ESERCIZIO: Si consideri $\Phi_h = \mathbb{R}^4 \to \mathbb{R}^3$ con $h \in \mathbb{R}$

definito da: $\Phi_h\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\Phi_h\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\h\\-1\end{pmatrix}$, $\Phi_h\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}h\\2\\h-2\end{pmatrix}$, $\Phi_h\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\h+2\\-2\end{pmatrix}$

a) determinare $\Phi_h\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}$:

$\mathcal{V} = \left\{\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ é una base di $\mathbb{R}^4$

$$\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix} = \alpha\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + \gamma\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \delta\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1&0&0&0&1\\-1&1&0&0&0\\0&-1&1&0&-1\\0&0&0&1&-1\end{array}\right) \longrightarrow \left(\begin{array}{cccc|c}1&0&0&0&1\\0&1&0&0&+1\\0&0&1&0&0\\0&0&0&1&-1\end{array}\right) \Rightarrow \begin{cases}\alpha = 1\\ \beta = 1\\ \gamma = 0\\ \delta = -1\end{cases}$$

$$\Rightarrow \Phi_h\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \begin{pmatrix}-1\\h\\-1\end{pmatrix} - \begin{pmatrix}0\\h+2\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

b) per $h=1$ trova $\ker\Phi_1$ e $\operatorname{Im}\Phi_1$:

$\Phi_1\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\Phi_1\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$, $\Phi_1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\Phi\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3\\-2\end{pmatrix}$

$\mathcal{V}$ é base di $\mathbb{R}^4 \Rightarrow \operatorname{Im}\Phi_1 = \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix}\begin{pmatrix}-1\\1\\-1\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}\begin{pmatrix}0\\3\\-2\end{pmatrix}\right\rangle$

$B_{\operatorname{Im}\Phi} = \left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}$ NB. $\begin{pmatrix}1\\2\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\-2\end{pmatrix}$

$\dim\mathbb{R}^4 = \dim\ker\Phi_1 + \dim\operatorname{Im}\Phi_1 \Rightarrow \dim\ker\Phi_1 = 2$

inoltre $\forall h \in \mathbb{R} \Rightarrow \Phi_1\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1\\0\\-1\\-1\end{pmatrix} \in \ker\Phi_1$

manca un vettore che deve essere l.i. rispetto a quello trovato

$$\Phi_1\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} - \Phi_1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} \in \ker\Phi_1$$

$B_{\ker\Phi_1} = \left\{\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}\right\}$

c) per quali $h \in \mathbb{R}$, Φ_h non è suriettivo? quando $\text{Im}\,\Phi_h < 3$

$$\begin{matrix} 1 & 2 & -1 \\ -1 & h & -1 \\ h & 2 & h-2 \\ 0 & h+2 & -2 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & -1 \\ 0 & h+2 & -2 \\ 0 & h+2 & -2 \\ 0 & 2-2h & 2h-2 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & -1 \\ 0 & h+2 & -2 \\ 0 & 2-2h & 2h-2 \\ 0 & 0 & 0 \end{matrix}$$

caso $h=1$ non è suriettiva $(rk = 2)$

per $h \neq 1$ divido per $2-2h$ $\Rightarrow$ $\begin{matrix} 1 & 2 & -1 \\ 0 & h+2 & -2 \\ 0 & 1 & -1 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & h \end{matrix}$

per $h = 0$ non è suriettiva $(rk = 2)$

per $h \neq 0, 1$ $\text{Im}\,\Phi_h = \mathbb{R}^3$, $\ker \Phi_h = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \right\rangle$

per $h = 0$ $B_{\text{Im}\,\Phi} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$\Phi_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ e $\Phi_0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

$\Rightarrow \Phi_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \Phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \in \ker \Phi_0$

