Calculus 2 — Final Exam

Prompt #1

Exercise 1. .

i) What does it mean that $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ is differentiable at point $\vec{x} \in D$?

ii) Let $f = f(x, y)$ be defined as

$$f(x,y) = \begin{cases} \frac{x^3 - yx^2}{\sqrt{x^2+y^2}}, & (x,y) \neq \vec{0}, \\ \\ 0, & (x,y) = \vec{0}. \end{cases}$$

Is f continuous at $(0,0)$? Is f differentiable at $(0,0)$?

Exercise 2. Let

$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1 + xy\right\}.$$

i) Show that $D \neq \emptyset$ is the zero set of a submersion.

ii) Is D compact?

iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 3. Let

$$D := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leqslant z \leqslant 1 - y^2\}.$$

i) Draw $D \cap \{x = 0\}$ and $D \cap \{y = 0\}$. Is D invariant by some rotation? Justify your answer. Draw D as best as you can.

ii) Compute the volume of D.

Exercise 4. Let

$$\vec{F} := \left(\frac{ax^2 + by^2}{(x^2+y^2)^2}, \frac{xy}{(x^2+y^2)^2}\right)$$

on $D = \mathbb{R}^2 \backslash \{(0,0)\}$. Here $a, b \in \mathbb{R}$ are constants.

i) Determine all possible values for a, b in such a way $\vec{F}$ be irrotational on D.

ii) Determine values of a, b, c in such a way $\vec{F}$ be conservative on D, in this case determining also all the possible potentials.

Exercise 5. .

i) What are the Cauchy–Riemann equations (or conditions)?

ii) Let $f = u + iv$ ($u = \text{Re } f$ and $v = \text{Im } f$) be a $\mathbb{C}$ differentiable function on the entire plane $\mathbb{C}$. Assume that also $\overline{f} = u - iv = u + i(-v)$ is $\mathbb{C}$ differentiable on $\mathbb{C}$. What conclusion can you draw on f?

Calculus 2 — Final Exam

Prompt #2

Exercise 6. Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 3, \ 0 \leqslant y \leqslant 2\}$ and

$$f(x, y) := x^2 - 2xy + 2y.$$

Determine min/max points of f on D (if any). Justify carefully.

Exercise 7. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (x^2 + y^2)^{1/4} \leqslant z \leqslant 2 - x^2 - y^2\right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 8. Consider the differential equation

$$y' = \frac{e^y}{e^y - t}.$$

i) Determine the domain D of local existence and uniqueness.
ii) Determine all the stationary solutions (if any) and plot the regions of D where solutions are increasing/decreasing.

Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $y(2) = 0$.

iii) Show that y is monotone and deduce that $\alpha > 1$.
iv) Determine the concavity of y. Is $\beta = +\infty$? Justify carerully.
v) Plot a qualitative graph of y.

Exercise 9. Let

$$v(x, y) := e^{-y} \left(y \cos x + x \sin x\right), \ \ (x, y) \in \mathbb{R}^2.$$

i) Determine all possible $u = u(x, y)$ in such a way that $f(x + iy) := u(x, y) + iv(x, y)$ be $\mathbb{C}-$differentiable on $\mathbb{R}^2$.
ii) Express the f found at i) as function of complex number z, that is $f = f(z)$.

Exercise 10. .

i) State Green's formula.
ii) Let $f \in \mathsf{C}^2(\mathbb{R}^2)$. Prove that

$$\oint_{\partial D} f \nabla f = 0.$$

Calculus 2 — Final Exam

Prompt #3

Exercise 11. Let

$$f(x,y) := (x^2+y^2)^3 - x^4 + y^4,\ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x,y)$.
ii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find the eventual min/max points of f. What about $f(\mathbb{R}^2)$?

Exercise 12. Consider the equation

$$y' = \frac{e^y}{e^y - t}.$$

i) Determine the domain D where local existence and uniqueness applies.
ii) Determine the constant solutions and regions of D where solutions are increasing/decreasing.
iii) Let now $y :]\alpha,\beta[\longrightarrow \mathbb{R}$ be the maximal solutions of the Cauchy problem $y(2) = 0$. Determine the monotonicity of y. What can you say about α?
iv) Determine the concavity of y. Is $\beta = +\infty$? Justify carefully.
v) Plot a qualitative graph of y.

Exercise 13. Let $D := \{(x,y,z) \in \mathbb{R}^3 \ :\ x^2 + 2y^2 \leqslant z \leqslant 4 - 3(x^2+2y^2)\}$.

i) Draw the set D. Is D a rotation volume with respect to the $z-$axis"?
ii) Compute the volume of D.

Exercise 14. Let

$$u(x,y) := x^2 + y^2.$$

i) Determine, if any, $v = v(x,y)$ in such a way that $f(x+iy) := u(x,y)+iv(x,y)$ be $\mathbb{C}-$differentiable on $\mathbb{C}$.
ii) For the f you found at i), write $f = f(z)$ as function of $z \in \mathbb{C}$.

Exercise 15. State the Lagrange multipliers theorem. Then, consider a curve $y = f(x)$ defined by a function $f = f(x) : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in \mathscr{C}^1(\mathbb{R})$. Let $P = (a,b)$ a point in the cartesian plane not belonging to the curve $y = f(x)$. Prove that if Q is a point of the curve $y = f(x)$ where the distance to P is minimum, then the segment $P - Q$ is perpendicular to the tangent to f.

Calculus 2 — Final Exam

Prompt #4

Exercise 16. Let

$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1, \ y^2 + z = 1\}.$$

i) Show that $D \neq \varnothing$ is the zero set of a submersion (g_1, g_2).
ii) Is D compact?
iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 17. Consider the vector field

$$\vec{F}(x, y) := \left(\frac{ax + by}{\sqrt{x^2 + y^2}}, \frac{cx + dy}{\sqrt{x^2 + y^2}} \right), \ (x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

i) Find all possible values of $a, b, c, d \in \mathbb{R}$ such that $\vec{F}$ is irrotational.
ii) Find all possible values for a, b, c, d such that $\vec{F}$ is conservative. For such values, determine the potentials of $\vec{F}$.

Exercise 18. Consider the system

$$\begin{cases} x' = y(x^2 - y^2), \\ \\ y' = (x - 1)(x^2 - y^2). \end{cases}$$

i) Find stationary solutions, and dhow that the system has a non constant first integral.
ii) Draw the phase portrait of the system. Are there periodic solutions? Are there non constant and non periodic global solutions?
iii) Find the x solution of the Cauchy problem $x(0) = 2$, $y(0) = 1$.

Exercise 19. Let $v(x, y) := y^3 - 3x^2y + 4xy - x$, $(x, y) \in \mathbb{R}^2$. Determine all possible $u = u(x, y)$ such that

$$f(x + iy) := u(x, y) + iv(x, y),$$

be holomorphic on $\mathbb{C}$. What is $f(z)$ as a function of z?

Exercise 20. .

i) State precisely the formula of change of variables for multiple integrals.
ii) Let $\vec{a}, \vec{b}, \vec{c}$ three linearly independent vectors of $\mathbb{R}^3$, $\alpha, \beta, \gamma > 0$ three positive constants, and let $E_{\alpha,\beta,\gamma} := \left\{ \vec{x} \in \mathbb{R}^3 \ : \ 0 \leqslant \vec{a} \cdot \vec{x} \leqslant \alpha, \ 0 \leqslant \vec{b} \cdot \vec{x} \leqslant \beta, \ 0 \leqslant \vec{c} \cdot \vec{x} \leqslant \gamma \right\}$. Prove the formula

$$\int_{E_{\alpha,\beta,\gamma}} (\vec{a} \cdot \vec{x})(\vec{b} \cdot \vec{x})(\vec{c} \cdot \vec{x}) \, d\vec{x} = \frac{(\alpha\beta\gamma)^2}{8 \det M}. \quad \left(\text{where } M = \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \right)$$

Calculus 2 — Final Exam

Prompt #5

Exercise 21. Let $\Gamma \subset \mathbb{R}^3$ the set described by equations

$$\Gamma \ : \ \begin{cases} x^2 + y^2 = 1, \\ \\ x^2 + z^2 = xz + 1. \end{cases}$$

i) Show that $\Gamma \neq \emptyset$ is the zero set of a submersion on Γ.
ii) Is Γ compact? Justify your answer.
iii) Determine points of Γ at minimum/maximum distance to $(0, 0, 0)$ (if any).

Exercise 22. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ 1 - (x^2 + y^2) \leqslant z \leqslant \sqrt{1 - (x^2 + y^2)}\}$.

i) Draw $D \cap \{y = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 23. Consider the equation

$$y' = (\log y)(\log(2 - y)).$$

i) Determine the domain D of local existence and uniqueness, constant solutions and regions of D where the solutions are increasing/decreasing.

Let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution with $y(0) = 1/2$.

ii) Discuss monotonicity and concavity of y.
iii) Is $\alpha = -\infty$? Is $\beta = +\infty$? Determine limits of $y(t)$ when $t \longrightarrow \alpha+, \beta-$.
iv) Plot a qualitative graph of y.

Exercise 24. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\sinh \frac{1}{z} = 0.$$

Exercise 25. .

i) What does it mean that a set $C \subset \mathbb{R}^d$ is closed? What is the Cantor characterization of closed sets?
ii) Given a set $S \subset \mathbb{R}^d$, we define the boundary of S the set

$$\partial S := \left\{ \vec{x} \in \mathbb{R}^d \ : \ \forall r > 0, \ B(\vec{x}, r] \cap S \neq \varnothing, \ B(\vec{x}, r] \cap S^c \neq \varnothing \right\}.$$

Is ∂S always closed? Justify your answer providing a proof if yes, a counterexample if no.

Calculus 2 — Final Exam

Prompt #6

Exercise 26. .

i) Provide a precise statement of the differentiability test says? Is this test a sufficient or necessary condition for differentiability? Justify your answer.

ii) Let

$$f(x,y) := x^2 \log(x^2+y^2),\ (x,y) \neq \vec{0}.$$

Is it possible to extend f at point $(0,0)$ in such a way that f is continuous at $(0,0)$? If yes,what about the differentiability at $(0,0)$ of the extended function?

Exercise 27. Let

$$f(x,y) := 3xy + x^2y + xy^2,\ \ (x,y) \in D := \{(x,y) \in \mathbb{R}^2\ :\ x \geqslant 0,\ 0 \leqslant y \leqslant 1-x\}.$$

i) Draw D. Is D closed? open? bounded? compact? Justify carefully.

ii) Discuss the problem of determining min/max (if any) of f on D.

Exercise 28. Let $a, b, c, d \in \mathbb{R}$ and

$$\vec{F}(x,y) := \left(\frac{ax+by}{(x^2+y^2)^2}, \frac{cx+dy}{(x^2+y^2)^2} \right),\ \ (x,y) \in D := \mathbb{R}^2 \backslash \{(0,0)\}.$$

i) Determine $a, b, c, d \in \mathbb{R}$ in such a way that $\vec{F}$ be irrotational on D.

ii) Determine a, b, c, d such that $\vec{D}$ be conservative on D. For these values (if any), determine all possible potentials of $\vec{F}$ on D.

iii) Let $\gamma = \gamma(t) \subset D$ be the segment joining $(1,0)$ to $(0,2)$. For $(a,b,c,d) = (2,0,0,2)$ compute

$$\int_\gamma \vec{F}.$$

Exercise 29. Let $D := \{(x,y) \in \mathbb{R}^2\ :\ x \geqslant 1,\ x^3 \leqslant y \leqslant 3\}$.

i) Draw D.

ii) By using the change of variables $u = y - x^3$, $v = y + x^3$, compute the integral

$$\int_D x^2(y-x^3)e^{y+x^3}\ dxdy.$$

Exercise 30. Let $f = u + iv$ be holomorphic on $D \subset \mathbb{C}$. Define

$$g(z) := \overline{f(\overline{z})},\ z \in \overline{D} := \{w \in \mathbb{C}\ :\ \overline{w} \in D\}$$

i) Express real and imaginary part of g in terms of real and imaginary parts u and v of f.

ii) Use i) to discuss whether g is holomorphic on $\overline{D}$ or not.

Calculus 2 — Final Exam

Prompt #7

Exercise 31. Let $D := \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 = y^2 + z^2, \quad x^2 + y^2 = xy + 1\}$.

i) Show that D is the zero set of a submersion on D itself.
ii) Is D compact? Justify your answer.
iii) Determine, if any, the points of D at the min / max distance to the origin.

Exercise 32. Consider the system

$$\begin{cases} x' = y(x^2 + y^2), \\ \\ y' = x(x^2 + y^2). \end{cases}$$

We accept that local existence and uniqueness holds.

i) Find stationary solutions, and show that the equation admits a first integral.
ii) Plot the phase portrait of the system. Are there periodic solutions? Are there non constant and non periodic global solutions?
iii) Solve the Cauchy problem $x(0) = 1$, $y(0) = 0$.

Exercise 33. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ 0 \leqslant z \leqslant \frac{1}{\cosh(x^2 + y^2)}\right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce the figure of D. Is D closed? Open? Bounded? Compact? Justify your answer.
ii) Determine the volume of D.
iii) Determine for which values of α the following integral has a finite value:

$$\int_D e^{\alpha(x^2+y^2)} \, dxdydz.$$

Exercise 34. Let

$$u(x, y) := x^3 + axy^2, \quad v(x, y) := bx^2y - y^3, \ \ (x, y) \in \mathbb{R}^2.$$

i) Determine $a, b \in \mathbb{R}$ in such a way that $f(x + iy) := u(x, y) + iv(x, y)$ be holomorphic on $\mathbb{C}$.
ii) For values of a, b found at i), express f as a function of the complex variable z.

Exercise 35. Let $\vec{a}_1, \ldots, \vec{a}_N \in \mathbb{R}^d$ be N fixed vectors, $\vec{a}_i \neq \vec{a}_j$ for $i \neq j$. Define

$$f(\vec{x}) := \sum_{j=1}^{N} \|\vec{x} - a_j\|^2.$$

Discuss the problem of determining, if any, points of min/max for f on $\mathbb{R}^d$. Justify carefully, state all general facts you use.

Calculus 2 —- Final Exam

Prompt #8

Exercise 36. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^2, \ y^2 + (z-2)^2 = 1\right\}.$$

i) Show that $D \neq \emptyset$ and it is the zero set of a submersion.
ii) Is D compact? Prove or disprove.
iii) Find points of D at min/max distance to $\vec{0}$.

Exercise 37. By using the change of variables $x = u^2$ and $y = v^2$, compute

$$\int_{[0,+\infty[^2} \frac{e^{-(x+y)}}{\sqrt{x^2y + xy^2}}\, dxdy.$$

Exercise 38. Let $f(x + iy) = u(x, y) + iv(x, y)$.

i) Which properties of u, v characterize $\mathbb{C}-$diffentiability of f at $z = x + iy$? Write a precise statement.
ii) Let
$$u(x, y) := ax^2 + bxy + cy^2, \quad v(x, y) := xy, \ \ x + iy \in \mathbb{C}.$$
(a, b, c are real constant). Determine all possible values for a, b, c in such a way f be $\mathbb{C}-$differentiable on $\mathbb{C}$.
iii) For the values of a, b, c at ii), determine explicitly $f(z)$ as a function of $z \in \mathbb{C}$.

