Proiezioni e simmetrie

$V = U \oplus W$

Proiezione di asse U e direzione W $\qquad p = p_U^W : V \longrightarrow V$

$\forall v \in V \qquad v = u + w \qquad\qquad p(v) = u$

① $p^2 = p \circ p = p$

② $\operatorname{Ker} p = W \qquad \operatorname{Im} p = U \qquad \dim \operatorname{Im} p = rg\, p = \dim U = h$

③ $\begin{cases} p(w) = \vec{0} & \forall w \in W \\ p(u) = u & \forall u \in U \end{cases}$

④ $B_U = \{u_1, \dots, u_h\} \qquad B_W = \{w_1, \dots, w_k\} \qquad h + k = \dim V$

$B_V = \{u_1, \dots, u_h, w_1, \dots, w_k\}$ è base di V perché $U \oplus W = V$

$$A_{B_V,B_V,f} = A_{B_V,B_V,p} = \begin{pmatrix} \boxed{I_h} & O \\ O & O \end{pmatrix} = \begin{array}{c} \overbrace{}^{h} \\ \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & & & & 0 \\ 0 & 1 & 0 & \cdots & & & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & & & & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & & & & \cdots & 0 \end{array}\right) \end{array} \begin{array}{l} \}\, h \\ \\ \\ \end{array}$$

$A_{B_V,B_V,p}$

$u_1 \qquad p(u_1) = u_1 = \underline{1} \cdot u_1 + \underline{0} u_2 + \cdots + \underline{0} w_k$

$u_2 \qquad p(u_2) = u_2 = \underline{0} \cdot u_1 + \underline{1} \cdot u_2 + \underline{0} u_3 + \dots + \underline{0} w_k$

$\vdots \qquad\qquad \vdots$

$u_h \qquad p(u_h) = u_h = \underline{0} \cdot u_1 + \dots + \underline{0} \cdot u_{h-1} + \underline{1} \cdot u_h + \underline{0} w_1 + \dots + \underline{0} w_k$

$w_1 \qquad p(w_1) = \vec{0} = \underline{0} \cdot u_1 + \dots + \underline{0} w_k$

$\vdots$

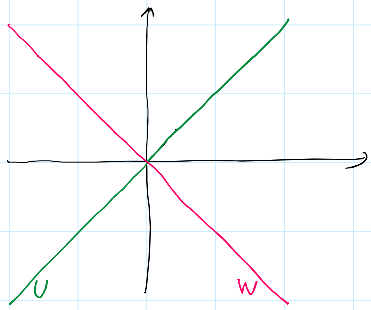
$w_k \qquad p(w_k) = \vec{0} = \underline{0} \cdot u_1 + \dots + \underline{0} \cdot w_k$

$$\begin{array}{cc} & \begin{array}{ccccc} 1^\circ & 2^\circ & & h^\circ & \end{array} \\ h^\circ \rightarrow & \begin{pmatrix} 1 & 0 & & 0 & \cdots & 0 \\ 0 & 1 & & 0 & & \vdots \\ \vdots & 0 & \ddots & 1 & & \vdots \\ 0 & 0 & & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

$$\left(\begin{array}{c|c} I_h & O \\ O & O \end{array}\right)$$

Esempio: calcolare la matrice $A_{\mathcal{E},\mathcal{E},p}$ $\quad A_{\mathcal{E},\mathcal{E},s}$ con p e s rispettivamente proiezione e simmetria di asse $U = \langle \binom{1}{1} \rangle$ e direzione $W = \langle \binom{1}{-1} \rangle$.

Svolg.:



Verifico che $U \oplus W = \mathbb{R}^2$

$\langle \binom{1}{1} \rangle \oplus \langle \binom{1}{-1} \rangle = \mathbb{R}^2$

$p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$v = \binom{x}{y} = u + w \qquad\qquad p(v) = u \qquad\qquad s(v) = u - w = 2u \overbrace{-u-w}^{-(u+w)} =$