$\Rightarrow B_{\ker \Phi} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \right\}$

d) determinare $\Phi_h^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\forall h \in \mathbb{R}$:

$$\begin{pmatrix}1\\-1\\1\end{pmatrix} = \frac{1}{h}\left(\begin{pmatrix}h\\2\\h-2\end{pmatrix} - \begin{pmatrix}0\\h+2\\-2\end{pmatrix}\right) \Rightarrow \Phi_h^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \frac{1}{h}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\-1\\-1\end{pmatrix} \right\rangle$$

caso $h=0$ $\rightarrow$ $\begin{pmatrix}1\\-1\\1\end{pmatrix} \notin \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle = \operatorname{Im}\Phi_0$

$$\Rightarrow \Phi_0^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \emptyset$$

caso $h=1$ $\rightarrow$ $\begin{pmatrix}1\\-1\\1\end{pmatrix} = -\begin{pmatrix}-1\\1\\-1\end{pmatrix} = -\Phi_1\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \Phi_1\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$

$$\Rightarrow \Phi_1^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} \right\rangle$$

metodo alternativo:

a) $$\left(\begin{array}{cccc|ccc} 1 & -1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & -1 & h & -1 \\ 0 & 0 & 1 & 0 & h & 2 & h-2 \\ 0 & 0 & 0 & 1 & 0 & h+2 & -2 \end{array}\right) \xrightarrow{II+III} \left(\begin{array}{cccc|ccc} 1 & -1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 & h-1 & h+2 & h-3 \\ 0 & 0 & 1 & 0 & h & 2 & h-2 \\ 0 & 0 & 0 & 1 & 0 & h+2 & -2 \end{array}\right)$$

$$\xrightarrow{I+II} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & h & h+4 & h-4 \\ 0 & 1 & 0 & 0 & h-1 & h+2 & h-3 \\ 0 & 0 & 1 & 0 & h & 2 & h-2 \\ 0 & 0 & 0 & 1 & 0 & h+2 & -2 \end{array}\right)$$

$$\Phi_h\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} h & h-1 & h & 0 \\ h+4 & h+2 & 2 & h+2 \\ h-4 & h-3 & h-2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} h\cdot x + (h-1)y + h\cdot z + 0\cdot t \\ (h+4)\cdot x + (h+2)y + 2\cdot z + (h+2)t \\ (h-4)x + (h-3)y + (h-2)z - 2\cdot t \end{pmatrix}$$

$$\Phi_h\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} h + 0 + (-h) + 0 \\ h+4 + 0 + 2\cdot(-1) - h - 2 \\ h-4 + 0 + (-h+2) + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b) $h = 1$

$$\Phi\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + z \\ 5x + 3y + 2z + 3t \\ -3x - 2y - z - 2t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 5 & 3 & 2 & 3 \\ -3 & -2 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$\ker \Phi_1$:
$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 5 & 3 & 2 & 3 & 0 \\ -3 & -2 & -1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -3 & 3 & 0 \\ 0 & -2 & 2 & -2 & 0 \end{array}\right)$$

$$\begin{array}{ccccc} x & y & z & t & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x = -z \\ y = +z - t \end{array}$$

$$\begin{pmatrix} -z \\ +z-t \\ z \\ t \end{pmatrix} = z\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$B\ker \Phi_1 = \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \in \ker \Phi_1 \quad \text{infatti} \quad \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \checkmark$$

$B\ Im\ \Phi$: $\dim \mathbb{R}^4 = \dim \ker \Phi_1 + \dim Im\ \Phi_1$

$\Rightarrow$ prendo 2 vettori l.i. dalla matrice di partenza!