Exercise 39. Consider the differential equation

$$y' = \frac{1}{y - \log t}.$$

i) Determine the domain D of local existence and uniqueness.
ii) Find stationary solutions (if any) and plot regions of D where solutions are increasing/decreasing.

Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $y(2) = 0$.

iii) Show that y is monotone. What is the $\lim_{t\to\alpha} y'(t)$?
iv) Discuss the concavity of y and say if $\beta = +\infty$ or less justifying carefully your argument.
v) With all the previous informations plot a qualitative graph of y.

Exercise 40. What does it mean that a set $S \subset \mathbb{R}^d$ is open? Let $\vec{F} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a continuous function on $\mathbb{R}^d$. Prove that the following property holds:

$$S \subset \mathbb{R}^m \text{ open}, \implies \vec{F}^{-1}(S) \text{ open}.$$

(recall that $\vec{F}^{-1}(S) = \{\vec{x} \in \mathbb{R}^d \ : \ \vec{F}(\vec{x}) \in S\}$). Hint: suppose that for some S open, $\vec{F}^{-1}(S)$ is not open. . .

Calculus 2 —- Final Exam

Prompt #9

Exercise 41. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = 1, \ x + y + z = 1\}$.

i) Show that D is the zero set of a submersion.
ii) Is D compact?
iii) Determine, if any, min/max points for $f(x, y, z) = x^2 - x + y^2 + yx + yz - y$ on D.

Exercise 42. .

i) What does it mean that a vector field $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ ($d \in \mathbb{N}$, $d \geqslant 2$) is conservative? And what does it mean that it is irrotational?
ii) Consider now the field

$$\vec{F}(x, y, z) := \left(\frac{ax + by}{1 + x^2 + y^2 + z^2}, \frac{cy + dz}{1 + x^2 + y^2 + z^2}, \frac{ex + fz}{1 + x^2 + y^2 + z^2} \right), \ (x, y, z) \in D := \mathbb{R}^3.$$

Determine all the possible values for $a, b, c, d, e, f \in \mathbb{R}$ such that $\vec{F}$ is irrotational on D.
iii) For which values a, b, c, d, e, f is $\vec{F}$ conservative on D? For these values, if any, determine the potentials of $\vec{F}$.

Exercise 43. Let

$$D := \left\{ (x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant 2x - \sqrt{x^2 + y^2} \right\}.$$

i) Is D closed? open? bounded? compact? Justify carefully.
ii) Compute the area of D.

Exercise 44. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined as

$$f(z) := \begin{cases} \frac{\overline{z}^2}{z}, & z \neq 0, \\ \\ 0, & z = 0. \end{cases}$$

i) Show that $u = \mathrm{re}\, f$ and $v = \mathrm{f}$ verify the Cauchy-Riemann equations at $(0, 0)$.
ii) Is f $\mathbb{C}-$differentiable at $z = 0$?

Exercise 45. Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, differentiable at $\vec{x}$.

i) What is the directional derivative $\partial_{\vec{v}} f$ at $\vec{x}$ along the direction $\vec{v}$? What is the relation between this derivative and the gradient vector $\nabla f(\vec{x})$?
ii) Determine

$$\max_{\vec{v} \neq \vec{0} \ : \ \|\vec{v}\| = 1} \partial_{\vec{v}} f(\vec{x}).$$

Exercise 46. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\cosh z + 4 = 0.$$

Exercise 47. Let

$$\vec{F}(x,y) := \left(axy + e^{by}, xe^{y} + 2y\right),\ (x,y) \in \mathbb{R}^2.$$

i) Determine a, b for which $\vec{F}$ is irrotational on $\mathbb{R}^2$
ii) Determine a, b for which $\vec{F}$ is conservative on $\mathbb{R}^2$, determining also its potentials.
iii) For the values of a, b such that $\vec{F}$ is irrotational, compute the path integral of $\vec{F}$ along the curve $\gamma(t) := (t^2, t^2 - t)$, $t \in [0,1]$.

Exercise 48. Let

$$D := \{(x,y,z) \in \mathbb{R}^3 \ :\ xy + z^2 = 1/2,\ x^2 + y^2 = z\}.$$

i) Check that $D \neq \varnothing$ is the zero set of a submersion on D.
ii) Is D compact?
iii) Determine (if any) point(s) of D at minimum and maximum distance to the z axis.

Exercise 49. Let $D := \{(x,y,z) \in \mathbb{R}^3 \ :\ x^2 + y^2 - z^2 \leqslant 1,\ x^2 + y^2 + z^2 \leqslant 9,\ z \geqslant 0\}$.

i) Draw the figure of D in the plane xz (that is, $D \cap \{y = 0\}$), and deduce a figure for D.
ii) Compute the volume of D (hint: use appropriate cylindrical coordinates)

Exercise 50. Let $f(x+iy) = u(x,y) + iv(x,y) : \mathbb{C} \longrightarrow \mathbb{C}$ be a $\mathbb{C}-$differentiable function on the complex plane $\mathbb{C}$.

i) What properties must verify u and v? Provide a precise statement (no proof is required).
ii) Assume that u is constant. Show that f is constant.
iii) Assume that $|f| = \sqrt{u^2 + v^2}$ is constant. What can be drawn on f? (hint: $|f|^2 = u^2 + v^2 \equiv k$, then $\partial_x(u^2 + v^2) = \ldots$)

January 2026 / Version B

Exercise 51. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\sinh z + 2i = 0.$$

Exercise 52. Let

$$\vec{F}(x, y) := \left(ye^x + x, axy + e^{bx})\right), \ (x, y) \in \mathbb{R}^2.$$

i) Determine a, b for which $\vec{F}$ is irrotational on $\mathbb{R}^2$
ii) Determine a, b for which $\vec{F}$ is conservative on $\mathbb{R}^2$, determining also its potentials.
iii) For the values of a, b such that $\vec{F}$ is irrotational, compute the path integral of $\vec{F}$ along the curve $\gamma(t) := (t^2 + 1, t^2)$, $t \in [0, 1]$.

Exercise 53. Let

$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ xz + 2y^2 = 1, \ x^2 + z^2 = 2y\}.$$

i) Check that $D \neq \varnothing$ is the zero set of a submersion on D.
ii) Is D compact?
iii) Determine (if any) point(s) of D at minimum and maximum distance to the y axis.

Exercise 54. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + z^2 - y^2 \leqslant 1, \ x^2 + y^2 + z^2 \leqslant 4, \ z \geqslant 0\}$.

i) Draw the figure of D in the plane xy (that is, $D \cap \{z = 0\}$), and deduce a figure for D.
ii) Compute the volume of D (hint: use appropriate cylindrical coordinates)

Exercise 55. Let $f(x + iy) = u(x, y) + iv(x, y) : \mathbb{C} \longrightarrow \mathbb{C}$ be a $\mathbb{C}-$differentiable function on the complex plane $\mathbb{C}$.

i) What properties must verify u and v? Provide a precise statement (no proof is required).
ii) Assume that v is constant. Show that f is constant.
iii) Assume that $|f| = \sqrt{u^2 + v^2}$ is constant. What can be drawn on f? (hint: $|f|^2 = u^2 + v^2 \equiv k$, then $\partial_x(u^2 + v^2) = \ldots$)

February 2026

Exercise 56. Let

$$u(x,y) := x^5 - 10x^3y^2 + 5xy^4.$$

i) Determine all possible $v = v(x,y)$ in such a way that $f(x+iy) := u(x,y) + iv(x,y)$ be $\mathbb{C}-$differentiable on $\mathbb{C}$.
ii) For the f found at i), determine the analytical expression of $f(z)$ as function of $z \in \mathbb{C}$.

Exercise 57. Let

$$f(x,y) := x^4 + y^4 - 2(x-y)^2,\ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x,y)$.
ii) Determine and classify the stationary points of f.
iii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find (if any) min/max points of f.

Exercise 58. Let

$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ :\ z \geqslant 1 - (x^2+y^2),\ x^2+y^2+z^2 \leqslant \tfrac{7}{4}\right\}$$

i) Draw $D \cap \{x = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 59. We consider the differential equation

$$y' = \log(1-ty).$$

i) Determine the domain $D \subset \mathbb{R}^2$ where local existence and uniqueness applies.
ii) Determine any constant solutions and the regions of D where solutions are increasing/decreasing.

Let now $y :]\alpha,\beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $y(0) = c$ with $c > 0$.

iii) Determine the monotonicity of y.
iv) Is $\beta = +\infty$? Determine $\lim_{t\to\beta} y(t)$. Is $\alpha = -\infty$? Determine $\lim_{t\to\alpha} y(t)$.
v) Plot a qualitative graph of $y(t)$.

Exercise 60. Let $\vec{a} \in \mathbb{R}^d \backslash \{\vec{0}\}$. Determine

$$\max_{\vec{x}\in\mathbb{R}^d \,:\, \|\vec{x}\|=1} (\vec{a}\cdot\vec{x})^2.$$

(here: $\vec{a}\cdot\vec{x} = \sum_{j=1}^d a_j x_j$, $\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_d^2}$)

June 2026

Exercise 61. Solve the equation

$$\sin z + \cos z = 2,$$

in the unknown $z \in \mathbb{C}$ (recall the Euler formulas for $\cos z$ and $\sin z$ ...).

Exercise 62. Let

$$D := \{(x, y) \in \mathbb{R}^2 \ : \ x \geqslant 0,\ 0 \leqslant x + y \leqslant 1\}$$

and

$$f(x, y) := xy^2 + x^2y - xy.$$

i) Draw D. Is D open? closed? bounded? compact?
ii) Discuss existence and value of min/max of f on D. Justify carefully your method.

Exercise 63. By using the change of variables $x^2 = r\cos\theta$ and $y^2 = r\sin\theta$, compute

$$\int_{[0,+\infty[^2} xye^{-x^4-y^4}\, dxdy.$$

Justify carefully the details.

Exercise 64. Let

$$\vec{F}(x, y) = \big(ax^2y + by^3 + x, cx^3 + dxy^2 + y\big).$$

i) Determine all $a, b, c, d \in \mathbb{R}$ such that $\vec{F}$ is irrotational on $\mathbb{R}^2$.
ii) For these values, decide whether $\vec{F}$ is conservative and, in this cases, determine all possible potentials of $\vec{F}$.
iii) For a, b, c, d for which $\vec{F}$ is conservative, compute

$$\oint_\gamma \vec{F},$$

where γ is the positively oriented boundary of the domain

$$D := \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ x^2 \leqslant y \leqslant x\}.$$

Exercise 65. Answer the following questions.

i) What does it mean that a set $K \subset \mathbb{R}^d$ is compact?
ii) Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuous function such that

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty.$$

Prove that the set $S := \{\vec{x} \in \mathbb{R}^d \ : \ f(\vec{x}) \leqslant M\}$ is compact for every $M \in \mathbb{R}$. Justify carefully your argument.

Solutions

Exercise 1. i) See LN for the definition.

ii) To check continuity, we have to check if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$$

For $(x,y) = (\rho\cos\theta, \rho\sin\theta) \neq 0$, in polar coordinates we have

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^3\left(\cos^3\theta - \cos^2\theta\sin\theta\right)}{\rho} = \rho^2\left(\cos^3\theta - \cos^2\theta\sin\theta\right),$$

from which

$$|f(x,y) - 0| = |f(x,y)| = \rho^2\left|\cos^3\theta - \cos^2\theta\sin\theta\right| \leqslant 2\rho^2 \longrightarrow 0,\ \rho\to 0, \iff (x,y)\to(0,0),$$

from which $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so f is continuous at $(0,0)$.

For differentiability, we start computing partial derivatives. We proceed by the definition:

$$\partial_x f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{\frac{t^3}{\sqrt{t^2}}}{t} = \lim_{t\to 0} t\,\operatorname{sgn} t = 0.$$

Similarly

$$\partial_y f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0.$$

Therefore, f is differentiable at $(0,0)$ iff

$$\lim_{\vec{h}\to\vec{0}}\frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - (0,0)\cdot\vec{h}}{\|\vec{h}\|} = 0,$$

that is

$$\lim_{\vec{h}\to\vec{0}}\frac{f(\vec{h})}{\|\vec{h}\|} = 0.$$

Setting $\vec{h} = (u,v)$, we have

$$\lim_{(u,v)\to(0,0)}\frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)}\frac{u^3 - vu^2}{u^2+v^2}.$$

Setting $(u,v) = (\rho\cos\theta, \rho\sin\theta)$ we have

$$\left|\frac{u^3 - vu^2}{u^2+v^2}\right| = \frac{\rho^3}{\rho^2}\left|\cos^3\theta - \cos^2\theta\sin\theta\right| \leqslant 2\rho \longrightarrow 0,\ \rho\to 0, \iff (u,v)\to(0,0).$$

This shows that f is differentiable at $(0,0)$ and $\nabla f(0,0) = (0,0)$. □

Exercise 2. i) For instance $(0,0,z) \in D$ iff $z^2 = 1$, thus $(0,0,\pm 1) \in D$ and $D \neq \emptyset$. D is also the zero set of $g(x,y,z) := x^2 + y^2 + z^2 - xy - 1$. This is a submersion on D iff

$$\nabla g \neq \vec{0},\ \text{on } D.$$

We have

$$\nabla g = \vec{0}, \iff \begin{cases} 2x - y = 0, \\ 2y - x = 0, \\ 2z = 0, \end{cases} \iff (x, y, z) = (0, 0, 0) \notin D,$$

from which it follows that g is a submersion on D.

ii) Certainly, $D = \{g = 0\}$ is closed ($g \in \mathscr{C}$). Is it also bounded? We may see this by using spherical coordinates:

$$\begin{cases} x = \rho \cos\theta \sin\varphi, \\ y = \rho \sin\theta \sin\varphi, \\ z = \rho \cos\varphi. \end{cases} \quad \rho^2 = x^2 + y^2 + z^2 = \|(x, y, z)\|^2.$$

Then, if $(x, y, z) \in D$ we have

$$\rho^2 = 1 + \rho^2 \cos\theta \sin\theta (\sin\varphi)^2 = 1 + \frac{1}{2}\rho^2 \sin(2\theta)(\sin\varphi)^2 \leqslant 1 + \frac{\rho^2}{2},$$

from which

$$\frac{\rho^2}{2} \leqslant 1, \iff \rho^2 = \|(x, y, z)\|^2 \leqslant 2.$$

Thus, D is bounded, hence compact.

iii) We have to minimize/maximize $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ or, which is equivalent (same min/max points), $f(x, y, z) = x^2 + y^2 + z^2$. According to i), we are in condition to apply Lagrange multipliers theorem. According to this result, at min/max points $(x, y, z) \in D$ we have

$$\nabla f = \lambda \nabla g, \iff \operatorname{rk} \begin{bmatrix} \nabla f(x, y, z) \\ \\ \nabla g(x, y, z) \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & 2y & 2z \\ \\ 2x - y & 2y - x & 2z \end{bmatrix} < 2.$$

This happens iff all 2×2 subdeterminats equal 0:

$$\begin{cases} 2x(2y - x) - 2y(2x - y) = 0, \\ \\ 2x2z - 2z(2x - y) = 0, \\ \\ 2y2z - 2z(2y - x) = 0, \end{cases} \iff \begin{cases} y^2 - x^2 = 0, \\ yz = 0, \\ xz = 0. \end{cases}$$

The first leads to $y = \pm x$, the second $y = 0$ (then $x = 0$) or $z = 0$. That is we have points $(0, 0, z)$ and $(x, \pm x, 0)$. Now

- $(0, 0, z) \in D$ iff $z^2 = 1$, that is $(0, 0, \pm 1)$.
- $(x, \pm x, 0) \in D$ iff $2x^2 = 1 \pm x^2$. If $+$, $2x^2 = 1 + x^2$, we get $x = \pm 1$, that is points $(1, 1, 0)$ and $(-1, -1, 0)$. It $-$, $x^2 = \frac{1}{3}$, thus points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$.

