Esercizio 3. In $\mathbb{R}^3$, siano $u = (1, 2, -1)$, $v = (0, -1, 1)$ e $w = (0, 0, 1)$. Sia f l'endomorfismo di $\mathbb{R}^3$ tale che

$$f(u) = u + v, \qquad f(v) = (-1, 0, 0) + u, \qquad f(w) = (1, 3, 0) - w.$$

(a) Dimostrare che esiste una unica f con tali proprietà.

(b) Determinare una base di $\mathrm{Ker}(f)$ ed una base di $\mathrm{Im}(f)$.

(c) Determinare la matrice $A_{\mathcal{E}_3,\mathcal{E}_3,f}$ associata a f rispetto alla base canonica di $\mathbb{R}^3$.

(d) Determinare tutti i vettori $z \in \mathbb{R}^3$ tali che $f(z) = (-1, 1, -1)$.

$$u = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$f(u) = u + v \qquad f(v) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + u \qquad f(w) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - w$$

$$f\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{c|c} u & f(u) \\ v & f(v) \\ w & f(w) \end{array}\right) \quad \text{Ridurre con Gauss} \quad \left(\begin{array}{c|c} e_1 & f(e_1) \\ e_2 & f(e_2) \\ e_3 & f(e_3) \end{array}\right) \quad \text{in riga}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 3 & -1 \end{array}\right) \xrightarrow[(-1)\,2^\circ R]{\text{Rid.}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \end{array}\right) \xrightarrow[2^\circ + 3^\circ]{1^\circ + 3^\circ}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 4 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & -1 \end{array}\right) \xrightarrow{1^\circ - 2 \cdot 2^\circ} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & -1 \end{array}\right) \begin{array}{l} f(e_1) \\ f(e_2) \\ f(e_3) \end{array}$$

$$A_{\mathcal{E},\mathcal{E},f} = \big(f(e_1) \;\; f(e_2) \;\; f(e_3) \big) = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 0 & -1 \end{pmatrix} = A$$

I vettori $\{u, v, w\}$ sono una base di $\mathbb{R}^3$ perché $rg\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 3$ e per il risultato visto in classe

Esiste un'unica applicazione lineare $f: \mathbb{R}^3 \to \mathbb{R}^3$ che è definita in modo tale da rispettare le condizioni precedenti.

$$f(au+bv+cw) = af(u)+bf(v)+cf(w).$$

b)

$$A_{\mathcal{E},\mathcal{E},f} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 0 & -1 \end{pmatrix}$$

Ker f

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{S_{13}} \left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[2^\circ+2\cdot 1^\circ]{(-1)\cdot I^\circ}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[3^\circ-2^\circ]{} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x+z=0 \\ y+z=0 \\ \cancel{0=0} \end{cases} \quad \begin{cases} x=-z \\ y=-z \end{cases}$$

$$\boxed{\begin{array}{l} \dim Ker f = 3-2 = 1 \\ B_{Ker f} = \left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} \end{array}}$$

$$Ker A = Ker f = \left\{\begin{pmatrix} -z \\ -z \\ z \end{pmatrix} \,\middle|\, z \in \mathbb{R}\right\} = \left\langle \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

$$Im f = \langle \underbrace{\overset{w_1}{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \overset{\cancel{w_3}}{\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}}}\rangle$$

$\to w_1 + w_2 = w_3$

$\boxed{w_1 + w_2 - w_3 = \vec{0} \text{ Rel. di dipendenza}}$

$$\dim \mathbb{R}^3 = \dim Ker f + \dim Im f \Rightarrow \dim Im f = 2$$
$$3 = 1 + 2$$

$$aw_1 + bw_2 + cw_3 = \vec{0}$$

$$a\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} b+c=0 \\ 2a+b+3c=0 \\ -a-c=0 \end{cases}$$