NON quella in forma a scala (le operazioni elementari hanno alterato il sottospazio!)

$$B\ Im\ \Phi = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}$$

c) vediamo per quali $k \in \mathbb{R}$ Φ_k non é suriettiva: $(\dim(\mathrm{Im}\,\Phi_k) \neq 3$ non é suriettiva$)$

$$\begin{matrix} k & k+4 & k-4 \\ k-1 & k+2 & k-3 \\ k & 2 & k-2 \\ 0 & k+2 & -2 \end{matrix} \xrightarrow{I-II} \begin{matrix} 1 & 2 & -1 \\ 0 & k+2 & -2 \\ k & 2 & k-2 \\ k-1 & k+2 & k-3 \end{matrix} \xrightarrow[IV-III]{III-kI} \begin{matrix} 1 & 2 & -1 \\ 0 & k+2 & -2 \\ 0 & 2-2k & 2k-2 \\ -1 & k & -1 \end{matrix}$$

$$\xrightarrow{IV+I} \begin{matrix} 1 & 2 & -1 \\ 0 & k+2 & -2 \\ 0 & 2(1-k) & -2(1-k) \\ 0 & k+2 & -2 \end{matrix} \xrightarrow{IV-II} \begin{matrix} 1 & 2 & -1 \\ 0 & k+2 & -2 \\ 0 & 2(1-k) & -2(1-k) \\ 0 & 0 & 0 \end{matrix}$$

per $k=1$: $\begin{matrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$ $\quad rk\,\Phi_1 = 2$ non é suriettiva

$\mathrm{Im}\,\Phi_1 = \langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \rangle \qquad \mathrm{Ker}\,\Phi_1 = \langle \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \rangle$

per $k \neq 1$ posso dividere per $(2-2k)$ $\Rightarrow \begin{matrix} 1 & 2 & -1 \\ 0 & k+2 & -2 \\ 0 & 1 & -1 \end{matrix} \longrightarrow \begin{matrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & k+2 & -2 \end{matrix}$

$$\xrightarrow{III-(k+2)II} \begin{matrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & k \end{matrix}$$

se $k=0$ $\quad \underset{\dim \mathrm{Im}}{rk\,\Phi_0} = 2 \Rightarrow \Phi_0$ non é suriettiva

$\mathrm{Im}\,\Phi_0 = \langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rangle \qquad \mathrm{Ker}\,\Phi_0 : \left(\begin{matrix} 0 & -1 & 0 & 0 \\ 4 & 2 & 2 & 2 \\ -4 & -3 & -2 & -2 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) \quad \dots$

$\downarrow$

$\dim \mathrm{Im}\,\Phi_0 = 2 \Rightarrow \dim \mathrm{Ker}\,\Phi_0 = 2 \qquad \mathrm{Ker}\,\Phi_0 = \langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \rangle$

$\underbrace{\text{per } k \neq 0,1}$ $\quad rk\,\Phi_k = 3 \Rightarrow \mathrm{Im}\,\Phi_k = \mathbb{R}^3 \quad \dim \mathrm{Im}\,\Phi_k = 3 \Rightarrow \dim \mathrm{Ker}\,\Phi_k = 1$

é suriettiva

$\Rightarrow \mathrm{Ker}\,\Phi_k = \langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \rangle$ perché abbiamo visto prima

che $\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \in \Phi_k \quad \forall k \in \mathbb{R}$

d) troviamo $\Phi_k^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\forall k \in \mathbb{R}$

caso $k=0$: noto che $\begin{pmatrix}1\\-1\\1\end{pmatrix} \notin \operatorname{Im}\Phi_0 \Rightarrow \Phi_0^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \emptyset$

oppure risolvo il sistema: $\left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 1 \\ 4 & 2 & 2 & 2 & -1 \\ -4 & -3 & -2 & -2 & 1 \end{array}\right)$ . . .

$$\left(\begin{array}{cccc|c} 4 & 2 & 2 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ -4 & -3 & -2 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 4 & 2 & 2 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 4 & 2 & 2 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\Rightarrow \operatorname{rk} A \neq \operatorname{rk} A|b \Rightarrow S_{A|b} = \emptyset \Rightarrow \Phi_0^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \emptyset$

caso $k=1$: noto che $\begin{pmatrix}1\\-1\\1\end{pmatrix} \in \operatorname{Im}\Phi_1$: $\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}0\\3\\-2\end{pmatrix}$

$$\Rightarrow \Phi_1^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = v' + \ker\Phi_1$$