Prom these we see that $(1, 1, 0)$ and $(-1, -1, 0)$ are points at max distance to $\vec{0}$ while $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$ are points of D at min distance to $\vec{0}$. □

Exercise 3. i) Figures are straightforward. D is not invariant by any rotation because one part of the inequality ($z \geqslant x^2 + y^2$) is invariant by rotations around $z-$axis while the second part ($z \leqslant 1 - y^2$) is not.

ii) We have

$$\lambda_3(D) = \int_D 1\ dxdydz \overset{RF}{=} \int_{x^2+y^2\leqslant 1-y^2} \int_{x^2+y^2}^{1-y^2} 1\ dz\ dxdy = \int_{x^2+2y^2\leqslant 1} \left(1-y^2-(x^2+y^2)\right)\ dxdy$$

$$= \int_{x^2+2y^2\leqslant 1} \left(1-(x^2+2y^2)\right)\ dxdy$$

$$\overset{CV\ x=\rho\cos\theta,\ \sqrt{2}y=\rho\sin\theta}{=} \int_{0\leqslant\rho\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi} \left(1-\rho^2\right)\frac{\rho}{\sqrt{2}}\ d\rho\ d\theta$$

$$\overset{RF}{=} \frac{2\pi}{\sqrt{2}}\int_0^1 \rho-\rho^3\ d\rho = \sqrt{2}\pi\left(\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} - \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\right) = \frac{\sqrt{2}\pi}{4}. \quad \square$$

Exercise 4. i) $\vec{F}$ is irrotational on D iff

$$\partial_y \frac{ax^2+by^2}{(x^2+y^2)^2} \equiv \partial_x \frac{xy}{(x^2+y^2)^2} \text{ on } D.$$

By computing derivatives, the previous is equivalent to

$$\frac{2by(x^2+y^2)-(ax^2+by^2)4y}{(x^2+y^2)^3} = \frac{y(x^2+y^2)-4x^2y}{(x^2+y^2)^3}$$

that is, iff

$$(2b-4a)yx^2-2by^3 = -3x^2y+y^3, \iff 2b=-1,\ -1-4a=-3, \iff b=-\frac{1}{2},\ a=\frac{1}{2}.$$

ii) To be conservative, $\vec{F}$ must be irrotational, hence, necessarily, $a=\frac{1}{2}=-b$. Thus,

$$\vec{F} = \left(\frac{1}{2}\frac{x^2-y^2}{(x^2+y^2)^2}, \frac{xy}{(x^2+y^2)^2}\right) = \nabla f, \iff \begin{cases} \partial_x f = \frac{1}{2}\frac{x^2-y^2}{(x^2+y^2)^2}, \\ \partial_y f = \frac{xy}{(x^2+y^2)^2}. \end{cases}$$

Looking at the second equation,

$$f(x,y) = \int \frac{xy}{(x^2+y^2)^2}\,dy + c(x) = \frac{x}{2}\int 2y(x^2+y^2)^{-2}\,dy + c(x) = \frac{x}{2}\frac{(x^2+y^2)^{-1}}{-1} + c(x) = -\frac{1}{2(x^2+y^2)} + c(x).$$

Now, by imposing also the first equation we get

$$c'(x) = 0, \iff c(x) \equiv \text{constant}.$$

Thus, all the potentials of $\vec{F}$ are

$$f(x,y) = -\frac{1}{2(x^2+y^2)} + c. \quad \square$$

Exercise 5. About the CR equations see the course notes. Assume that $f = u + iv$ is $\mathbb{C}$ differentiable on $\mathbb{C}$. Then, u, v are $\mathbb{R}$ differentiable and the CR eqns hold,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

If also $\overline{f} = u - iv = u + i(-v)$ is $\mathbb{C}$ differentiable, $u, -v$ fulfill the CR eqns,

$$\begin{cases} \partial_x u = \partial_y(-v) = -\partial_y v, \\ \\ \partial_y u = -\partial_x(-v) = +\partial_x v. \end{cases}$$

But then, combining the two CR eqns, we get

$$\partial_x u = -\partial_y v = -\partial_x u, \implies 2\partial_x u \equiv 0,$$

and, similarly, $\partial_y u \equiv 0$. From this $\nabla u \equiv 0$ hence u is constant. Similar conclusion holds for v. We conclude that both u and v must be constant, hence also f must be constant. □

Exercise 6. The domain D is defined by large inequalities involving continuous functions: by a known fact, D is closed. Moreover, by the constraints, D is manifestly bounded. Therefore D is compact, and since f is continuous on $\mathbb{R}^2$, it is $f \in \mathscr{C}(D)$. So, the Weierstrass theorem applies and this ensures existence on global min/max points for f on D.

To determine these points we argue as follows. If (x, y) is any min/max point then

- either $(x, y) \in \operatorname{Int} D$, so, being f differentiable (polynomial), Fermat theorem applies and $\nabla f(x, y) = 0$. Now,

  $$\nabla f(x, y) = (2x - y, -2x + 2) = (0, 0), \quad \iff \quad (x, y) = (1, 2).$$

  Since $(1, 2) \notin \operatorname{Int} D$, we conclude that there aren't stationary points for f in $\operatorname{Int} D$. In particular, any min/max point cannot lie into $\operatorname{Int} D$.
- or $(x, y) \in \partial D$. Now, since D is a rectangle,

$$\partial D = \{(x, 0) \,:\, 0 \leqslant x \leqslant 3\} \cup \{(x, 2) \,:\, 0 \leqslant x \leqslant 3\} \cup \{(0, y) \,:\, 0 \leqslant y \leqslant 2\} \cup \{(3, y) \,:\, 0 \leqslant y \leqslant 2\}$$

$$= E_1 \cup E_2 \cup E_3 \cup E_4.$$

We have

$$f(x, 0) = x^2,$$

so, on E_1 the minimum point is for $x = 0$ (point $(0, 0)$) while the maximum point is for $x = 3$ (point $(3, 0)$). Then,

$$f(x, 2) = x^2 - 4x + 4 = (x - 2)^2,$$

so, on E_2 the min pt is for $x = 2$ (point $(2, 2)$), while max points are $x = 0$ (point $(0, 2)$) and $x = 3$ (point $(3, 2)$).

On E_3 we have

$$f(0, y) = 2y,$$

so the min point is at $y = 0$ (point $(0, 0)$) and the max point is at $y = 2$ (point $(0, 2)$). Finally, on E_4,

$$f(3, y) = 9 - 6y + 2y = 9 - 4y,$$

so min point is at $y = 2$ (point $(3, 2)$) and max point is at $y = 0$ (point $(3, 0)$).

From the previous analysis we conclude that:

- candidates to be min points are $(0, 0)$, $(2, 2)$, $(3, 2)$, and since

$$f(0, 0) = 0, \quad f(2, 2) = 0, \ f(3, 2) = 1,$$

we conclude that $(0, 0)$ and $(2, 2)$ are both min points for f on D.
- candidates to be max points are $(3, 0)$, $(3, 2)$, $(0, 2)$, and since

$$f(3, 0) = 9, \quad f(3, 2) = 1, \ f(0, 2) = 4,$$

we conclude that $(3, 0)$ is the unique max point for f on D. □

Exercise 7. i) $D \cap \{x = 0\} = \{(0, y, z) \ : \ \sqrt{|y|} \leqslant z \leqslant 2 - y^2\}$. Thus, in the plane yz, $D \cap \{x = 0\}$ is the plane region between $z = \sqrt{|y|}$ and the parabola $z = 2 - y^2$ (see figure). Since $(x, y, z) \in D$ depends on (x, y) through $x^2 + y^2$, D is invariant by rotations around the $z-$axis.

ii) We have

$$\begin{aligned}
\lambda_3(D) &= \int_D 1 \, dxdydz = \int_{\sqrt[4]{x^2+y^2} \leqslant z \leqslant 2-(x^2+y^2)} 1 \, dxdydz \overset{RF}{=} \int_{\sqrt[4]{x^2+y^2} \leqslant 2-(x^2+y^2)} \int_{\sqrt[4]{x^2+y^2}}^{2-(x^2+y^2)} 1 \, dz \, dxdy \\
&= \int_{\sqrt[4]{x^2+y^2} \leqslant 2-(x^2+y^2)} \left(2 - (x^2 + y^2) - \sqrt[4]{x^2 + y^2}\right) \, dxdy \\
&\overset{CV}{=} \int_{\sqrt{\rho} \leqslant 2-\rho^2, \ \theta \in [0, 2\pi]} \left(\sqrt{\rho} - \left(2 - \rho^2\right)\right) \rho \, d\rho d\theta.
\end{aligned}$$

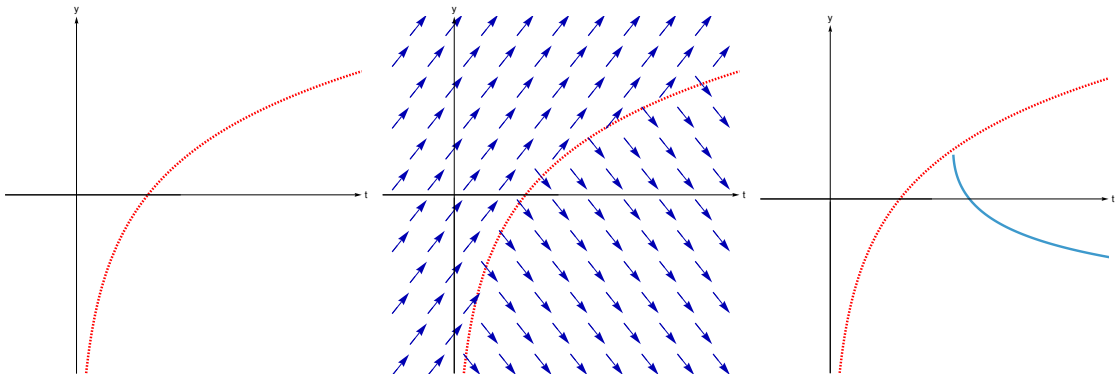
Now, $\sqrt{\rho} \leqslant 2 - \rho^2$ might be hard to solve. However, here $\rho \geqslant 0$; $\sqrt{\rho}$ is increasing while $2 - \rho^2$ decreases. Since at $\rho = 1$ they are equal, we conclude that $\sqrt{\rho} \leqslant 2 - \rho^2$ iff $0 \leqslant \rho \leqslant 1$. We can continue previous chain by the RF:

$$\begin{aligned}
&\overset{RF}{=} \int_0^1 \int_0^{2\pi} \left(2\rho - \rho^3 - \rho^{3/2}\right) \, d\theta \, d\rho = 2\pi \left(-[\rho^2]_{\rho=0}^{\rho=1} - \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1} - \left[\frac{\rho^{5/2}}{5/2}\right]_{\rho=0}^{\rho=1}\right) \\
&= 2\pi \left(1 - \tfrac{1}{4} - \tfrac{2}{5}\right) = \tfrac{7\pi}{10}. \quad \square
\end{aligned}$$

Exercise 8. i) Let $f(t, y) := \frac{e^y}{e^y - t}$. Then, f is defined on

$$D = \{(t, y) \in \mathbb{R}^2 \ : \ e^y - t \neq 0\}.$$

Now, $e^y - t = 0$ iff $e^y = t$, that is $y = \log t$. Therefore $D = \{(t, y) \ : \ y \neq \log t\}$. Clearly, $f, \partial_y f \in \mathscr{C}(D)$, so local existence and uniqueness applies on D.



ii) $y \equiv C$ is a constant solution iff $0 = \frac{e^C}{e^C - t}$ iff $e^C = 0$, which is impossible. Conclusion: no stationary solutions. Furthermore,

$$y \nearrow, \iff 0 \leqslant y' = \frac{e^y}{e^y - t}, \overset{e^y > 0}{\iff} e^y - t > 0, \iff e^y > t.$$

This is always true if $t \leqslant 0$, while, for $t > 0$, it is true iff $e^y > t$, that is $y > \log t$.

iii) Claim: $y \searrow$. This follows once we prove that $y(t) < \log t$ for all t. This is indeed true at $t = 2$: $y(2) = 0 < \log 2$. If the conclusion were false, there would be a $\widehat{t}$ such that $y(\widehat{t}) \geqslant \log \widehat{t}$. But,

- if $y(\widehat{t}) = \log \widehat{t}$, $(\widehat{t}, y(\widehat{t})) \notin D$, which is impossible for a solution.
- if $y(\widehat{t}) > \log \widehat{t}$ then, by continuity, there would be another $\widehat{\widehat{t}}$ such that $y(\widehat{\widehat{t}}) = \log \widehat{\widehat{t}}$, so we fall again into a contradiction.

In any case, we get a contradiction, so $y(t) < \log t$ must be always true and the conclusion follows.

$\alpha > 1$: indeed, if $\alpha \leqslant 1$ we would have $y(\alpha) > y(2) = 0 = \log 1 \leqslant \log \alpha$, contradicting $y(t) < \log t$ always.

iv) For the concavity we notice that

$$y'' = \left(\frac{e^y}{e^y - t}\right)' = \frac{e^y y'(e^y - t) - e^y(e^y y' - 1)}{(e^y - t)^2} = \frac{e^y(-ty' + 1)}{(e^y - t)^2}.$$

Therefore,

$$y \uparrow, \iff y'' \geqslant 0, \iff 1 - ty' \geqslant 0.$$

Since $t \in]\alpha, \beta[$, $t > \alpha > 1 > 0$ and $y' \leqslant 0$, we conclude that $1 - ty' \geqslant 1 > 0$ for all t, thus $y'' > 0$ for all t and y is convex.

