Determinanti

Teorema: il $\det A$ con $A \in M_{n,n}(\mathbb{R})$ non dipende dalla riga o dalla colonna usate nello sviluppo.

Definizione: una matrice $A \in M_{n,n}(\mathbb{R})$ si dice **Triangolare superiore** se $a_{ij} = 0 \quad \forall\, i > j$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$A \in M_{n,n}(\mathbb{R})$ si dice **Triangolare inferiore** se $a_{ij} = 0 \quad \forall\, i < j$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$

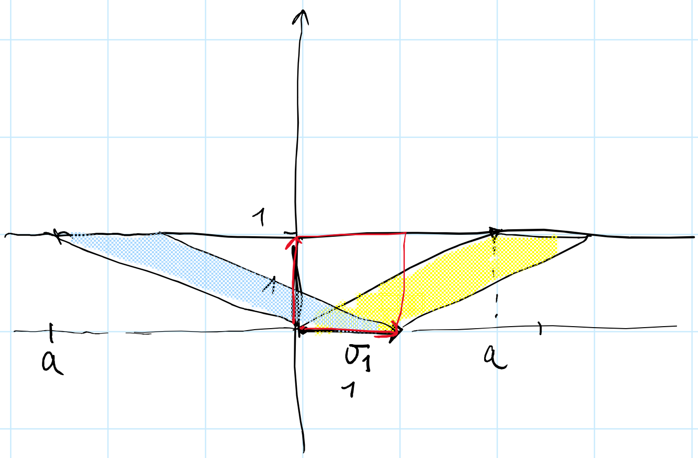
A è **matrice diagonale** $a_{ij} = 0$
$\forall\, i \neq j$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Esempio:

$$\det \begin{array}{c} \begin{matrix} v_1 & \end{matrix} \\ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{array} = 1$$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$



$$\det \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = 1$$

$a = b = c = 0$

Prop: $\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11} a_{22} \cdots a_{nn}$

A Triangolare superiore

Dim: $n=1$ $\quad \det(a_{11}) = a_{11}$ $\quad$ VERO

Supponiamo vero per $A \in M_{n,n}(\mathbb{R})$ e dimostriamolo per $n+1$

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} & a_{1n+1} \\ 0 & a_{22} \cdots & a_{2n} & a_{2n+1} \\ \vdots & \ddots & & \\ 0 & \cdots & 0 \; a_{nn} & a_{nn+1} \\ 0 & \cdots & 0 & \boxed{a_{n+1n+1}} \end{pmatrix} = a_{n+1n+1} \; a_{11} a_{22} \cdots a_{nn}$$

$(-1)^{n+1+n+1} = +$

Esempio: $\det\begin{pmatrix} 1 & 8 & -13 & 5 \\ 0 & 2 & 18 & 21 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 2$

Definizione: data $A \in M_{m,n}(\mathbb{R})$ si dice **Trasposta di A**

$A^t \in M_{n,m}(\mathbb{R})$ $\qquad {}^tA$

$(A^t)_{ij} = A_{ji}$

Esempio: $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \in M_{2,4}(\mathbb{R})$

$\boxed{(A^t)^t = A}$

$A^t = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \in M_{4,2}(\mathbb{R})$

Osservazione: se $A \in M_{n,n}(\mathbb{R})$ $\quad \boxed{\det(A) = \det(A^t)}$

Se $A \in M_{n,n}(\mathbb{R})$ è triang. superiore, oppure triang. inferiore oppure diagonale $\det(A) = a_{11}\, a_{22} \cdots a_{nn}$.

Teorema di Binet: date A e B matrici in $M_{n,n}(\mathbb{R})$

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Oss:

$$\det\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} = 1\cdot 2 - 3\cdot 0 = 2$$

$$\begin{array}{c} 1^\circ \\ 2^\circ + 2\cdot 1^\circ \end{array} \quad \det\begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix} = 1\cdot 8 - 2\cdot 3 = 8 - 6 = 2$$

operazione elem. del 3° tipo $\det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 1$

$$\begin{array}{c} 2^\circ \\ 1^\circ \end{array} \quad \det\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = 0\cdot 3 - 1\cdot 2 = -2$$

cambia il determ. moltiplicato (-1)

$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

$$\begin{array}{c} 3\cdot 1^\circ \\ 2^\circ \end{array} \quad \det\begin{pmatrix} 3 & 9 \\ 0 & 2 \end{pmatrix} = 6 = 3 \det\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$$

Esempio:

$$\det\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \qquad \begin{array}{l} 1^\circ - 2\cdot 2^\circ \\ 2^\circ \\ 3^\circ - 2^\circ \\ 4^\circ - 3^\circ \end{array}$$

$$\det\begin{pmatrix} 0 & -3 & -1 & -1 \\ \cancel{1} & \cancel{2} & \cancel{1} & \cancel{1} \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \qquad \begin{array}{c} + \\ - \\ + \\ - \end{array}$$

$$= - \det\begin{pmatrix} -3 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = -\det\begin{pmatrix} -3 & -1 & -1 \\ -1 & 1 & 0 \\ -3 & -2 & 0 \end{pmatrix} = \qquad \begin{array}{l} 1^\circ \\ 2^\circ \\ 3^\circ + 1^\circ \end{array}$$