$\Rightarrow$ risolvo il sistema: $\begin{cases} x + z = 1 \\ 5x + 3y + 2z + 3t = -1 \\ -3x - 2y - z - 2t = 1 \end{cases}$ (*)

oppure per linearità $\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}0\\3\\-2\end{pmatrix} = \Phi(e_3) - \Phi(e_2) = \Phi(v')$

$$\Rightarrow v' = e_3 - e_2 = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$$

$$\Phi_1^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}-1\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} \right\rangle$$

(*) v' é una soluzione particolare del sistema

$\Rightarrow$ fisso in maniera indipendente 2 parametri $z=1$ e $t=0$

2 $=$ dim $\ker\Phi_1$

$$\Rightarrow v' = \begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$$

caso $h \neq 0,1$: $\begin{pmatrix}1\\-1\\1\end{pmatrix} \in \mathrm{Im}\,\Phi_h = \mathbb{R}^3 \Rightarrow \Phi_h^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = v' + \ker \Phi_h$

risolvo il sistema : $\begin{cases} hx + (h-1)y + hz = 1 \\ (h+4)x + (h+2)y + 2z + (h+2)t = -1 \\ (h-4)x + (h-3)y + (h-2)z - 2t = 1 \end{cases}$ . . .

oppure $\begin{pmatrix}1\\-1\\1\end{pmatrix} = \left[\begin{pmatrix}h\\2\\h-2\end{pmatrix} - \begin{pmatrix}0\\h+2\\-2\end{pmatrix}\right] \cdot \frac{1}{h} = \left(\Phi(e_3) - \Phi(e_4)\right)\frac{1}{h} = \Phi(v')$

$$v' = (e_3 - e_4)\frac{1}{h} = \left[\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]\frac{1}{h} = \frac{1}{h}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$$\Rightarrow \Phi_h^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \frac{1}{h}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} + \left\langle \begin{pmatrix}-1\\0\\1\\1\end{pmatrix} \right\rangle$$

Foglio di esercizi 5

Esercizio 2

a) per quali valori di k $\exists$ un'applicazione lineare $\phi: \mathbb{R}^3 \to \mathbb{R}^4$ t.c.

$$\phi\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\k\\0\end{pmatrix} \quad \phi\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \quad \phi\begin{pmatrix}3\\0\\-1\end{pmatrix} = \begin{pmatrix}-1\\k-3\\2-k\\1-k\end{pmatrix} \quad \phi\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\k+1\\0\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|cccc} 1 & 2 & 1 & 1 & 0 & k & 0 \\ 0 & 1 & -1 & 2 & k+1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & -1 & -1 & k-3 & 2-k & 1-k \end{array}\right) \xrightarrow[\text{IV}-3\text{I}]{\text{I}+\text{III}} \left(\begin{array}{ccc|cccc} 1 & 2 & 1 & 1 & 0 & k & 0 \\ 0 & 1 & -1 & 2 & k+1 & 0 & 0 \\ 0 & 3 & 2 & 2 & 1 & k & 0 \\ 0 & -6 & -4 & -4 & k-3 & 2-4k & 1-k \end{array}\right) \xrightarrow{\text{IV}+2\text{III}}$$

$$\left(\begin{array}{ccc|cccc} 1 & 2 & 1 & 1 & 0 & k & 0 \\ 0 & 1 & -1 & 2 & k+1 & 0 & 0 \\ 0 & 3 & 2 & 2 & 1 & k & 0 \\ 0 & 0 & 0 & 0 & k-1 & 2-2k & 1-k \end{array}\right) \xrightarrow{\text{III}-3\text{II}} \left(\begin{array}{ccc|cccc} 1 & 2 & 1 & 1 & 0 & k & 0 \\ 0 & 1 & -1 & 2 & k+1 & 0 & 0 \\ 0 & 0 & 5 & -4 & -3k-2 & k & 0 \\ 0 & 0 & 0 & 0 & k-1 & 2(1-k) & 1-k \end{array}\right)$$