$\beta = +\infty$? We have two possibilities: $\beta < +\infty$, $\beta = +\infty$. If $\beta < +\infty$ then, defined $y(\beta) = \lim_{t \to \beta} y(t) \geqslant -\infty$ (the limit exists being y decreasing), we would have

- either $y(\beta) > -\infty$: in this case, the solution would remain into the compact box $[2, \beta] \times [y(\beta), 0]$ for all future times, violating the exit from compact sets, impossible;
- or $y(\beta) = -\infty$: also this is impossible because, by convexity, y would be above each of its tangents, in particular to that one for $t = 2$:

  $$y(t) \geqslant y(2) + y'(2)(t - 2) \implies y(\beta) \geqslant y(2) + y'(2)(\beta - 2) > -\infty,$$

  which is impossible being $y(\beta) = -\infty$.

In any case, we got a contradiction. This means $\beta = +\infty$. □

Exercise 9. i) $f = u + iv$ is $\mathbb{C}-$differentiable on $\mathbb{C}$ iff u, v are $\mathbb{R}-$differentiable on $\mathbb{R}^2$ and u, v fulfill the CR conditions. Clearly v is differentiable. Thus we have to look at $u = u(x, y)$ $\mathbb{R}-$differentiable such that

$$\begin{cases} \partial_x u &= \partial_y v = -e^{-y}(y\cos x + x\sin x) + e^{-y}\cos x, \\ \partial_y u &= -\partial_x v = -e^{-y}\left(-y\sin x + \sin x + x\cos x\right). \end{cases}$$

From the first equation,

$$u(x, y) = \int \partial_x u(x, y)\, dx + c(y) = -e^{-y}\left(y\sin x - x\cos x\right) + c(y).$$

We have

$$\partial_y u = e^{-y}\left(y\sin x - x\cos x\right) - e^{-y}\sin x + c'(y) = e^{-y}\left(y\sin x - x\cos x + \sin x\right) + c'(y)$$

thus $\partial_y u = -\partial_x v$ iff $c'(y) = 0$, that is $c(y)$ is constant. We conclude that

$$u(x, y) = -e^{-y}\left(y\sin x - x\cos x\right) + c + e^{-y}\left(y\cos x + x\sin x\right).$$

ii) We have

$$\begin{aligned} f &= u + iv = -e^{-y}\left(y\sin x - x\cos x\right) + ie^{-y}\left(y\cos x + x\sin x\right) \\ &= e^{-y}\left(y\left(-\sin x + i\cos x\right) + x\left(\cos x + i\sin x\right)\right) \\ &= e^{-y}\left(iye^{ix} + xe^{ix}\right) \\ &= e^{ix-y}(iy + x) = e^{i(x+iy)}(x + iy) = e^{iz}z. \quad \square \end{aligned}$$

Exercise 10. Let $\vec{F} := f\nabla f = (f\partial_x f, f\partial_y f) =: (F_1, F_2)$. According to Green formula,

$$\oint_{\partial D} f\nabla f = \oint_{\partial D} \vec{F} = \int_D \left(\partial_y F_1 - \partial_x F_2\right)\, dxdy.$$

Now, since

$$\partial_y F_1 = \partial_y(f\partial_x f) = \partial_y f\partial_x f + f\partial_{yx} f, \;\; \partial_x F_2 = \partial_x(f\partial_y f) = \partial_x f\partial_y f + f\partial_{xy} f$$

we easily deduce that $\partial_y F_1 - \partial_x F_2 \equiv 0$ being $f \in \mathscr{C}^2(\mathbb{R}^2)$. □

Exercise 11. i) Clearly $f(x, 0) = x^6 - x^4 \longrightarrow +\infty$ for $|x| \longrightarrow +\infty$. So, if a limit exists it must be $= +\infty$. We check this changing coordinates and using polar coords:

$$f(x, y) = \rho^6 - (\rho\cos\theta)^4 + (\rho\sin\theta)^4 \geqslant \rho^6 - 2\rho^4 \longrightarrow +\infty, \text{ if } \rho = \|(x, y)\| \longrightarrow +\infty.$$

ii) By i) and a consequence of Weierstrass theorem, f has global minimum on $\mathbb{R}^2$ but not any global maximum. Since every point of $\mathbb{R}^2$ lies in its interior, according to Fermat theorem (clearly

$\partial_x f = 6x(x^2+y^2)^2 - 4x^3$ and $\partial_y f = 6y(x^2+y^2)2 + 4y^3$ are both continuous on $\mathbb{R}^2$, hence f is differentiable on $\mathbb{R}^2$ according to the differentiability test), at min we have $\nabla f = \vec{0}$. Now,

$$\nabla f = \vec{0}, \iff \begin{cases} 6x(x^2+y^2)^2 - 4x^3 = 0, \\ \\ 6y(x^2+y^2)^2 + 4y^3 = 0 \end{cases} \iff \begin{cases} x\left(6(x^2+y^2)^2 - 4x^2\right) = 0, \\ \\ y\left(6(x^2+y^2)^2 + 4y^2\right) = 0, \end{cases}$$

Now, looking at second equation, we see that either $y = 0$ or $6(x^2+y^2)^2 + 4y^2 = 0$. In the second case we obtain trivially $x = 0$ and $y = 0$, thus the point $(0,0)$. Plugging $y = 0$ into the first equation we get

$$x\left(6x^4 - 4x^2\right) = 0, \iff x^3(3x^2-2) = 0, \iff x = 0, \vee x = \pm\sqrt{\frac{2}{3}}.$$

Thus we have again $(0,0)$ and two more points $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$. Since $f(0,0) = 0$ while

$$f\left(\pm\sqrt{\frac{2}{3}}, 0\right) = \frac{8}{27} - \frac{4}{9} = -\frac{28}{27} < f(0,0) = 0,$$

we conclude that $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$ are global minimums. Finally, since $\mathbb{R}^2$ is connected,

$$f(\mathbb{R}^2) = \left[-\frac{28}{27}, +\infty\right[. \quad \square$$

Exercise 12.

Exercise 13. ii) We have,

$$\begin{aligned} \lambda_3(D) &= \int_{x^2+2y^2 \leqslant z \leqslant 4-3(x^2+2y^2)} 1 \; dxdydz \\ &\overset{RF}{=} \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} \int_{x^2+2y^2}^{4-3(x^2+2y^2)} 1 \; dz \; dxdy \\ &= \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} 4\left(1 - (x^2+2y^2)\right) \; dxdy. \end{aligned}$$

Noticed that $x^2 + 2y^2 \leqslant 4 - 3(x^2+2y^2)$ iff $x^2 + 2y^2 \leqslant 1$, we have

$$\lambda_3(D) = \int_{x^2+2y^2 \leqslant 1} 4\left(1 - (x^2+2y^2)\right) \; dxdy.$$

Changing variables to adapted polar coordinates

$$x = \rho\cos\theta, \quad \sqrt{2}y = \rho\sin\theta,$$

we have

$$\lambda_3(D) = \int_{0\leqslant\rho\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi} 4\left(1-\rho^2\right)\frac{\rho}{\sqrt{2}}\, d\rho d\theta \overset{RF}{=} \frac{8\pi}{\sqrt{2}}\int_0^1 (\rho - \rho^3)\, d\rho = \frac{8\pi}{\sqrt{2}}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{4\pi}{\sqrt{2}}. \quad \square$$

Exercise 14. i) Let $u = x^2 + y^2$. Clearly, u is $\mathbb{R}-$differentiable on $\mathbb{R}^2$. Therefore, $f = u + iv$ is $\mathbb{C}-$differentiable iff v is $\mathbb{R}-$differentiable and CR equations hold,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases} \iff \begin{cases} \partial_x v = -\partial_y u = -2y, \\ \\ \partial_y v = \partial_x u = 2x. \end{cases}$$

From the first equation $v(x,y) = -\int 2y\, dx + c(y) = -2xy + c(y)$. Plugging this into the second equation we have $\partial_y v = -2x + c'(y) = 2x$, that is $c'(y) = 4x$, which is impossible since c does not depend on y. We conclude that such v does not exist.

ii) Since there is no v such that $f = u + iv$ is $\mathbb{C}-$differentiable, there is no f to be found. □

Exercise 15. See notes for the statement. We may formally set the optimization problem in the following way. The set $y = f(x)$ is also $f(x) - y = 0$. Setting $g(x,y) := f(x) - y$ we see that

$$\{(x,y) \ : \ y = f(x)\} = \{(x,y) \ : \ g(x,y) = 0\}.$$

Let's check that g is a submersion on $\{g = 0\}$. Indeed $\nabla g = (\partial_x g, \partial_y g) = (f'(x), -1) \neq 0$, whatever is x. Let now

$$d(x,y) := (x-a)^2 + (y-b)^2,$$

the square of distance from (a,b) to (x,y). At minimum (x,y) on the curve, that is $y = f(x)$, according to Lagrange theorem we have

$$\nabla d = \lambda \nabla g = \lambda (f'(x), -1).$$

Since

$$\nabla d = (2(x-a), 2(y-b)) = 2(x-a, y-b) = 2(Q-P),$$

we have

$$Q - P = \frac{\lambda}{2}(f'(x), -1).$$

Now, since the tangent direction to $y = f(x)$ at point $(x, f(x))$ is $(1, f'(x))$, and clearly $(f'(x), -1) \perp (1, f'(x))$, we have that

$$Q - P \parallel (f'(x), -1) \perp (1, f'(x)) \parallel \text{ tangent to } f,$$

from which the conclusion follows. □

Exercise 16. i) Point $(0, y, 0) \in D$ iff $y^2 = 1$ and $y^2 = 1$, that is $y = \pm 1$, so $(0, \pm 1, 0) \in D$. D is the zero set of $(g_1, g_2) = (x^2 + y^2 - z^2 - 1, y^2 + z - 1)$. According to the Definition,

$$(g_1, g_2) \text{ is a submersion on } D \iff \operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} = 2 \text{ on } D.$$

Since this is a 2×3 matrix, its rank is < 2 iff all 2×2 sub determinant equal 0, or

$$\begin{cases} 4xy = 0, \\ 2x = 0, \\ 2y(-1 + 2z) = 0, \end{cases} \iff \begin{cases} x = 0, \\ \\ y(1 + 2z) = 0. \end{cases} \iff \begin{cases} x = 0, \\ y = 0, \end{cases} \iff (0, 0, z),$$

$$\iff \begin{cases} x = 0, \\ \\ z = -\frac{1}{2}, \end{cases} \iff (0, y, -\tfrac{1}{2}).$$

Now,

- $(0,0,z) \in D$ iff $-z^2 = 1$ and $z = 1$, impossible;
- $(0, y, -\frac{1}{2}) \in D$ iff $y^2 = \frac{5}{4}$ and $y^2 = \frac{3}{2}$, impossible.

Conclusion: at no point of D the rank of the matrix $\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix}$ is less than 2, thus (g_1, g_2) is a submersion on D.

ii) D is certainly closed being defined by equations involving continuous functions. Is it also bounded? From the second equation $y^2 = 1 - z$, thus $y = \pm\sqrt{1-z}$ for $z \leqslant 1$. Plugging this into the first equation

$$x^2 = z^2 - (1-z) + 1 = z^2 + z = z(z+1), \implies x = \pm\sqrt{z^2+z} \text{ for } z \leqslant 0 \ \vee \ z \geqslant 1.$$

In particular, for $z \leqslant 0$ points

$$(\pm\sqrt{z^2+z}, \pm\sqrt{1-z}, z) \in D, \ \forall z \leqslant 0.$$

These points are unbounded because

$$\|(\pm\sqrt{z^2+z}, \pm\sqrt{1-z}, z)\|^2 = z^2 + z + (1-z) + z^2 = 2z^2 + 1 \longrightarrow +\infty, \ z \longrightarrow -\infty.$$

We conclude that D is unbounded.

iii) By ii) D is closed and unbounded. We have to min/max $\sqrt{x^2+y^2+z^2}$ or, equivalently, $f := x^2 + y^2 + z^2$, which is continuous on D and such that $\lim_{\infty_3} f = +\infty$. We conclude f has no max point on D while it has min points. By i) and according to the Lagrange multipliers theorem, at min point we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} < 3.$$

This happens iff the determinant of the previous jacobian matrix equals 0, that is

$$8xy(x+z) = 0, \quad \iff \quad x = 0, \ \vee \ y = 0, \ \vee \ z = -x.$$

This leads to points $(0, y, z)$, $(x, 0, z)$ and $(x, y, -x)$. Now,

- $(0, y, z) \in D$ iff $y^2 - z^2 = 1$ and $y^2 + z = 1$. From these, $z^2 + z = 0$ that is, $z = 0$ or $z = -1$, thus we have points $(0, \pm 1, 0)$ and $(0, \pm\sqrt{2}, -1)$;
- $(x, 0, z) \in D$ iff $x^2 - z^2 = 1$ and $z = 1$, that is $(\pm\sqrt{2}, 0, 1)$.
- $(x, y, -x) \in D$ iff $x^2 + y^2 - x^2 = 1$ and $y^2 - x = 1$, that is $y^2 = 1$ and $x = 0$, from which we have points $(0, \pm 1, 0)$.

Conclusion: min points are among $(0, \pm 1, 0)$, $(0, \pm\sqrt{2}, -1)$, $(\pm\sqrt{2}, 0, 1)$, and clearly thos at min distance to $\vec{0}$ are $(0, \pm 1, 0)$. □

Exercise 17. i) To be irrotational, the field must verify

$$\partial_y \frac{ax + by}{\sqrt{x^2+y^2}} \equiv \partial_x \frac{cx + dy}{\sqrt{x^2+y^2}}, \ \ \forall (x, y) \in D = \mathbb{R}^2 \backslash \{\vec{0}\}.$$

We have

$$\partial_y \frac{ax+by}{\sqrt{x^2+y^2}} = \frac{b\sqrt{x^2+y^2} - (ax+by)\frac{2y}{2\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{b(x^2+y^2) - y(ax+by)}{(x^2+y^2)^{3/2}} = \frac{bx^2 - axy}{(x^2+y^2)^{3/2}},$$

and, similarly

$$\partial_x \frac{cx+dy}{\sqrt{x^2+y^2}} = \frac{cy^2 - dxy}{(x^2+y^2)^{3/2}}.$$

Thus, the field is irrotational iff

$$\frac{bx^2 - axy}{(x^2+y^2)^{3/2}} \equiv \frac{cy^2 - dxy}{(x^2+y^2)^{3/2}}, \quad \iff \quad bx^2 - axy = cy^2 - dxy, \ \ \forall (x,y) \in \mathbb{R}^2 \backslash \{\vec{0}\}.$$

Since the identity is trivally verified at $(x,y) = \vec{0}$, we may say that the field is irrotational iff

$$bx^2 - axy \equiv cy^2 - dxy, \quad \iff \quad b = c = 0, \ a = d.$$

ii) By i), to be conservative $\vec{F}$ must have the form

$$\vec{F} = \left(\frac{ax}{\sqrt{x^2+y^2}}, \frac{ay}{\sqrt{x^2+y^2}} \right)$$

Now, such a $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = \frac{ax}{\sqrt{x^2+y^2}}, \\ \\ \partial_y f = \frac{ay}{\sqrt{x^2+y^2}}. \end{cases}$$

From first equation,

$$f(x,y) = \int \frac{ax}{\sqrt{x^2+y^2}}\, dx + k(y) = \frac{a}{2} \int (x^2+y^2)^{-1/2}(2x)\, dx + k(y) = a(x^2+y^2)^{1/2} + k(y).$$

Plugging this into the second equation we have

$$\partial_y f = a\frac{1}{2}(x^2+y^2)^{-1/2}2y + k'(y) = \frac{ay}{\sqrt{x^2+y^2}}, \quad \iff \quad k'(y) = 0.$$

Thus, we deduce that

$$f(x,y) = a\sqrt{x^2+y^2} + k, \ \ k \in \mathbb{R},$$

are all the potentials for $\vec{F}$. □

Exercise 18.