$u \in U = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$ $\quad w \in W = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$ $\quad = 2p(v) - v = (2p - id_V)(v)$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \overset{u}{\begin{pmatrix} a \\ a \end{pmatrix}} + \begin{pmatrix} b \\ -b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix} \qquad \begin{cases} x = a+b \\ y = a-b \end{cases}$$

incognite sono a e b

$$\begin{cases} a = x-b \\ y = x-b-b \end{cases} \qquad \begin{cases} a = x-b \\ 2b = x-y \end{cases} \qquad \begin{cases} a = x - \frac{x-y}{2} = \frac{2x-x+y}{2} = \frac{x+y}{2} \\ b = \frac{x-y}{2} \end{cases}$$

$$u = \begin{pmatrix} a \\ a \end{pmatrix} \qquad w = \begin{pmatrix} b \\ -b \end{pmatrix}$$

$$P\begin{pmatrix} x \\ y \end{pmatrix} = u = \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{pmatrix} \qquad P = A_{\mathcal{E},\mathcal{E},P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\boxed{\begin{matrix} P\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ P\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{matrix}}$$

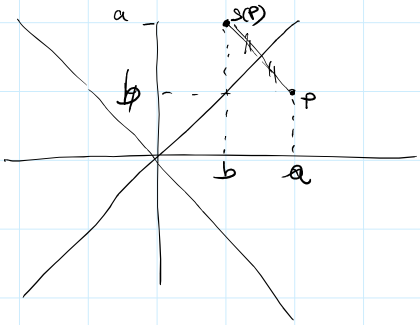
$P\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $P\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \boxed{P \cdot P = P}$

$\mathrm{Im}\,p = \langle \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \rangle = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle = U$ $\qquad \mathrm{Ker}\,p$ $\left(\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right)$ $\quad x+y=0 \quad y=-x \quad \begin{pmatrix} x \\ -x \end{pmatrix}$

$$\mathrm{Ker}\,p = \{ v \in \mathbb{R}^2 \mid Av = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = W$$

1° $\quad S = 2P - Id = 2\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$



$S(v) = u - w \qquad S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} - \begin{pmatrix} b \\ -b \end{pmatrix} = \begin{pmatrix} a-b \\ a+b \end{pmatrix} =$ $\qquad a = \frac{x+y}{2}$, $\quad b = \frac{x-y}{2}$

$$= \begin{pmatrix} \frac{x+y}{2} - \frac{x-y}{2} \\ \frac{x+y}{2} + \frac{x-y}{2} \end{pmatrix} = \begin{pmatrix} \frac{\cancel{x}+y-\cancel{x}+y}{2} \\ \frac{x+\cancel{y}+x-\cancel{y}}{2} \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\boxed{\begin{matrix} p(v) = u \\ S(v) = u - w \end{matrix}} \qquad v = u + w$$

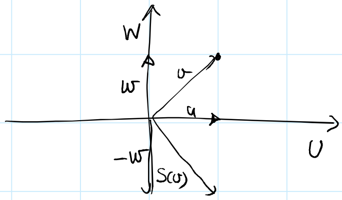
$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proprietà:

1 $\quad S^2 = S \circ S = id_V \qquad S \cdot S = I$

② $\operatorname{Ker} s = \{\vec{0}\}$ $\quad s$ è biiettiva $\quad$ l'inversa di s è s.

$\operatorname{Im} s = V$



③ $\begin{cases} S(u) = u & \forall u \in U \\ S(w) = -w & \forall w \in W \end{cases}$

④ $B_V = \{\underbrace{u_1, \dots, u_h}_{U} / \underbrace{w_1, \dots, w_k}_{W}\}$ $\qquad A_{B_V, B_V, S} = \begin{pmatrix} \mathbb{I}_h & 0 \\ 0 & -\mathbb{I}_k \end{pmatrix}$ $\qquad S$

$$\begin{array}{cccc} 1° & 2° & & h° \end{array}$$
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$u_1 \quad S(u_1) = u_1 = 1 u_1 + 0 u_2 + \dots + 0 w_k$

$u_2 \quad S(u_2) = u_2 = 0 \cdot u_1 + 1 \cdot u_2 + 0 u_3 + \dots$

$\vdots$

$u_h \quad S(u_h) = u_h = 0 \cdot u_1 + \dots + 1 \cdot u_h + 0 w_1 + \dots$

$w_1 \quad S(w_1) = -w_1 = 0 \cdot u_1 + \dots + 0 \cdot u_h + (-1) w_1 + 0 \dots$ $\qquad S^2 = \mathbb{I}_n$