$= 1 \det\begin{pmatrix} -1 & 1 \\ -3 & -2 \end{pmatrix} = 2 - (-3)(1) =$

$= 2 + 3 = 5$

$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

Conseguenza: $A \in M_{n,n}(\mathbb{R})$ invertibile $\Longleftrightarrow$ $\det A \neq 0$

$\det(AB) = \det(A)\det(B)$ $\quad A \quad B = A^{-1}$

$1 = \det(I_n) = \det(\underline{A A^{-1}}) = \det(A)\det(A^{-1})$

$1 = \det(A)\det(A^{-1})$ $\quad \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$

Esempi di uso del determinante

Esercizio: si consideri l'endomorfismo f_k di $\mathbb{R}^3$ con matrice associata rispetto alla base canonica $\mathcal{E}$

$$A_k = \begin{pmatrix} k+1 & 5 & 8 \\ 0 & k & 3 \\ 0 & -2k & 1 \end{pmatrix}$$

Determinare i valori del parametro $k \in \mathbb{R}$ tali che $\overset{f_k}{A_k}$ sia biiettiva

Svolg.: f_k è biiettiva $\Longleftrightarrow \det(A_k) \neq 0 \Longleftrightarrow rg(A_k) = 3$

$\det A_k = (k+1)\det\begin{pmatrix} k & 3 \\ -2k & 1 \end{pmatrix}$ $\quad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$= (k+1)(k+6k) = 7\underset{k=0}{k}\,\underset{k=-1}{(k+1)}$

f_k è biiettiva $\forall\, k \in \mathbb{R} \setminus \{0, -1\}$

Definizione

$GL_n(\mathbb{R})$

$$GL_n(\mathbb{R}) = \{A \in M_{n,n}(\mathbb{R}) \mid \det(A) \neq 0\} =$$

gruppo lineare $= \{A \in M_{n,n}(\mathbb{R}) \mid A \text{ è invertibile}\} =$

$$= \{A \in M_{n,n}(\mathbb{R}) \mid rg(A) = n\}$$

Uso del determinante per calcolare matrici inverse.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad\begin{matrix} + & - \\ - & + \end{matrix}$ $\quad$ costruiamo una matrice

$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$S_{11} = +\det\begin{pmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{pmatrix} = d$

$S_{12} = -\det\begin{pmatrix} \cancel{a} & \cancel{b} \\ c & \cancel{d} \end{pmatrix} = -c$

$S_{21} = -\det\begin{pmatrix} \cancel{a} & b \\ \cancel{c} & \cancel{d} \end{pmatrix} = -b$

$S_{22} = +\det\begin{pmatrix} a & \cancel{b} \\ \cancel{c} & \cancel{d} \end{pmatrix} = a$

$$\boxed{A \cdot S^t = \det(A)\, I_2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -\cancel{ab}+\cancel{ab} \\ \cancel{cd}-\cancel{cd} & -bc+ad \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \det(A)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Se $A \in GL_2(\mathbb{R}) \iff \det A \neq 0$ $\quad \boxed{A^{-1} = \frac{1}{\det(A)} S^t}$

Esempio: $A = \begin{pmatrix} 0 & -1 & 0 \\ 3 & 1 & 5 \\ 1 & 0 & 2 \end{pmatrix}$ calcoliamo la sua inversa.

$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$\det A = 1 \det\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 6 - 5 = 1$ $\qquad A^{-1} = \frac{1}{\det A} S^t = S^t$

$S_{11} = +\det\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix} = 2$ $\qquad S_{21} = -\det\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 2$ $\qquad S_{31} = +\det\begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix} = -5$

$S_{12} = -\det\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = -1$ $\qquad S_{22} = +\det\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = 0$ $\qquad S_{32} = -\det\begin{pmatrix} 0 & 0 \\ 3 & 5 \end{pmatrix} = 0$

$S_{13} = +\det\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = -1$ $\qquad S_{23} = -\det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -1$ $\qquad S_{33} = +\det\begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix} = 3$

$$S = \begin{pmatrix} 2 & -1 & -1 \\ 2 & 0 & -1 \\ -5 & 0 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 & 0 \\ 3 & 1 & 5 \\ 1 & 0 & 2 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 2 & 2 & -5 \\ -1 & 0 & 0 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Fate la verifica $A^{-1}A = \mathbb{I}_3$

$$\begin{pmatrix} 2 & 2 & -5 \\ -1 & 0 & 0 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 3 & 1 & 5 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrici di cambio di base.