Exercise 19. In order $f = u + iv$ be holomorphic, we need that u, v are both $\mathbb{R}-$differentiable (and certainly v it is), and they verify the CR equations,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

Thus we have to look for an $\mathbb{R}-$differentiable u such that

$$\begin{cases} \partial_x u = 3y^2 - 3x^2 + 4x, \\ \\ \partial_y u = -(-6xy + 4y - 1). \end{cases}$$

From the first equation we get,

$$u(x,y) = \int (3y^2 - 3x^2 + 4x)\, dx + k(y) = 3y^2x - x^3 + 2x^2 + k(y).$$

Plugging this into the second equation we have

$$6xy + k'(y) = 6xy - 4y + 1, \quad \iff \quad k'(y) = -4y + 1, \quad \iff \quad k(y) = -2y^2 + y + k,\ k \in \mathbb{R}.$$

Thus, all the possible u that verify the CR eqns together with v are

$$u(x,y) = 3y^2x - x^3 + 2x^2 - 2y^2 + y + k.$$

Since such u are clearly $\mathbb{R}-$differentiable, $f = u + iv$ is $\mathbb{C}-$differentiable (holomorphic) on $\mathbb{R}^2$.

To determine the analytical expression for f as a function of complex variable $z = x + iy$, we may notice that

$$\begin{aligned} f \quad &= u + iv = 3y^2x - x^3 + 2x^2 - 2y^2 + y + k + i\left(y^3 - 3x^2y + 4xy - x\right) \\ &= -i\underbrace{(x+iy)}_{z} + 2\underbrace{(x^2 - y^2 + i2xy)}_{z^2} - \underbrace{\left(x^3 - iy^3 - 3y^2x + i3x^2y\right)}_{z^3} + k \\ &= -z^3 + 2z^2 - iz + k. \quad \square \end{aligned}$$

Exercise 20. i) See LN.

ii) We introduce the change of variable

$$(u,v,w) := (\vec{a}\cdot\vec{x}, \vec{b}\cdot\vec{x}, \vec{c}\cdot\vec{x}) = \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \vec{x} =: M\vec{x} =: \Psi(\vec{x})$$

where M is the 3×3 matrix made by $\vec{a}, \vec{b}, \vec{c}$ as lines. Since $\vec{a}, \vec{b}, \vec{c}$ are linearly independent, M has rank 3, thus it is invertible and the previous is a true change of variable. Moreover, $\Phi'(\vec{x}) = M$ so

$$|\det(\Psi^{-1})'| = \frac{1}{|\det \Psi'|} = \frac{1}{|\det M|}.$$

therefore, according to the change of variable formula

$$\begin{aligned}\int_{E_{\alpha,\beta,\gamma}} (\vec{a}\cdot\vec{x})(\vec{b}\cdot\vec{x})(\vec{c}\cdot\vec{x})\,d\vec{x} &= \int_{0\leqslant u\leqslant\alpha,\ 0\leqslant v\leqslant\beta,\ 0\leqslant w\leqslant\gamma} uvw\frac{1}{|\det M|}\,dudvdw\\ &\overset{RF}{=} \frac{1}{|\det M|}\left(\int_0^\alpha u\,du\right)\left(\int_0^\beta v\,dv\right)\left(\int_0^\gamma w\,dw\right)\\ &= \frac{1}{|\det M|}\frac{\alpha^2}{2}\frac{\beta^2}{2}\frac{\gamma^2}{2} = \frac{(\alpha\beta\gamma)^2}{8|\det M|}.\end{aligned}$$

(clearly, there is a typo in the text of the exercise, $\det M$ must be replaced by $|\det M|$). □

Exercise 21. i) We have $(x,y,0)\in\Gamma$ iff $x^2+y^2=1$ and $x^2=1$, thus $x=\pm1$ and $y^2=0$, hence $(\pm1,0,0)\in\Gamma$. Now, $\Gamma=\{g_1=0,\ g_2=0\}$, where $g_1=x^2+y^2-1$, and $g_2=x^2+z^2-xz-1$. Clearly $g_1,g_2\in\mathscr{C}^1$ and (g_1,g_2) is a submersion on Γ iff

$$\operatorname{rank}\begin{bmatrix}\nabla g_1\\ \nabla g_2\end{bmatrix} = \operatorname{rank}\begin{bmatrix}2x & 2y & 0\\ 2x-z & 0 & 2z-x\end{bmatrix} = 2,\ \forall(x,y,z)\in\Gamma.$$

This is false iff all 2×2 submatrices have determinant =0, that is

$$\begin{cases}2y(2x-z)=0,\\ 2x(2z-x)=0,\\ 2y(2z-x)=0.\end{cases}$$

Working on the first equation, we have the alternatives $y=0$ or $2x-z=0$. In the first case, the system reduces to $x(2z-x)=0$ that is $x=0$ (points $(0,0,z)$) or $x=2z$ (points $(2z,0,z)$). In the second case, the system reduces to

$$\begin{cases}z=2x,\\ 3x^2=0,\\ 3yx=0,\end{cases}\quad\Longleftrightarrow\quad(0,y,0).$$

Thus, rank is less than 2 at points $(0,0,z)$, $(2z,0,z)$ and $(0,y,0)$. Now:

- $(0,0,z)\in\Gamma$ iff $0=1$ (first condition), impossible;
- $(2z,0,z)\in\Gamma$ iff $4z^2=1$ and $5z^2=2z^2+1$, that is $z^2=\frac{1}{4}$ and $z^2=\frac{1}{3}$ which are impossible together.
- $(0,y,0)\in\Gamma$ iff $y^2=1$ and $0=1$, which is, again, impossible.

Conclusion: none of points where rank is ¡2 belong to Γ, this meaning that rank =2 on Γ, hence (g_1,g_2) is a submersion on Γ.

ii) Clearly Γ is closed because defined by equations involving continuous functions. Boundedness: from first equation we deduce $x^2,y^2\leqslant1$. From second equation, recalling that $ab\leqslant\frac{a^2+b^2}{2}$ we have

$$x^2+z^2=xz+1\leqslant\frac{x^2+z^2}{2}+1,\ \Longrightarrow\ \frac{x^2+z^2}{2}\leqslant1,$$

from which, in particular, $z^2 \leqslant 2$. Therefore $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2} \leqslant \sqrt{1+1+2} = \sqrt{4} = 2$, for every $(x,y,z) \in \Gamma$. Conclusion: Γ is bounded, hence compact.

iii) We have to minimize/maximize $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ or, equivalently, $f(x,y,z) = x^2+y^2+z^2$. By ii), Γ is compact and obviously $f \in \mathscr{C}$, thus existence of min and max for f is ensured by Weierstrass' theorem. To determine min/max points we apply Lagrange's thm. According to i), this thm can be applied on Γ. We deduce that, at min/max points $(x,y,z) \in \Gamma$,

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \Longleftrightarrow \quad \operatorname{rank} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 2x-z & 0 & 2z-x \end{bmatrix} = 2,$$

or, equivalently, the determinant of this last matrix equals 0. We obtain

$$2z \cdot (-2y(2x-z)) = 0, \quad \Longleftrightarrow \quad yz(2x-z) = 0, \quad \Longleftrightarrow \quad y = 0, \ \vee\, z = 0, \ \vee\, z = 2x.$$

Thus possible min/max points are among points $(x,0,z)$, $(x,y,0)$ and $(x,y,2x)$. Now,

- $(x,0,z) \in \Gamma$ iff $x^2 = 1$ and $x^2+z^2 = xz+1$, or, equivalently, $x^2 = 1$ and $z^2 = xz+1$. For $x = 1$ we get $z^2 = z+1$, that is $z = \frac{1\pm\sqrt{5}}{2}$, namely points $(1,0,\frac{1\pm\sqrt{5}}{2})$. For $x = -1$ we get $z^2 = -z+1$, that is $z = \frac{-1\pm\sqrt{5}}{2}$, namely points $(-1,0,\frac{-1\pm\sqrt{5}}{2})$.
- $(x,y,0) \in \Gamma$ iff $x^2+y^2 = 1$ and $x^2 = 1$, that is $x = \pm 1$ and $y^2 = 0$, namely points $(\pm 1,0,0)$.
- $(x,y,2x) \in \Gamma$ iff $x^2+y^2 = 1$ and $x^2+4x^2 = 2x^2+1$, from which $x^2 = \frac{1}{3}$, $x = \pm\frac{1}{\sqrt{3}}$ and $y^2 = \frac{2}{3}$, $y = \pm\sqrt{\frac{2}{3}}$, thus we get points $\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right)$ (4 points).

We have

- $f(1,0,\frac{1\pm\sqrt{5}}{2}) = 1 + \left(\frac{1\pm\sqrt{5}}{2}\right)^2 = \frac{10\pm 2\sqrt{5}}{4}$, $f(-1,0,\frac{-1\pm\sqrt{5}}{2}) = 1 + \left(\frac{-1\pm\sqrt{5}}{2}\right)^2 = \frac{10\pm 2\sqrt{5}}{4}$;
- $f(\pm 1,0,0) = 1$;
- $f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = \frac{1}{3}+\frac{2}{3}+\frac{4}{3} = \frac{7}{3}$ and $f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{1}{3}+\frac{2}{3}+\frac{4}{3} = \frac{7}{3}$.

From this we see that $(1,0,\frac{1+\sqrt{5}}{2})$ and $(-1,0,\frac{-1-\sqrt{5}}{2})$ are maximum points while $(\pm 1,0,0)$ are min points. □

Exercise 22. ii) D is closed (because defined by large inequalities involving continuous functions) and bounded (the root imposes $x^2+y^2 \leqslant 1$ and, consequently, $0 \leqslant 1-(x^2+y^2) \leqslant z \leqslant \sqrt{1-(x^2+y^2)} \leqslant \sqrt{1}$, that is $0 \leqslant z \leqslant 1$). Thus D is compact, hence 1_D is integrable on D. Furthermore, noticed that, calling $\rho^2 = x^2+y^2$,

$$1-\rho^2 \leqslant \sqrt{1-\rho^2}, \quad \Longleftrightarrow \quad \sqrt{1-\rho^2} \leqslant 1,$$

which is always true, thus $1-(x^2+y^2) \leqslant \sqrt{1-(x^2+y^2)}$ always when defined. Then

$$\begin{aligned}
\text{Vol } D &= \int_D 1\, dxdydz \overset{RF}{=} \int_{x^2+y^2\leqslant 1} \int_{1-(x^2+y^2)}^{\sqrt{1-(x^2+y^2)}} 1\, dz\, dxdy \\
&= \int_{x^2+y^2\leqslant 1} \left(\sqrt{1-(x^2+y^2)} - (1-(x^2+y^2))\right)\, dxdy \\
&\overset{pol.\ coords}{=} \int_{0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\rho\leqslant 1} \left(\sqrt{1-\rho^2} - 1 + \rho^2\right) \rho\, d\rho d\theta \\
&\overset{RF}{=} 2\pi \int_0^1 \rho(1-\rho^2)^{1/2} - \rho + \rho^3\, d\rho = 2\pi \left[\left[-\tfrac{1}{3}(1-\rho^2)^{3/2}\right]_{\rho=0}^{\rho=1} - \left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} + \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\right] \\
&= 2\pi \left[+\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4}\right] = \frac{\pi}{6}. \quad \square
\end{aligned}$$

Exercise 23.

Exercise 24. Let $z \neq 0$. Setting $w = \frac{1}{z}$, we have to solve

$$\sinh w = 0, \iff \frac{e^w - e^{-w}}{2} = 0, \iff e^{2w} = 1, \iff 2w = \log|1| + i\,(0 + k2\pi) = ik2\pi,\ k \in \mathbb{Z}.$$

Thus

$$\frac{1}{z} = w = ik\pi, \iff z = \frac{1}{ik\pi} = \frac{-i}{k\pi} = \frac{i}{k\pi},\ k \in \mathbb{Z}\backslash\{0\}. \quad \square$$

Exercise 25. i) See notes for definitions and characterizations.

ii) If $\partial S = \varnothing, \partial S$ it is closed. Let $\partial S \neq \varnothing$. To verify that ∂S is closed, we use the Cantor characterization. Let $(\vec{x}_n) \subset \partial S$ be such that $\vec{x}_n \longrightarrow \vec{x} \in \mathbb{R}^d$. We prove that $\vec{x} \in \partial S$. Fix $r > 0$. Since $\vec{x}_n \longrightarrow \vec{x}$, we have that for $n \geqslant N$ $\|\vec{x}_n - \vec{x}\| \leqslant \frac{r}{2}$. Now, since $\vec{x}_n \in \partial S$,

$$B(\vec{x}_n, r/2] \cap S \neq \varnothing, \ \wedge B(\vec{x}_n, r/2] \cap S^c \neq \varnothing.$$

Since $\|\vec{x}_n - \vec{x}\| \leqslant \frac{r}{2}$, we have that

$$B(\vec{x}_n, r/2] \subset B(\vec{x}, r],$$

therefore

$$B(\vec{x}, r] \cap S \supset B(\vec{x}_n, r/2] \cap S \neq \varnothing,$$

and, similarly, $B(\vec{x}, r] \cap S^c \neq \varnothing$. We conclude that $\vec{x} \in \partial S$, thus ∂S is closed. $\square$

Exercise 26. i) See LN for statements.

ii) To extend f at $(0,0)$ as a continuous function, we need to compute the limit

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x^2 \log(x^2+y^2).$$

In polar coordinates

$$f(x,y) = \rho^2 \cos^2\theta \log\rho^2, \implies |f(x,y)| \leqslant 2\rho^2 |\log\rho| \longrightarrow 0,\ \rho \to 0+.$$

We conclude that

$$\exists \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Thus, setting

$$f(x,y) := \begin{cases} x^2\log(x^2+y^2), & (x,y)\neq(0,0), \\ \\ 0, & (x,y)=(0,0), \end{cases}$$

we get a continuous function at $(0,0)$. To check differentiability we notice that, for $(x,y)\neq(0,0)$,

$$\partial_x f(x,y) = 2x\log(x^2+y^2) + x^2\frac{2x}{x^2+y^2},\ \ \partial_y f(x,y) = x^2\frac{2y}{x^2+y^2}.$$