$\vdots$

$w_k \quad S(w_k) = -w_k = 0 u_1 + \dots + 0 \cdot w_{k-1} + (-1) w_k$ $\qquad n = \dim V = h + k$

Esempio: $P \qquad U = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \qquad W = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$

Determinare una base di $V = \mathbb{R}^2$ B_V tale che $\qquad A_{B_V, B_V, P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

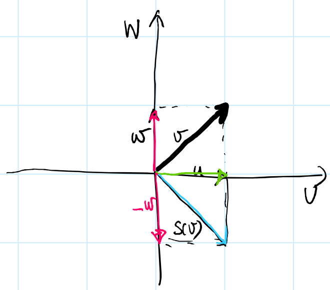
$A_{B_V, B_V, S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

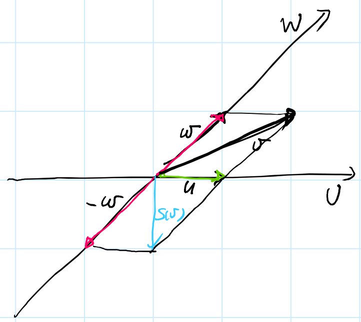
$B_V = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \qquad A_{B_V, B_V, S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad s\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad s\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$





Fatto: data un'applicazione lineare $f: V \longrightarrow W$ con

$\begin{cases} \dim \operatorname{Im} f = r \\ \dim V = n \\ \dim W = m \end{cases}$ esistono una base B_V di V e una base B_W di W tali che

$$A_{B_V,B_W,f} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \qquad M_{m,n}(\mathbb{R})$$

$$= \begin{array}{c} \begin{array}{ccccccc} 1^\circ & 2^\circ & \dots & r^\circ & r+1^\circ & & n^\circ \end{array} \\ \begin{pmatrix} 1 & 0 & & 0 & 0 & \dots & 0 \\ 0 & 1 & & 0 & & & \\ \vdots & 0 & & \vdots & & & \\ & \vdots & & 0 & & & \\ & & & 1 & & & \\ & & & 0 & \vdots & & \vdots \\ 0 & 0 & & 0 & 0 & \dots & 0 \end{pmatrix} \end{array} \begin{array}{c} 1^\circ \\ 2^\circ \\ r^\circ \\ \\ \\ \\ \\ \end{array}$$

$B_{\ker f} = \{v_{r+1},\dots,v_n\}$

$n-r$ vettori

$n = \dim V = \dim \ker f + \dim \operatorname{Im} f$

$n = \underbrace{\dim \ker f}_{n-r} + r$

Completate a base di V $\boxed{B_V = \{v_1,\dots,v_r,v_{r+1},\dots,v_n\}}$

$$\operatorname{Im} f = \langle f(v_1),\dots,f(v_r),\underbrace{\cancel{f(v_{r+1})}}_{\vec 0},\dots,\underbrace{\cancel{f(v_n)}}_{\vec 0}\rangle = \langle f(v_1),\dots,f(v_r)\rangle$$

sono lin. indipendenti

$w_1 = f(v_1)$, $w_2 = f(v_2)$, $\dots$, $w_r = f(v_r)$ completate a base di W

$\boxed{B_W = \{w_1,\dots,w_r,w_{r+1},\dots,w_m\}}$

v_1 $\quad f(v_1) = 1\cdot w_1 + 0 w_2 + \dots + 0 w_n$

$\vdots$

v_r $\quad f(v_r) = w_r = 0 w_1 + \dots + 1 w_r + \dots 0 w_n$

$v_{r+1} \in \ker f$ $\quad f(v_{r+1}) = \vec 0 = 0\cdot w_1 + \dots + 0 w_n$

$v_n \in \ker f$ $\quad f(v_n) = \vec 0 = 0\cdot w_1 + \dots + 0\cdot w_n$

$$A_{B_V,B_W,f} = \begin{array}{c} \begin{array}{cccc} 1^\circ & r^\circ & & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots \\ \vdots & 1 & & \\ \hline 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