Definizione: sia V uno spazio vettoriale e

$B_1 = \{v_1, \dots, v_n\}$
$B_2 = \{v_1', \dots, v_n'\}$ due basi di V.

Si dice **matrice di cambio di base** dalla base B_1 alla B_2 la matrice: $T_{B_1}^{B_2} = A_{B_1, B_2, id_V}$

Esempi: $V = \mathbb{R}^3$ $\quad B_1 = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$$\mathcal{E} = B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

dimostriamo che B_1 è base di $\mathbb{R}^3$ e calcoliamo $T_{B_1}^{B_2}$.

$$T_{B_1}^{B_2} = A_{B_1, B_2, id_{\mathbb{R}^3}} \qquad T_{B_1}^{\mathcal{E}} = T_{B_1}^{B_2} = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 3 & 0 \\ 8 & 1 & 1 \end{pmatrix}$$

$$T_{B_1}^{B_2} = A_{B_1, B_2, id_{\mathbb{R}^3}} \qquad T_{B_1}^{\mathcal{E}} = T_{B_1}^{B_2} = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 3 & 0 \\ 8 & 1 & 1 \end{pmatrix}$$

$v_1 \quad id(v_1) = v_1 = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} = \underline{1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{5}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underline{8}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$v_2 \quad id(v_2) = v_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \underline{1} \cdot e_1 + \underline{3} e_2 + \underline{1} \cdot e_3$

$v_3 \quad id(v_3) = v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{0} \cdot e_1 + \underline{0} e_2 + \underline{1} \cdot e_3$

B_1 è base perché $\det\left(T_{B_1}^{B_2}\right) \neq 0$

quindi le colonne sono lin. indip.

Esercizio: $B = \left\{ \begin{pmatrix} 18 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \right\}$ $\mathcal{E} = \{ e_1, e_2, e_3 \}$ $T_B^{\mathcal{E}} = \begin{pmatrix} 18 & 2 & -1 \\ 1 & 0 & 0 \\ 2 & 1 & 5 \end{pmatrix}$

Osservazione: $\boxed{T_{B_2}^{B_1} = \left(T_{B_1}^{B_2}\right)^{-1}}$

$$(V, B_1) \xrightarrow{T_{B_1}^{B_2}} (V, B_2) \xrightarrow{T_{B_2}^{B_1}} (V, B_1)$$

$$\boxed{T_{B_2}^{B_1} \, T_{B_1}^{B_2} = A_{B_1, B_1, id_V} = \mathbb{I}_n}$$

Esercizio: calcolare la matrice di cambio di base $T_{\mathcal{E}}^{B}$

con $\mathcal{E} = \{ e_1, e_2 \}$ $\quad B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$.

Svolg.: $T_B^{\mathcal{E}} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \quad T_{\mathcal{E}}^{B} = \left(T_B^{\mathcal{E}}\right)^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{1} \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

$\det\left(T_B^{\mathcal{E}}\right) = 1 \cdot (-1) - 2 \cdot (-1) = -1 + 2 = 1$

Ora sappiamo calcolare se $V=\mathbb{R}^n$

$T^{\mathcal{E}}_{B_1} \quad T^{\mathcal{E}}_{B_2} \quad T^{B_1}_{\mathcal{E}} \quad T^{B_2}_{\mathcal{E}}$ cerchiamo $T^{B_2}_{B_1}$

$$\begin{tikzcd}
(\mathbb{R}^n, B_1) \arrow[rr, "T^{B_2}_{B_1}"] \arrow[dr, "T^{\mathcal{E}}_{B_1}"'] & & (\mathbb{R}^n, B_2) \\
& (\mathbb{R}^n, \mathcal{E}) \arrow[ur, "T^{B_2}_{\mathcal{E}}"'] &
\end{tikzcd}$$

$$T^{B_2}_{B_1} = T^{B_2}_{\mathcal{E}} \underset{\substack{\uparrow \\ \text{dopo}}}{\cdot} T^{\mathcal{E}}_{B_1}$$

Esempio: $B_1 = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$B_2 = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Calcoliamo $T^{B_2}_{B_1} = T^{B_2}_{\mathcal{E}} \, T^{\mathcal{E}}_{B_1} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

$$\begin{tikzcd}
(\mathbb{R}^2, B_1) \arrow[rr, "T^{B_2}_{B_1}"] \arrow[dr, "T^{\mathcal{E}}_{B_1}"'] & & (\mathbb{R}^2, B_2) \\
& (\mathbb{R}^2, \mathcal{E}) \arrow[ur, "T^{B_2}_{\mathcal{E}}"'] &
\end{tikzcd}$$

$T^{\mathcal{E}}_{B_1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

$T^{B_2}_{\mathcal{E}} = \left(\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1}$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow[2^\circ + 2 \cdot 1^\circ]{} \left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$T^{B_2}_{B_1} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix}$$