Clearly $\partial_x f, \partial_y f \in \mathscr{C}(\mathbb{R}^2\backslash\{(0,0)\})$ so, according to the differentiability test, f is differentiable on $\mathbb{R}^2\backslash\{(0,0)\}$. Let's discuss differentiability at $(0,0)$. Previous formulas do not allow to compute $\partial_x f(0,0)$ and $\partial_y f(0,0)$. However, we notice that

$$\partial_x f(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{t^2\log t^2}{t} = \lim_{t\to 0} t\log t^2 = 0,$$

and,

$$\partial_y f(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0.$$

We can now easily check that $\partial_x f$ and $\partial_y f$ are continuous at $(0,0)$ too. Indeed, for example

$$|\partial_x f(x,y)| = \left|2\rho\cos\theta\log\rho^2 + 2\frac{\rho^3\cos^3\theta}{\rho^2}\right| \leqslant 2\rho|\log\rho^2| + 2\rho \longrightarrow 0,\ \rho\to 0+.$$

Therefore

$$\lim_{(x,y)\to(0,0)}\partial_x f(x,y) = 0 = \partial_x f(0,0),$$

thus $\partial_x f$ is continuous at $(0,0)$. Similarly, $\partial_y f$ is continuous at $(0,0)$. We conclude that f is differentiable at $(0,0)$. □

Exercise 27. i) D is closed being defined by large inequalities involving continuous functions of (x,y). It is not open since $D\neq\varnothing,\mathbb{R}^2$. It is bounded because $x\geqslant 0$ and from $0\leqslant y\leqslant 1-x$, in particular $1-x\geqslant 0$, that is $x\leqslant 1$, so $0\leqslant x\leqslant 1$ and, at same time, $0\leqslant y\leqslant 1-x\leqslant 1$. Thus $0\leqslant x,y\leqslant 1$ and this implies that D is bounded. Since D is closed and bounded it is also compact.

ii) Since f is clearly continuous on D and D is compact, f admits both global min/max on D. To determine min/max points, we may argue in the following way. If $(x,y)\in D$ is a min/max point for f then

- either $(x,y)\in \text{Int } D$
- or $(x,y)\in D\backslash\text{Int } D = \partial D$.

In the first case, since

$$\partial_x f = 3y + 2xy + y^2,\ \ \partial_y f = 3x + x^2 + 2xy$$

so $\partial_x f, \partial_y f\in\mathscr{C}(D)$, f is then differentiable on D, according to Fermat theorem, at min/max points

$$\nabla f(x,y) = \vec{0},\ \ \Longleftrightarrow\ \ \begin{cases} 3y+2xy+y^2 = 0, \\ \\ 3x+x^2+2xy = 0. \end{cases}\ \ \Longleftrightarrow\ \ \begin{cases} y(3+2x+y) = 0, \\ \\ x(3+2y+x) = 0. \end{cases}$$

The first equation leads to the alternative $y = 0$ or $3 + 2x + y = 0$. In the first case, the second equation becomes $x(3 + x) = 0$. whose solutions are $x = 0$ and $x = -3$. This produces points $(0, 0)$ and $(-3, 0)$. In any case these do not belong to Int D. In the second case, $y = -2x - 3$, from the second equation we obtain $x(-3 - 3x) = 0$, that is $x = 0$ or $x = -1$. This yields points $(0, -3), (-1, -1) \notin D$. In conclusion, no stationary point for f is in the interior of D.

Thus, min/max points for f are on $\partial D = A \cup B \cup C$ where $A = \{(0, y) \ : \ 0 \leqslant y \leqslant 1\}$, $B = \{(x, 0) \ : \ 0 \leqslant x \leqslant 1\}$ and, finally, $C = \{(x, 1 - x) \ : \ 0 \leqslant x \leqslant 1\}$. On A we have

$$f(0, y) \equiv 0,$$

thus every point is min/max point for f on A. On B, similarly, we have $f(x, 0) \equiv 0$, thus every point of B is at same time min/max for f on B. Finally, on C

$$f(x, 1 - x) = 3x(1 - x) + x^2(1 - x) + x(1 - x)^2 = 3x - 3x^2 + x^2 - x^3 + x - 2x^2 + x^3 = -4x^2 + 4x =: g(x).$$

Let's determine min/max points for g with $x \in [0, 1]$. We have $g'(x) = -8x + 4 \geqslant 0$ iff $x \leqslant \frac{1}{2}$. Thus $x = \frac{1}{2}$ is max point for g and $x = 0, 1$ are min points for g. This means that

- $\left(\frac{1}{2}, \frac{1}{2}\right)$ is max point for f on C
- $(0, 1)$, $(1, 0)$ are min points for f on C.

We can now draw the conclusion:

- for minimum, candidates are points $(x, 0)$, $(0, y)$ with $0 \leqslant x, y \leqslant 1$ where $f = 0$. All these are min points for f on D;
- for maximum, candidates are points $\left(\frac{1}{2}, \frac{1}{2}\right)$ (where $f = 1$) and $(x, 0)$ and $(0, y)$ with $0 \leqslant x, y \leqslant 1$ (where $f = 0$). Thus, the max point is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

□

Exercise 28. i) Let $\vec{F} = (\phi, \psi)$. In order $\vec{F}$ be irrotational on D we need

$$\partial_y \phi \equiv \partial_x \psi, \text{ on } D.$$

We have

$$\partial_y \phi = \frac{b(x^2+y^2)^2 - (ax+by)2(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{b(x^2+y^2) - 4y(ax+by)}{(x^2+y^2)^2} = \frac{bx^2 - 4axy - 3by^2}{(x^2+y^2)^2},$$

$$\partial_x \psi = \frac{c(x^2+y^2)^2 - (cx+dy)2(x^2+y^2)2x}{(x^2+y^2)^4} = \frac{c(x^2+y^2) - 4x(cx+dy)}{(x^2+y^2)^2} = \frac{-3cx^2 - 4dxy + cy^2}{(x^2+y^2)^2}.$$

Hence,

$$\partial_y \phi \equiv \partial_x \psi, \iff bx^2 - 4axy - 3by^2 \equiv -3cx^2 - 4dxy + cy^2, \iff \begin{cases} b = -3c, \\ a = d, \\ -3b = c \end{cases}$$

from which $b = c = 0$ and $a = d \in \mathbb{R}$. Thus

$$\vec{F} = \left(\frac{ax}{(x^2 + y^2)^2}, \frac{ay}{(x^2 + y^2)^2}\right), \ \forall (x, y) \in D.$$

ii) Necessary condition to be conservative is that $\vec{F}$ be irrotational, thus $\vec{F}$ is given as at the end of i). Now, such $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = \frac{ax}{(x^2+y^2)^2}, \\ \\ \partial_y f = \frac{ay}{(x^2+y^2)^2}. \end{cases}$$

From the first equation

$$f(x,y) = \int \frac{ax}{(x^2+y^2)^2}\, dx + k(y) = \frac{a}{2}\int \partial_x - (x^2+y^2)^{-1}\, dx + k(y) = -\frac{a}{2}(x^2+y^2)^{-1} + k(y).$$

Plugging this into the second equation we have

$$\partial_y f = \frac{ay}{(x^2+y^2)^2}, \iff \frac{ay}{(x^2+y^2)^2} + k'(y) = \frac{ay}{(x^2+y^2)^2}, \iff k'(y) = 0, \iff . .k(y) = k \in \mathbb{R}.$$

Thus, $\vec{F}$ is conservative with potentials

$$f(x,y) = -\frac{a}{2}(x^2+y^2)^{-1} + k,\ k \in \mathbb{R}.$$

iii) By previous discussion, when $(a,b,c,d) = (2,0,0,2)$, field $\vec{F}$ is conservative. Thus

$$\int_\gamma \vec{F} = f(0,2) - f(1,0) = -\frac{1}{4} - (-1) = \frac{3}{4}. \quad \square$$

Exercise 29. ii) The change or variable is given in the form $(u,v) = \Phi(x,y) = (y - x^3, y + x^3)$. According to the change of variable formula,

$$\int_D f(x,y)\, dxdy = \int_{\Phi(D)} f(\Phi^{-1}(u,v))|\det(\Phi^{-1})'(u,v)|\, dudv.$$

We need to determine Φ^{-1}. Notice that

$$\begin{cases} u = y - x^3, \\ \\ v = y + x^3, \end{cases} \iff \begin{cases} u + v = 2y, \\ \\ v - u = 2x^3, \end{cases} \iff \begin{cases} y = \frac{u+v}{2}, \\ \\ x^3 = \frac{v-u}{2}, \end{cases} \iff \begin{cases} y = \frac{u+v}{2}, \\ \\ x = \left(\frac{v-u}{2}\right)^{1/3}, \end{cases}$$

Therefore

$$\Phi^{-1}(u,v) = \left(\left(\frac{v-u}{2}\right)^{1/3}, \frac{u+v}{2}\right).$$

Moreover,

$$(x,y) \in D, \iff \begin{cases} x \geqslant 1, \\ \\ x^3 \leqslant y \leqslant 3, \end{cases} \iff \begin{cases} \left(\frac{v-u}{2}\right)^{1/3} \geqslant 1, \\ \\ \frac{v-u}{2} \leqslant \frac{u+v}{2} \leqslant 3 \end{cases} \iff \begin{cases} v - u \geqslant 2, \\ \\ v - u \leqslant v + u \leqslant 6 \end{cases}$$

that is

$$\Phi(D) = \{(u,v)\ :\ 2 \leqslant v - u \leqslant v + u \leqslant 6\}.$$

Now, to be $v-u \leqslant v+u$ it must be $u \geqslant 0$, and from $2 \leqslant v-u \leqslant v+u \leqslant 6$ we get $2+u \leqslant v \leqslant 6-u$ provided $2+u \leqslant 6-u$, that is $u \leqslant 2$. In conclusion

$$\Phi(D) = \{(u,v) \ : \ 0 \leqslant u \leqslant 2, \ 2+u \leqslant v \leqslant 6-u\}.$$

About f, in coordinates (u,v) we have

$$f(\Phi^{-1}(u,v)) = \left(\frac{v-u}{2}\right)^{2/3} ue^v,$$

while

$$\det(\Phi^{-1})' = \det \begin{bmatrix} \frac{1}{3}\left(\frac{v-u}{2}\right)^{-2/3}\left(-\frac{1}{2}\right) & \frac{1}{3}\left(\frac{v-u}{2}\right)^{-2/3}\left(+\frac{1}{2}\right) \\ \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = -\frac{1}{6}\left(\frac{v-u}{2}\right)^{-2/3}.$$

In conclusion

$$\begin{aligned}
\int_D f \, dxdy &= \int_{0\leqslant u\leqslant 2, \ 2+u\leqslant v\leqslant 6-u} \left(\tfrac{v-u}{2}\right)^{2/3} ue^v \tfrac{1}{6}\left(\tfrac{v-u}{2}\right)^{-2/3} \, dudv = \tfrac{1}{6}\int_{0\leqslant u\leqslant 2, \ 2+u\leqslant v\leqslant 6-u} ue^v \, dudv \\
&\overset{RF}{=} \tfrac{1}{6}\int_0^2 \int_{2+u}^{6-u} ue^v \, dv \, du = \tfrac{1}{6}\int_0^2 u \int_{2+u}^{6-u} e^v \, dv \, du = \tfrac{1}{6}\int_0^2 u \, [e^v]_{v=2+u}^{v=6-u} \, du \\
&= \tfrac{1}{6}\int_0^2 u\left(e^{6-u} - e^{2+u}\right) \, du = \tfrac{1}{6}\left(e^6 \int_0^2 ue^{-u} \, du - e^2 \int_0^2 ue^u \, du\right) \\
&= \tfrac{1}{6}\left(e^6\left([-ue^{-u}]_{u=0}^{u=2} + \int_0^2 e^{-u} \, du\right) - e^2\left([ue^u]_{u=0}^{u=2} - \int_0^2 e^u \, du\right)\right) \\
&= \tfrac{1}{6}\left(e^6\left(-2e^{-2} - (e^{-2}-1)\right) - e^2\left(2e^2 - (e^2-1)\right)\right) \\
&= \tfrac{e^2}{6}\left(-2e^2 + e^4 - 1\right). \quad \square
\end{aligned}$$

Exercise 30. i) If $u(x,y) = \text{Re } f(x+iy)$ and $v(x,y) = \text{Im } f(x+iy)$, then

$$g(x+iy) = \overline{f(x-iy)} = \overline{u(x,-y)+iv(x,-y)} = u(x,-y) - iv(x,-y),$$

from which we see that

$$U(x,y) = \text{Re } g(x+iy) = u(x,-y), \ \ V(x,y) = \text{Im } g(x+iy) = -v(x,-y).$$

ii) g is holomorphic iff U, V are $\mathbb{R}-$differentiable and they verify CR equations. Clearly, sunce f is holomorphic, u, v are $\mathbb{R}-$differentiable, hence also U, V are $\mathbb{R}-$differentiable. Therefore, we have to verify if U, V fulfil also the CR equations, that is

$$\begin{cases} \partial_x U \equiv \partial_y V, \\ \\ \partial_y U \equiv -\partial_x V. \end{cases}$$

We have,

$$\partial_x U = \partial_x(u(x,-y)) = \partial_x u(x,-y), \ \ \partial_y V = \partial_y\left(-v(x,-y)\right) = -\partial_y v(x,-y)(-1) = \partial_y v(x,-y).$$

And since $\partial_x u \equiv \partial_y v$ we deduce that also $\partial_x U = \partial_y V$. Similarly, $\partial_y U = -\partial_x V$ and the check is completed. □

Exercise 31. i) Let $g_1 := x^2 - y^2 - z^2$ and $g_2 := x^2 + y^2 - xy - 1$. Then, $\vec{g} = (g_1, g_2)$ is a submersion on D iff $\mathrm{rk}\vec{g}'(x, y, z) = 2$ for all $(x, y, z) \in D$. Now,

$$\mathrm{rk}\,\vec{g}'(x,y,z) = \mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & -2y & -2z \\ \\ 2x-y & 2y-x & 0 \end{bmatrix} < 2, \iff \begin{cases} 2x(2y-x) + 2y(2x-y) = 0, \\ 2z(2x-y) = 0, \\ 2z(2y-x) = 0. \end{cases}$$

Simplifying, we get the system

$$\begin{cases} x^2 + y^2 - 4xy = 0, \\ z(2x-y) = 0, \\ z(2y-x) = 0. \end{cases}$$

Choosing the second equation, we have the alternative $z = 0$ or $2x - y = 0$. In the first case the system reduces to

$$\begin{cases} z = 0, \\ \\ x^2 + y^2 - 4xy = 0. \end{cases}$$

These points belong to D iff

$$\begin{cases} x^2 = y^2, \\ \\ 4xy = xy + 1, \end{cases} \iff \begin{cases} y = \pm x, \\ \\ 3xy = 1. \end{cases}$$

However, since $x^2 + y^2 = 4xy$ implies that, for $y = \pm x$, that $x = 0 = y$, it is impossible that $3xy = 1$, thus no solutions are in D.

