}"']
\end{tikzcd}$$

$$\underset{4\times 3}{M} = \underset{4\times 4}{T_{\mathcal{E}_{M_{2,2}(\mathbb{R})}}^{\mathcal{C}}} \; \underset{4\times 3}{A} \; \underset{3\times 3}{T_{\mathcal{B}}^{\mathcal{E}_3}}$$

$$T_{\mathcal{B}}^{\mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad T_{\mathcal{E}_{M_{2,2}(\mathbb{R})}}^{\mathcal{C}} = \left(T_{\mathcal{C}}^{\mathcal{E}_{M_{2,2}(\mathbb{R})}} \right)^{-1}$$

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$T_{\mathcal{C}}^{\mathcal{E}_{M_{2,2}(\mathbb{R})}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} 2^\circ R \\ 1^\circ R \\ 4^\circ R \\ 3^\circ R \end{array} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \qquad \left(\mathbb{I}_4 \;\middle|\; T_{\mathcal{E}_{M_{2,2}(\mathbb{R})}}^{\mathcal{C}} \right)$$

$$M = \underset{4\times 4}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \left[\underset{4\times 3}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}} \right]$$

4×3

2) Determinare $B_{Ker f}$ e $B_{Im f}$.

$$f: \mathbb{R}^3 \longrightarrow M_{2,2}(\mathbb{R}) \qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z & y+z \\ y-2z & x+5z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Ker} f = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\begin{cases} x+2z=0 \\ y+z=0 \\ y-2z=0 \\ x+5z=0 \end{cases} \quad \begin{cases} x=-2z \\ 2z+z=0 \\ y=2z \\ x=-5z \end{cases} \quad \begin{cases} x=0 \\ 3z=0 \\ y=0 \\ x=0 \end{cases}$$

$$\boxed{\begin{array}{ll} \text{Ker} f = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} & \dim \text{Ker} f = 0 \\ f \text{ è iniettiva} & B_{Ker f} = \emptyset \end{array}}$$

$$3 = \dim \mathbb{R}^3 = \dim \text{Ker} f + \dim \text{Im} f \Longrightarrow \boxed{\dim \text{Im} f = 3}$$
$$3 = 0 + 3$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z & y+z \\ y-2z & x+5z \end{pmatrix} \qquad \text{Im} f = \langle f(e_1), f(e_2), f(e_3) \rangle =$$

$$= \langle \underset{\text{coeff. di } x}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{\text{coeff. di } y}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \underset{\text{coeff. di } z}{\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}} \rangle$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}$$

$$\boxed{B_{Im f} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix} \right\}}$$

f non è suriettiva perché $\dim \text{Im} f = 3$ ma $\dim M_{2,2}(\mathbb{R}) = 4$.

3) Determinare, se esiste, un'applicazione lineare $g: M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ tale che $\boxed{g \circ f = id_{\mathbb{R}^3}}$.

Se esiste è unica?

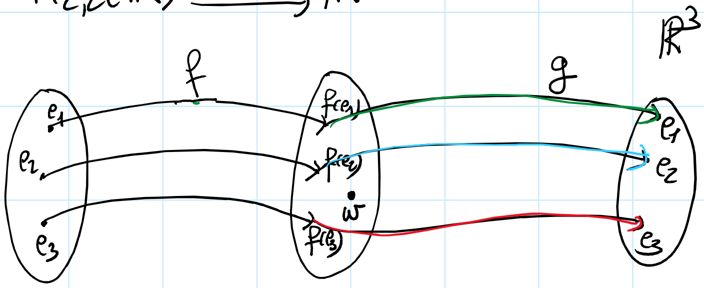
$$\mathbb{R}^3 \xrightarrow{f} M_{2,2}(\mathbb{R}) \xrightarrow{g} \mathbb{R}^3$$

$$\mathbb{R}^3 \xrightarrow{\quad id \quad} \mathbb{R}^3$$

$g \circ f = id_{\mathbb{R}^3}$ è equivalente a richiedere

$$\boxed{\begin{aligned} g(f(e_1)) &= e_1 \\ g(f(e_2)) &= e_2 \\ g(f(e_3)) &= e_3 \end{aligned}}$$

$g : M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$



$$B_{M_{2,2}(\mathbb{R})} = \{ f(e_1), f(e_2), f(e_3), w \} =$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & -2 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 5 \end{pmatrix} \quad rg \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4$$

Esiste un'applicazione lineare $g : M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ tale che

1. $\boxed{g(f(e_1)) = g\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

2. $\boxed{g(f(e_2)) = g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

3. $\boxed{g(f(e_3)) = g\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

4. $\boxed{g(w) = g\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}}$ $\quad a, b, c \in \mathbb{R}$

$$\mathbb{R}^3 \xrightarrow{\ f\ } M_{2,2}(\mathbb{R}) \xrightarrow{\ g\ } \mathbb{R}^3 \qquad \boxed{g \circ f = id_{\mathbb{R}^3}}$$

$$(\mathbb{R}^3, \mathcal{E}_3) \xrightarrow{\ A\ } (M_{2,2}(\mathbb{R}), \mathcal{E}_{M_{2,2}(\mathbb{R})}) \xrightarrow{\ P\ } (\mathbb{R}^3, \mathcal{E}_3)$$

$$P = A_{\mathcal{E}_{M_{2,2}(\mathbb{R})}, \mathcal{E}_3, g} = M_{3,4}(\mathbb{R}) \qquad g \circ f = id_{\mathbb{R}^3}$$

$$\boxed{P \cdot A = I_3}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

12 incognite p_{ij} con $1 \leq i \leq 3$, $1 \leq j \leq 4$

9 equazioni

---

4) Esiste un'applicazione lineare $h: M_{2,2}(\mathbb{R}) \to \mathbb{R}^3$ tale che $f \circ h = id_{M_{2,2}(\mathbb{R})}$?

$$f\left(h\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$f: \mathbb{R}^3 \to M_{2,2}(\mathbb{R})$

$\forall\, w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Se h esistesse $f\left(h\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow$ f dovrebbe essere suriettiva

$f(h(w)) = w$ ma non lo era.

$$M_{2,2}(\mathbb{R}) \xrightarrow{h} \mathbb{R}^3 \xrightarrow{f} M_{2,2}(\mathbb{R})$$

$$\boxed{\underset{4\times 3}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix}} \underset{3\times 4}{\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{pmatrix}} = A\,P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

12 incognite
16 equazione

impossibile

Definizione di relazione di equivalenza:

$A, B \in M_{n,n}(\mathbb{R})$ $\qquad A \sim B$

① Riflessiva $\quad A \sim A$

② Simmetrica $\quad A \sim B$ allora $B \sim A$

③ Transitiva $\quad A \sim B$ e $B \sim C$ allora $A \sim C$