Esercizio 2 Sia f_a l'endomorfismo di $\mathbb{R}^3$ avente come matrice rispetto alle basi canoniche la matrice:

$$A_a = \begin{pmatrix} 0 & 1 & a-1 \\ 1-a & -1 & 0 \\ 2-2a & 2a & 0 \end{pmatrix}$$

a) Per ogni $a \in \mathbb{R}$ calcolare $f_a^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$.

b) Per ogni $a \in \mathbb{R}$ determinare una base di $\ker f_a$ e una base di $\operatorname{Im} f_a$.

c) Esistono dei valori di $a \in \mathbb{R}$ tali che f_a sia biiettiva?

d) Esistono dei valori di $a \in \mathbb{R}$ tali che $\ker f_a$ e $\operatorname{Im} f_a$ siano in somma diretta?

e) Posto $a = 1$ determinare B_1 e B_2 basi di $\mathbb{R}^3$ tali che $A_{B_1,B_2,f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Svolgimento: $\quad f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad a \in \mathbb{R}$

a) $\underbrace{f_a^{-1}}\left\{\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{w}\right\}$

$f_a^{-1}\{w\} = \{v \in \mathbb{R}^3 \mid A_a v = w\}$

$$\boxed{\left(\begin{array}{ccc|c} 0 & 1 & a-1 & 1 \\ 1-a & -1 & 0 & 0 \\ 2-2a & 2a & 0 & 0 \end{array}\right)} \qquad A_a | w$$

$f_a\{w\} = \{v \in \mathbb{R} \mid A_a v = w\}$

Riduciamo con Gauss.

$$\xrightarrow{S_{12}} \left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 2-2a & 2a & 0 & 0 \end{array}\right) \xrightarrow{3^\circ R - 2\cdot 1^\circ R} \left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & \boxed{2a+2} & 0 & 0 \end{array}\right) \xrightarrow{3^\circ R - (2a+2)\cdot 2^\circ R}$$

$2a+2-(2a+2)\cdot 1 = 0$

$0-(2a+2)(a-1)$

$0-(2a+2)\cdot 1$

$$\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & -2(a+1)(a-1) & -2(a+1) \end{array}\right)$$

$-2(a+1)(a-1)$

$$\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & -2(a+1)(a-1) & -2(a+1) \end{array}\right)$$

A_a

$\text{rg}\, A_a = 3$ se e solo se $a \in \mathbb{R} \setminus \{1,-1\}$ $\quad \text{rg}(A_a|b) = 3$

per R-C il sistema ha soluzioni $\quad \dim \text{Ker}\, A_a = 3-3=0$

$$\begin{cases} (1-a)x - y = 0 \\ y + (a-1)z = 1 \\ +\cancel{2(a+1)}(a-1)z = +\cancel{2(a+1)} \end{cases}$$

$(a-1)z = 1$

$$\begin{cases} (1-a)x = y = 0 & x = 0 \\ y = 1 - (a-1)z = 1 - \cancel{(a-1)}\frac{1}{\cancel{a-1}} = 1-1 = 0 \\ z = \frac{1}{a-1} \end{cases}$$

$$\boxed{f_a^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\0\\ 1/(a-1)\end{pmatrix}\right\} \quad \text{per } a \in \mathbb{R} \setminus \{1,-1\}}$$

Se $a=1$

$$\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & -2(a+1)(a-1) & -2(a+1) \end{array}\right) = \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right) =$$

$$\xrightarrow{2^\circ R + 1^\circ R} \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right) \xrightarrow{3^\circ + 2 \cdot 2^\circ} \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{rg}\, A_1 = 1$

$\text{rg}(A_1|b) = 2$

$\text{rg}\, A_1 \neq \text{rg}(A_1|b)$ per R-C il sistema non ha soluzioni quindi

$$\boxed{f_1^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \emptyset \qquad a=1}$$

Se $a=-1$ $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $rg\, A_{-1} = 2 = rg(A_{-1}|b)$ per R-C il sistema ha soluzioni e

$\dim Ker\, A_{-1} = 3-2 = 1$

$$\begin{cases} 2x-y=0 \\ y-2z=1 \\ 0=0 \end{cases} \quad \begin{cases} 2x-\underbrace{2z-1}=0 \\ y=2z+1 \end{cases} \quad \begin{cases} x=\frac{2z+1}{2}=z+\frac{1}{2} \\ y=2z+1 \end{cases}$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\left\{ \begin{pmatrix} z+1/2 \\ 2z+1 \\ z \end{pmatrix} \middle| z \in \mathbb{R} \right\}$ $\downarrow z=0$