esiste se e solo se $(0 \ 0 \ 0) = (0 \ \ k-1 \ \ 2(1-k) \ \ 1-k)$

cioè per $k=1$

b) per tali valori $(k=1)$ calcolare la matrice associata rispetto alle basi canoniche di dominio e codominio

$$\left(\begin{array}{ccc|cccc} 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 5 & -4 & -5 & 1 & 0 \end{array}\right) \xrightarrow{\text{I}-2\text{II}} \left(\begin{array}{ccc|cccc} 1 & 0 & 3 & -3 & -4 & 1 & 0 \\ 0 & 1 & -1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & -1 & \frac{1}{5} & 0 \end{array}\right)$$

$$\xrightarrow[\text{II}+\text{III}]{\text{I}-3\text{III}} \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & -\frac{3}{5} & -1 & \frac{2}{5} & 0 \\ 0 & 1 & 0 & 6/5 & 1 & 1/5 & 0 \\ 0 & 0 & 1 & -4/5 & -1 & 1/5 & 0 \end{array}\right)$$

$$A_{\mathcal{E}_3\mathcal{E}_4}\phi = \frac{1}{5}\begin{pmatrix} -3 & 6 & -4 \\ -5 & 5 & -5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

c) calcolare il rango di ϕ

$$\begin{matrix} -3 & 6 & -4 \\ -5 & 5 & -5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{matrix} \longrightarrow \begin{matrix} -3 & 6 & -4 \\ 0 & -5 & 5/3 \\ 0 & 5 & -5/3 \\ 0 & 0 & 0 \end{matrix} \longrightarrow \begin{matrix} -3 & 6 & -4 \\ 0 & 5 & -5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$\mathrm{rk}\,\phi = 2$

Foglio di Esercizi 6 :

Esercizio 2

a) determinare l'endomorfismo f di $\mathbb{R}^3$ t.c. $\operatorname{Ker} f = \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

$\operatorname{Im} f = \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

tale f è unica? Perchè?

$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ $\quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ tale f è unica $\quad 1^a$ caso

tuttavia $\exists$ infinite f che soddisfano alle proprietà richieste

un'altra è : $f\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ $\quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\quad 2^a$ caso

b) Se ne dia la matrice associata rispetto alla base canonica

$$A_{\mathcal{E}_{\mathbb{R}^3}\mathcal{E}_{\mathbb{R}^3}} f = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 0 \\ -1 & 2 & -2 \end{pmatrix} \quad 1^a \text{ caso}$$

2^a caso : $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ $\Rightarrow$ per linearità :

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\left(f\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$$

$$\Rightarrow A_{\mathcal{E}_{\mathbb{R}^3}\mathcal{E}_{\mathbb{R}^3}} f = \begin{pmatrix} 1 & 2 & 2 \\ 1/2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

c) per la f al punto a) si determini l'antiimmagine di $\begin{pmatrix}1\\1\\1\end{pmatrix}$ e $\begin{pmatrix}0\\-1\\-3\end{pmatrix}$

$$\det\begin{pmatrix}1 & 2 & 2\\1 & 0 & 1\\1 & -1 & 2\end{pmatrix} = -3 \Rightarrow \begin{pmatrix}1\\1\\1\end{pmatrix} \notin \left\langle \begin{pmatrix}2\\0\\-1\end{pmatrix} \begin{pmatrix}2\\1\\2\end{pmatrix} \right\rangle = \mathrm{Im} f$$

$$\Rightarrow f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \emptyset$$

$$\begin{pmatrix}0\\-1\\-3\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} \Rightarrow f^{-1}\begin{pmatrix}0\\-1\\-3\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} + \left\langle \begin{pmatrix}2\\0\\-1\end{pmatrix} \right\rangle$$