$$B_{Im f} = \left\{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array}\begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix} \xrightarrow{S_{1,2}} \begin{array}{c} w_2 \\ w_1 \\ w_3 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 1 & 3 & -1 \end{pmatrix} \xrightarrow[3^\circ-1^\circ]{} \begin{array}{c} w_2 \\ w_1 \\ w_3-w_2 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\xrightarrow[3^\circ-2^\circ]{} \begin{array}{c} w_2 \\ w_1 \\ w_3-w_2-w_1 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad B'_{\mathrm{Im} f} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$$

c)

$$\boxed{\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 0 & -1 \end{pmatrix} = A}$$

d) $$\left\{ z \in \mathbb{R}^3 \,\middle|\, f(z) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} = f^{-1}\left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} \qquad (A|b)$$

$$b = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \qquad A|b = \left(\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 2 & 1 & 3 & 1 \\ -1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{S_{1,3}}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & -1 & -1 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \xrightarrow[2^\circ+2\cdot1^\circ]{} \left(\begin{array}{ccc|c} -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right) \xrightarrow[3^\circ-2^\circ]{}$$

$$\left(\begin{array}{ccc|c} +1 & 0 & +1 & +1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x + z = 1 \\ y + z = -1 \end{cases} \qquad \begin{cases} x = 1 - z \\ y = -1 - z \end{cases}$$

$$\boxed{\mathrm{Ker} f = \left\langle \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle}$$

$$f^{-1}\left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1-z \\ -1-z \\ z \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\} =$$

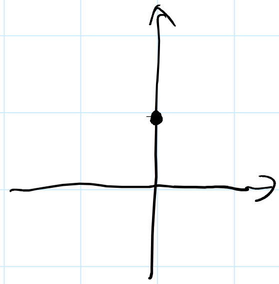
$$= \underset{z=0}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} + \left\langle \underline{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}} \right\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \underline{\mathrm{Ker} f}$$

$z^5 = i \qquad z \in \mathbb{C} \qquad (z-1)^5 = 0 \qquad 1$ con molt. 5

$z^5 - i = 0$

$(z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)=0$

$z_i \qquad i=0,\dots,4$



Si consideri il sottospazio vettoriale

$$W = \left\langle \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}8\\-7\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\4\\2\end{pmatrix} \right\rangle$$

di $\mathbb{R}^4$.

Scegli una o più alternative:

a. $W = \left\langle \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\4\\2\end{pmatrix} \right\rangle$ NO

b. $\dim(W) = 4.$ NO

c. $W = \left\langle \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}8\\-7\\0\\1\end{pmatrix} \right\rangle$ Sì

d. $\dim(W) = 3.$ Sì

$$\begin{array}{c} w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 2 & 1 \\ 8 & -7 & 0 & 1 \\ 1 & 1 & 4 & 2 \end{pmatrix} \xrightarrow{\begin{array}{c} 2^\circ - 2\cdot 1^\circ \\ 3^\circ - 8\cdot 1^\circ \\ 4^\circ - 1^\circ \end{array}} \begin{array}{l} w_1 \\ w_2 - 2w_1 \\ w_3 - 8w_1 \\ w_4 - w_1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & -7 & -16 & -7 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{\begin{array}{c} 3^\circ - 7\cdot 2^\circ \\ \boxed{4^\circ + 2^\circ} \end{array}}$$

$$\begin{array}{l} w_1 \\ w_2 - 2w_1 \\ (w_3 - 8w_1) - 7(w_2 - 2w_1) \\ \boxed{(w_4 - w_1) + (w_2 - 2w_1)} \end{array} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \rightarrow \begin{array}{l} w_1 \\ w_2 - 2w_1 \\ w_3 - 8w_1 \\ w_4 - w_1 \end{array} \rightarrow \begin{array}{l} w_1 \\ w_2 - 2w_1 \\ (w_3 - 8w_1) - 7(w_2 - 2w_1) \\ \boxed{w_4 - w_1 + w_2 - 2w_1 = \vec{0}} \end{array}$$

$$-3w_1 + w_2 + w_4 = \vec{0}$$

cioè $w_2 + w_4 = 3w_1$