$Ker f_{-1}$

$$f_{-1}^{-1}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle \quad a=-1.$$

2) $B_{Ker\, fa}$

$$\boxed{3 = \dim \mathbb{R}^3 = \dim Ker\, f_a + \dim Im\, f_a}$$

$B_{Im\, fa}$

$$Ker\, f_a := \{ v \in \mathbb{R}^3 \mid A_a v = \vec{0} \}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & a-1 & 0 \\ 1-a & -1 & 0 & 0 \\ 2-2a & 2a & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 0 \\ 0 & 0 & -2(a+1)(a-1) & 0 \end{array}\right)$$

Se $a \in \mathbb{R} \setminus \{1,-1\}$ $rg\, A_a = \dim Im\, f_a = 3 \Rightarrow Im\, f_a = \mathbb{R}^3$ è suriettiva

$\dim Ker\, f_a = 3-3=0 \rightarrow Ker\, f_a = \{\vec{0}\}$ è iniettiva

$B_{Im\, fa} = \{e_1, e_2, e_3\}$ $B_{Ker\, fa} = \emptyset$ $\Downarrow$

$\dim Im\, f_a = 3$ $\dim Ker\, f_a = 0$ è biiettiva

isomorfismo

Se $a=1$ $B_{Ker\, f_1}$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad y=0$$

$$Ker\, f_1 = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle| x,z \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$\dim Ker\, f_1 = 3-1=2$

$\dim Im\, f_1 = rg\, A_1 = 1$

Nota bene:

l'immagine è generata dalle colonne della matrice nelle basi canoniche (Non la ridotta)

Se $a=1$ $B_{\ker f_1}=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ $\dim\ker f_1=2$

non è iniettiva

$B_{\mathrm{Im} f_1}=\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}\right\}$ $\dim \mathrm{Im} f_1=1$

non è suriettiva

$$A_a=\begin{pmatrix}0 & 1 & a-1\\ 1-a & -1 & 0\\ 2-2a & 2a & 0\end{pmatrix}$$

$a=1$ $$A_1=\begin{pmatrix}0 & 1 & 0\\ 0 & -1 & 0\\ 0 & 2 & 0\end{pmatrix}$$

Se $a=-1$ $$A_{-1}=\begin{pmatrix}0 & 1 & -2\\ 2 & -1 & 0\\ 4 & -2 & 0\end{pmatrix}$$

$$\mathrm{Im} f_{-1}=\left\langle\begin{pmatrix}0\\2\\4\end{pmatrix},\begin{pmatrix}1\\-1\\-2\end{pmatrix},\begin{pmatrix}-2\\0\\0\end{pmatrix}\right\rangle=$$

$$=\left\langle\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\-1\\-2\end{pmatrix},\begin{pmatrix}-1\\0\\0\end{pmatrix}\right\rangle$$

$w_1+w_2+w_3=\vec{0}$

$B_{\mathrm{Im} f_{-1}}=\left\{\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ $\dim \mathrm{Im} f_{-1}=2$

non è suriettiva

$\ker f_{-1}$

Se $a=-1$

$$\left(\begin{array}{ccc|c}1-a & -1 & 0 & 0\\ 0 & 1 & a-1 & 0\\ 0 & 0 & -2(a+1)(a-1) & 0\end{array}\right)=\left(\begin{array}{ccc|c}+2 & -1 & 0 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

$$\begin{cases}+2x-y=0\\ y-2z=0\\ 0=0\end{cases}\quad\begin{cases}+2x-2z=0\\ y=2z\end{cases}\quad\begin{cases}\cancel{2}x=+\cancel{2}z\\ y=2z\end{cases}\quad\begin{cases}x=+z\\ y=2z\end{cases}\quad\begin{pmatrix}x\\y\\z\end{pmatrix}\ \begin{pmatrix}+z\\2z\\z\end{pmatrix}$$

$$\ker f_{-1}=\left\langle\begin{pmatrix}1\\2\\1\end{pmatrix}\right\rangle$$

$B_{\ker f_1}=\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ non è iniettiva