In the second case, namely, $z \neq 0$ and $2x - y = 0$ or $y = 2x$, condition $\mathrm{rk}\vec{g}'(x, y, z) < 2$ reduces to

$$\begin{cases} y - 2x, \\ x(2y-x) = 0, \\ 2y - x = 0, \end{cases}$$

we easily get $x = y = 0$, that is a point of type $(0, 0, z)$. Now,

$$(0,0,z) \in D, \iff \begin{cases} z = 0, \\ \\ 0 = 1, \end{cases}$$

clearly impossible. Conclusion: rank of $\vec{g}'(x, y, z)$ is never less than 2 on D, that is $\vec{g}$ is a submersion on D.

ii) D is clearly closed being defined by equalities involving continuous functions. To determine whether D is bounded or less, we look first at constraint $x^2 + y^2 = xy + 1$. Writing $x = \rho\cos\theta$ and $y = \rho\sin\theta$, this reads as

$$\rho^2 = \rho^2\cos\theta\sin\theta + 1 = \frac{\rho^2}{2}\sin(2\theta) + 1, \leqslant \frac{\rho^2}{2} + 1, \implies \frac{\rho^2}{2} \leqslant 1, \implies x^2 + y^2 \leqslant 2, \ \forall (x, y, z) \in D.$$

But then, by the first equation,

$$z^2 = x^2 - y^2 \leqslant x^2 \leqslant x^2 + y^2 \leqslant 2, \implies x^2 + y^2 + z^2 \leqslant 4, \implies \|(x, y, z)\| \leqslant 2, \ \forall (x, y, z) \in D.$$

This means that D is bounded, hence compact.

iii) We have to miunimize/maximize $f(x, y, z) = \|(x, y, z)\|$ or, which is the same, $f(x, y, z) = \|(x, y, z)\|^2 = x^2 + y^2 + z^2$. The existence of min and max is ensured by the Weierstrass theorem being D compact by ii).

To determine min/max points, we apply Lagrange multipliers theorem. By i), assumptions of this theorem are verified. Thus, at min/max point $(x, y, z) \in D$ we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \quad \iff \quad \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 0.$$

Now,

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & -2y & -2z \\ 2x - y & 2y - x & 0 \end{bmatrix} = -(2y - z)(-8xz) = 8xz(2y - z),$$

iff $x = 0$, or $z = 0$ or $2y - z = 0$. Thus, we have points $(0, y, z)$, $(x, y, 0)$ and $(x, y, 2y)$. Now:

- $(0, y, z) \in D$ iff $0 = y^2 + z^2$ and $y^2 = 1$, and of course this is impossible.
- $(x, y, 0) \in D$ iff $x^2 = y^2$ and $x^2 + y^2 = xy + 1$. From the first we have $y = \pm x$. For $y = x$, second condition becomes $2x^2 = x^2 = 1$, thus $x^2 = 1$, so $x = \pm 1$ and we have points $(\pm 1, \pm 1, 0)$ (same sign). For $y = -x$, second condition becomes $2x^2 = -x^2 + 1$, that is $x^2 = \frac{1}{3}$, that is $x = \pm\frac{1}{\sqrt{3}}$, from which we have points $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, 0\right)$ (opposite sign).
- $(x, y, 2y) \in D$ iff $x^2 = y^2 + 4y^2 = 5y^2$ and $x^2 + y^2 = xy + 1$. From first equation we get $x = \pm\sqrt{5}y$. In the case $x = \sqrt{5}y$, from second eqn we have $5y^2 + y^2 = \sqrt{5}y^2 + 1$, that is $(6 - \sqrt{5})y^2 = 1$, that is $y = \pm\frac{1}{\sqrt{6-\sqrt{5}}}$, this yielding to points $\left(\pm\frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm\frac{1}{\sqrt{6-\sqrt{5}}}, 0\right)$ (same sign). In the case $x = -\sqrt{5}y$, second condition yields to $5y^2 + y^2 = -\sqrt{5}y^1$, that is $y^2 = \frac{1}{5+\sqrt{5}}$, or $y = \pm\frac{1}{\sqrt{5+\sqrt{5}}}$, from which we get points $\left(\mp\frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm\frac{1}{\sqrt{5+\sqrt{5}}}, 0\right)$ (opposite sign).

Previous analysis figured out possible min/max points. To decide which are min and which max it suffices to compute f at these points. We have:

- $f(\pm 1, \pm 1, 0) = 2$;
- $f\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, 0\right) = \frac{2}{3} = 0,\bar{6}$;
- $f\left(\pm\frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm\frac{1}{\sqrt{6-\sqrt{5}}}, 0\right) = \frac{6}{6-\sqrt{5}} \approx 1,59\ldots$
- $f\left(\mp\frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm\frac{1}{\sqrt{5+\sqrt{5}}}, 0\right) = \frac{6}{5+\sqrt{5}} \approx 0,83\ldots$

From this it is clear that $(\pm 1, \pm 1, 0)$ are points of D at max distance to $\vec{0}$, while $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, 0\right)$ are poitns of D at min distance to $\vec{0}$. □

Exercise 32.

Exercise 33. i) D is closed because it is defined by large inequalities. It is not open because $D \neq \emptyset, \mathbb{R}^3$. It is unbounded since $(n, n, \frac{1}{\cosh(2n^2)}) \in D$ for every $n \in \mathbb{N}$, therefore it is not compact.

ii) We have

$$\lambda_3(D) = \int_D 1 \, dxdydz \overset{RF}{=} \int_{\mathbb{R}^2} \left(\int_0^{1/\cosh(x^2+y^2)} dz \right) dxdy = \int_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)} \, dxdy.$$

By introducing polar coordinates,

$$\lambda_3(D) = \int_{\rho \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi} \frac{1}{\cosh \rho^2} \rho \, d\rho d\theta = 2\pi \int_0^{+\infty} \frac{\rho}{\cosh \rho^2} \, d\rho.$$

Notice that

$$\frac{\rho}{\cosh \rho^2} = \frac{2\rho}{e^{\rho^2} + e^{-\rho^2}} = \frac{2\rho e^{\rho^2}}{1 + e^{2\rho^2}} = \partial_\rho \arctan(e^{\rho^2}),$$

thus

$$\lambda_3(D) = 2\pi \left[\arctan(e^{\rho^2}) \right]_{\rho=0}^{\rho=+\infty} = 2\pi \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi^2}{2}.$$

iii) Proceeding as in ii), we have

$$I_\alpha := \int_D e^{\alpha(x^2+y^2)} \, dxdydz \overset{RF}{=} \int_{\mathbb{R}^2} \left(\int_0^{1/\cosh(x^2+y^2)} e^{\alpha(x^2+y^2)} \, dz \right) dxdy = \int_{\mathbb{R}^2} \frac{e^{\alpha(x^2+y^2)}}{\cosh(x^2+y^2)} \, dxdy.$$

Changing vars to polar coords,

$$I_\alpha = \int_{\rho \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi} \frac{e^{\alpha\rho^2}}{\cosh \rho^2} \rho \, d\rho d\theta = 2\pi \int_0^{+\infty} \frac{2\rho e^{(\alpha+1)\rho^2}}{1 + e^{2\rho^2}} \, d\rho.$$

Notice that

$$\frac{2\rho e^{(\alpha+1)\rho^2}}{1 + e^{2\rho^2}} \sim_{+\infty} 2\rho \frac{e^{(\alpha+1)\rho^2}}{e^{2\rho^2}} = 2\rho e^{(\alpha-1)\rho^2}$$

and

$$\exists \int_0^{+\infty} \rho e^{(\alpha-1)\rho^2} \, d\rho \iff \alpha - 1 < 0, \iff \alpha < 1. \quad \square$$

Exercise 34. i) In order $f = u + iv$ be $\mathbb{C}-$differentiable on $\mathbb{C}$ we need 1. that u, v are $\mathbb{R}$ differentiable on $\mathbb{R}^2$ (which is true, being u, v polynomials) and 2. u, v fulfil the CR equations, namely

$$\begin{cases} \partial_x u \equiv \partial_y v, \\ \partial_y u \equiv -\partial_x v, \end{cases} \iff \begin{cases} 3x^2 + ay^2 \equiv bx^2 - 3y^2, \\ 2axy \equiv -2bxy, \end{cases} \iff b = 3,\ a = -3.$$

ii) We have

$$f = (x^3 - 3xy^2) + i(3x^2y - y^3) = (x + iy)^3 = z^3. \quad \square$$

Exercise 35. Clearly $f \in \mathscr{C}(\mathbb{R}^d)$ and moreover $f \geqslant 0$ (trivial) and

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty.$$

Just notice that $f(\vec{x}) \geqslant \|\vec{x} - \vec{a}_1\|^2 \longrightarrow +\infty$ when $\vec{x} \longrightarrow \infty_d$. Thus f cannot have a maximum but it has a minimum according to Weierstrass' thm. Now, f is differentiable on $\mathbb{R}^d$,

$$\nabla f = \sum_{j=1}^{N} \nabla\|\vec{x} - \vec{a}_j\|^2$$

and

$$\nabla\|\vec{x} - \vec{a}_j\|^2 = \left(\partial_1\|\vec{x} - \vec{a}_j\|^2, \ldots, \partial_d\|\vec{x} - \vec{a}_j\|^2\right),$$

so, writing

$$\|\vec{x} - \vec{a}_j\|^2 = \sum_{k=1}^{d}(x_k - a_{j,k})^2, \implies \partial_i\|\vec{x} - \vec{a}_j\|^2 = \partial_i \sum_{k=1}^{d}(x_k - a_{j,k})^2 = 2(x_i - a_{j,i}),$$

we deduce

$$\nabla\|\vec{x} - \vec{a}_j\|^2 = \big(2(x_1 - a_{j,1}), 2(x_2 - a_{j,2}, \ldots, 2(x_d - a_{j,d})\big) = 2(\vec{x} - \vec{a}_j).$$

Therefore, $\nabla f \in \mathscr{C}$ and f is differentiable. According to Fermat thm, at min point we must have

$$\nabla f = \vec{0}, \iff \sum_{j=1}^{N} 2(\vec{x} - \vec{a}_j) = 0, \iff N\vec{x} - \sum_{j=1}^{N} \vec{a}_j = \vec{0}, \iff \vec{x} = \frac{1}{N}\sum_{j=1}^{N} \vec{a}_j. \quad \square$$

Exercise 36. i) For $D \neq \varnothing$ we consider a point of type $(x, y, 2)$. Then $(x, y, 2) \in D$ iff $x^2 + y^2 = 4$ and $y^2 = 1$, thus $y = \pm 1$ and $x^2 = 3$, that is $x = \pm\sqrt{3}$. We conclude that points $(\pm\sqrt{3}, \pm 1, 2)$ (four points, all possible combinations of sign) belong to D.

We have that $D = \{g_1 = 0,\ g_2 = 0\}$ where $g_1 = x^2 + y^2 - z^2$, and $g_2 = y^2 + (z-2)^2 - 1$. Clearly, both g_1 and g_2 are differentiable functions (they are polynomials). In order $\vec{g} = (g_1, g_2)$ be a submersion on D we need to verify that

$$\operatorname{rk} \vec{g}' = \operatorname{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \operatorname{rk}\begin{bmatrix} 2x & 2y & -2z \\ \\ 0 & 2y & 2(z-2) \end{bmatrix} = 2,\ \forall (x, y, z) \in D.$$

Now, this is false iff all 2×2 sub-determinants of the Jacobian matrix $\vec{g}'$ vanish, that is iff

$$\begin{cases} 4xy = 0, \\ 4x(z-2) = 0, \\ 8y(z-1) = 0. \end{cases} \iff \begin{cases} x = 0, \\ \\ y(z-1) = 0, \end{cases} \vee \begin{cases} y = 0, \\ \\ x(z-2) = 0, \end{cases}$$

The first subsystem has solutions $(0, 0, z)$ and $(0, y, 1)$ $(x, y \in \mathbb{R})$; the second, $(0, 0, z)$ and $(x, 0, 2)$, $(x, z \in \mathbb{R})$. Now:

- $(0, 0, z) \in D$ iff $z^2 = 0$ and $(z-2)^2 = 1$, impossible;
- $(0, y, 1) \in D$ iff $y^2 = 1$ and $y^2 + 1 = 1$, again impossible;
- $(x, 0, 2) \in D$ iff $x^2 = 4$ and $0 = 1$, impossible.

Cocnlusion: there is no point on D at which rank of $\vec{g}'$ is less than 2, therefore rank of $\vec{g}'(x,y,z)$ is 2 for every $(x,y,z) \in D$, that is $\vec{g}$ is a submersion on D.

ii) D is defined by equalities involving continuous functions, it is therefore closed. From the second equation

$$y^2 + (z-2)^2 = 1, \implies y^2 \leqslant 1,\ (z-2)^2 \leqslant 1.$$

In particular, $-1 \leqslant z-2 \leqslant 1$, that is $1 \leqslant z \leqslant 3$, thus $z^2 \leqslant 9$. Plugging this into the first equation,

$$x^2 + y^2 = z^2, \ \leqslant \ x^2 + y^2 \leqslant 9, \implies x^2 \leqslant 9.$$

In conclusion $x^2+y^2+z^2 \leqslant 9+1+9 = 19$, for every $(x,y,z) \in D$, from which we see that D is bounded. We conclude that D is compact.

iii) Points at min/max distance to $\vec{0}$ minimize/maximize the function $f = x^2+y^2+z^2$. Since f is continuous and D is compact, according to the Weierstrass theorem, f has both min and max on D.

To determine these points, we apply the Lagrange multipliers' theorem. By i), hypotheses of the theorem are fulfilled. Thus, at every $(x,y,z) \in D$ min/max point for f in D we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \iff \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 2(z-2) \end{bmatrix} = 0.$$

By computing the determinant we get

$$0 = 2x \cdot 4y(z-2+z) - 2x \cdot 4y(z-2-z) = 16xyz,$$

whose solutions are points $(0,y,z)$, $(x,0,z)$ and $(x,y,0)$. Now,

- $(0,y,z) \in D$ iff $y^2 = z^2$ and $y^2+(z-2)^2 = 1$, from which $z^2+(z-2)^2 = 1$, or $2z^2-2z+3 = 0$, and since $\Delta < 0$ there are no solutions to this equation;
- $(x,0,z) \in D$ iff $x^2 = z^2$ and $(z-2)^2 = 1$, from which $z = 1, 3$ and $x^2 = 1$ (that is $x = \pm 1$), or $x^2 = 9$ (that is $x = \pm 3$). We obtain points $(\pm 1, 0, 1)$ and $(\pm 3, 0, 3)$;
- $(x,y,0) \in D$ iff $x^2+y^2 = 0$, $y^2+4 = 1$ which is impossible.

Since $f(\pm 1,0,1) = 2$ and $f(\pm 3,0,3) = 18$ we deduce that $(\pm 1,0,1)$ are points of D at min distance to $\vec{0}$, $(\pm 3,0,3)$ are points of D at max distance to $\vec{0}$. □

Exercise 37. Following the hint, we set $(x,y) = (u^2, v^2)$ in such a way that, with notations of the change of variables formula, $(x,y) = \Phi^{-1}(u,v)$. To have a true change of variable, Φ, Φ^{-1} must be bijections, that is injective and surjective. Now, as defined, Φ^{-1} is not injective on $\mathbb{R}^2$. However, since

$$(x,y) \in [0,+\infty[^2, \begin{cases} x = u^2, \\ \\ y = v^2, \end{cases} \iff \begin{cases} u = \pm\sqrt{x}, \\ \\ v = \pm\sqrt{y} \end{cases}$$

we see that Φ^{-1} becomes injective (and bijective) if also $(u,v) \in [0,+\infty[^2$. Therefore, Φ, Φ^{-1} are a bijection from $[0,+\infty[^2$ into itself:

$$(x,y) \in [0,+\infty[^2, \ \Phi^{-1} \begin{cases} x = u^2, \\ \\ y = v^2, \end{cases} \iff \Phi \begin{cases} u = \sqrt{x}, \\ \\ v = \sqrt{y} \end{cases}$$

According to the change of variables formula

$$\int_D f(x,y)\,dxdy = \int_{\Phi(D)} f(\Phi^{-1}(u,v))|\det(\Phi^{-1})'(u,v)|\,dudv$$

Here $D = [0,+\infty[^2$, and $\Phi(D) = [0,+\infty[^2$.

$$f(\Phi^{-1}(u,v)) = f(u^2,v^2) = \frac{e^{-(u^2+v^2)}}{\sqrt{u^4v^2+u^2v^4}} = \frac{e^{-(u^2+y^2)}}{\sqrt{u^2v^2(u^2+v^2)}}.$$

and since

$$\det(\Phi^{-1})'(u,v) = \det\begin{bmatrix} 2u & 0 \\ 0 & 2v \end{bmatrix} = 4uv,$$

we have

$$\begin{aligned}
\int_{[0,+\infty[^2} \frac{e^{-(x+y)}}{\sqrt{x^2y+xy^2}}\,dxdy &= \int_{[0,+\infty[^2} \frac{e^{-(u^2+y^2)}}{\sqrt{u^2v^2(u^2+v^2)}}|4uv|\,dudv = \\
&= 4\int_{[0,+\infty[^2} \frac{e^{-(u^2+v^2)}}{\sqrt{u^2+v^2}}\,dudv \\
&= 4\int_{\rho\geqslant 0,\ 0\leqslant\theta\leqslant\frac{\pi}{2}} \frac{e^{-\rho^2}}{\rho}\rho\,d\rho d\theta \quad (u=\rho\cos\theta,\ v=\rho\sin\theta) \\
&\overset{RF}{=} 2\pi\int_0^{+\infty} e^{-\rho^2}\,d\rho = 2\pi\frac{\sqrt{\pi}}{2} = \pi\sqrt{\pi}. \quad \square
\end{aligned}$$

Exercise 38. i) In order $f = u+iv$ is holomorphic on $\mathbb{C}$ we need that $u,v \in \mathscr{C}^1$ (true, u and v are polynomials) and they fulfill the CR equations:

$$\begin{cases} \partial_x u = \partial_y v, \\ \partial_y u = -\partial_x v, \end{cases} \iff \begin{cases} 2ax+by = x, \\ bx+2cy = -y, \end{cases} \forall (x,y)\in\mathbb{R}^2., \iff \begin{cases} 2a = 1,\ b = 0, \\ b = 0,\ 2c = -1. \end{cases}$$

Thus,

$$u = \frac{1}{2}x^2 - \frac{1}{2}y^2,\ \ v = xy,$$

and $f = u+iv$ is holomorphic on $\mathbb{C}$.

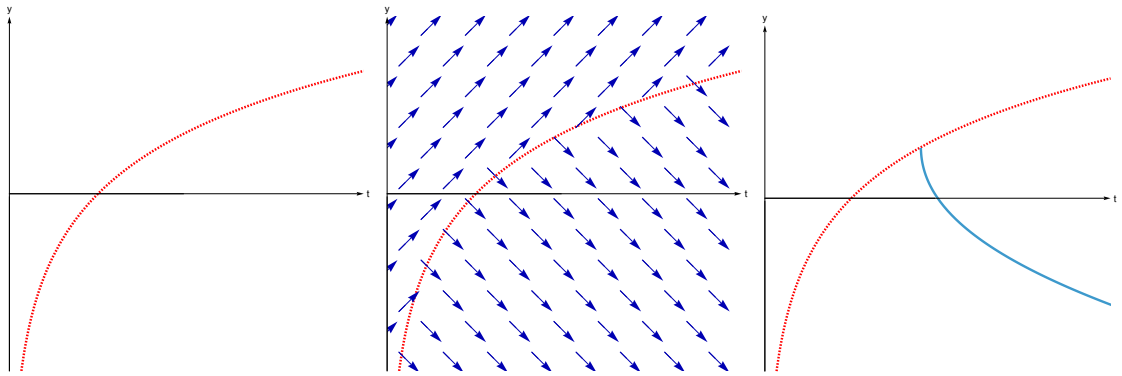
ii) Notice that

$$f = u+iv = \frac{1}{2}x^2 - \frac{1}{2}y^2 + ixy = \frac{1}{2}\left(x^2-y^2+i2xy\right) = \frac{1}{2}(x+iy)^2 \equiv \frac{z^2}{2},\ z\in\mathbb{C}. \quad \square$$

Exercise 39. i) Let $f(t,y) := \frac{1}{y-\log t}$. Clearly, f is well defined on

$$D := \{(t,y)\in\mathbb{R}^2\ :\ y-\log t \neq 0\} = \{(t,y)\ :\ t>0,\ y\neq\log t\}.$$

Clearly $f, \partial_y f \in \mathscr{C}(D)$, so local existence and uniqueness applies on D.



ii) $y \equiv C$ is a constant solution iff $0 = \frac{1}{C - \log t}$, which is impossible. Conclusion: no stationary solutions. Furthermore,

$$y \nearrow, \iff 0 \leqslant y' = \frac{1}{y - \log t}, \iff y - \log t > 0, \iff y > \log t.$$

iii) Claim: $y \searrow$. This follows once we prove that $y(t) < \log t$ for all t. This is indeed true at $t = 2$: $y(2) = 0 < \log 2$. If the conclusion were false, there would be a $\widehat{t}$ such that $y(\widehat{t}) \geqslant \log \widehat{t}$. But,

- if $y(\widehat{t}) = \log \widehat{t}$, $(\widehat{t}, y(\widehat{t})) \notin D$, which is impossible for a solution.
- if $y(\widehat{t}) > \log \widehat{t}$ then, by continuity, there would be another $\widehat{\widehat{t}}$ such that $y(\widehat{\widehat{t}}) = \log \widehat{\widehat{t}}$, so we fall again into a contradiction.

In any case, we get a contradiction, so $y(t) < \log t$ must be always true and the conclusion follows.

We have

$$\lim_{t \to \alpha} y'(t) = \lim_{t \to \alpha} \frac{1}{y(t) - \log t}.$$

Claim: $y(t) \longrightarrow \log \alpha$. Indeed, we know $y(t) < \log t$ for every t, so letting $t \to \alpha$ we have $y(\alpha) \leqslant \log \alpha$. If $y(\alpha) < \log \alpha$ then

$$\forall t \in]\alpha, 2], \ (t, y(t)) \in [\alpha, 2] \times [0, y(\alpha)] =: K \text{ compact } \subset D,$$

so the solution never leaves the compact K in the past. This is in conflict with the exit from compact sets, so $y(\alpha) = \log \alpha$, from which $y(t) - \log t \longrightarrow 0$. Since $y(t) < \log t$, this implies $y(t) - \log t \longrightarrow 0-$, so

$$\lim_{t \to \alpha} y'(t) = \frac{1}{0-} = -\infty.$$

iv) For the concavity we notice that

$$y'' = \left(\frac{1}{y - \log t} \right)' = -\frac{y' - \frac{1}{t}}{(y - \log t)^2}.$$

Therefore,

$$y \uparrow, \iff y'' \geqslant 0, \iff y' - \frac{1}{t} \leqslant 0.$$

Since $t \in]\alpha, \beta[$, $t > \alpha > 0$ and $y' \leqslant 0$, we conclude that $y' - \frac{1}{t} \leqslant 0$ for all $t > 0$, thus $y'' > 0$ for all t and y is convex.

$\beta = +\infty$? We have two possibilities: $\beta < +\infty$, $\beta = +\infty$. If $\beta < +\infty$ then, defined $y(\beta) = \lim_{t\to\beta} y(t) \geqslant -\infty$ (the limit exists being y decreasing), we would have

- either $y(\beta) > -\infty$: in this case, the solution would remain into the compact box $[2, \beta] \times [y(\beta), 0]$ for all future times, violating the exit from compact sets, impossible;
- or $y(\beta) = -\infty$: also this is impossible because, by convexity, y would be above each of its tangents, in particular to that one for $t = 2$:

$$y(t) \geqslant y(2) + y'(2)(t-2) \implies y(\beta) \geqslant y(2) + y'(2)(\beta - 2) > -\infty,$$

which is impossible being $y(\beta) = -\infty$.

In any case, we got a contradiction. This means $\beta = +\infty$. □

Exercise 40. See notes for definitions. We aim to prove that $\vec{F}^{-1}(S)$ is open if S it is. Suppose this is false. There exists then a point $\vec{x} \in \vec{F}^{-1}(S)$ for which

$$\nexists B(\vec{x}, r] \subset \vec{F}^{-1}(S).$$

This means that:

$$\forall r > 0,\ B(\vec{x}, r] \cap \vec{F}^{-1}(S)^c \neq \varnothing.$$

Taking $r = \frac{1}{n}$

$$\forall n \in \mathbb{N},\ n \geqslant 1,\ \exists \vec{x}_n \in B(\vec{x}, 1/n] \cap \vec{F}^{-1}(S)^c.$$

This means that $\|\vec{x}_n - \vec{x}\| \leqslant \frac{1}{n}$, thus $\vec{x}_n \longrightarrow \vec{x}$. By continuity, $\vec{F}(\vec{x}_n) \longrightarrow \vec{F}(\vec{x})$. Furthermore, by construction of $(\vec{x}_n)$, we have that $\vec{x}_n \notin \vec{F}^{-1}(S)$, that is $\vec{F}(\vec{x}_n) \notin S$ for every n. Since $\vec{F}(\vec{x}) \in S$ (recall that $\vec{x} \in \vec{F}^{-1}(S)$), and S is supposed to be open,

$$\exists B(\vec{F}(\vec{x}), \rho] \subset S.$$

And since $\vec{F}(\vec{x}_n) \longrightarrow \vec{F}(\vec{x})$, we have that

$$\exists N\ :\ \vec{F}(\vec{x}_n) \in B(\vec{F}(\vec{x}), \rho] \subset S,\ \forall n \geqslant N,$$

which is in contradiction with the construction of $(\vec{x}_n)$. We deduce that the initial assumption must be false, that is $\vec{F}^{-1}(S)$ is open. □

Exercise 41. i) Let $(g_1, g_2) := (x^2 + y^2 - 1, x + y + z - 1)$ in such a way $D = \{g_1 = 0,\ g_2 = 0\}$. To check that (g_1, g_2) is a submersion on D we have to verify that

$$\operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & 2y & 0 \\ \\ 1 & 1 & 1 \end{bmatrix} = 2,\ \forall (x, y, z) \in D.$$

Now, rank is < 2 iff the two gradients are linearly dependent. This is manifestly impossible because of their third component.

ii) D is closed being defined by equalities involving continuous functions. D is also bounded: indeed, by first equation we have $x^2, y^2 \leqslant 1$, thus $-1 \leqslant x, y \leqslant 1$, and by the secon

$$-1 \geqslant z = 1 - (x + y) \leqslant 3,$$

thus $z^2 \leqslant 9$ and $x^2 + y^2 + z^2 \leqslant 11$.

iii) Function f is continuous on D compact: existence of min/max is ensured by Weierstrass thm. To determine these points, we apply Lagrange multipliers thm. By i), D fulfils the assumption of the thm. Thus, at (x, y, z) min/max point for f on D we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \Longleftrightarrow \quad \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \mathrm{rk} \begin{bmatrix} 2x+y-1 & 2y+x+z-1 & y \\ 2x & 2y & 0 \\ 1 & 1 & 1 \end{bmatrix} < 3,$$

that is iff the determinant of previous matrix vanishes. We get the condition

$$2y(x-y) + 2(y(2x+y-1) - x(2y+x+z-1)) = 0,$$

from which, simplifying,

$$y(y-x) + (y^2 - y - x^2 + x - xz) = 0.$$

Since we are looking for solutions $(x, y, z) \in D$, we must have $z = 1 - x - y$, and plugging this into previous equation yields,

$$y\,(2y-1) = 0, \quad \Longleftrightarrow \quad y = 0, \ \vee\ y = \frac{1}{2}.$$

Thus we get points $(x, 0, 1-x)$ and $(x, \frac{1}{2}, \frac{1}{2}-x)$, to which we have still to impose the condition $x^2+y^2 = 1$. In the first case $x^2 + 0^2 = 1$, thus $x = \pm 1$, that is points $(\pm 1, 0, \mp 1)$ (two points). In the second case, $x^2 + \frac{1}{4} = 1$, thus $x^2 = \frac{3}{4}$ and $x = \pm\frac{\sqrt{3}}{2}$, that is points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1+\sqrt{3}}{2}\right)$. We have

- $f(1, 0, -1) = 0$
- $f(-1, 0, 1) = 2$
- $f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} - 2)$
- $f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1+\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} + 2)$

From this we see that $(-1, 0, 1)$ is max point, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right)$ is min point. □

Exercise 42. i) See LN for the definitions and general properties.

ii) To be irrotational, $\vec{F}$ must verify

$$\begin{cases} \partial_y \frac{ax+by}{1+x^2+y^2+z^2} = \partial_x \frac{cy+dz}{1+x^2+y^2+z^2}, \\ \partial_z \frac{ax+by}{1+x^2+y^2+z^2} = \partial_x \frac{ex+fz}{1+x^2+y^2+z^2}, \\ \partial_z \frac{cy+dz}{1+x^2+y^2+z^2} = \partial_y \frac{ex+fz}{1+x^2+y^2+z^2}, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \frac{b(1+x^2+y^2+z^2)-(ax+by)2y}{(1+x^2+y^2+z^2)^2} = \frac{-(cy+dz)2x}{(1+x^2+y^2+z^2)^2}, \\ \frac{-(ax+by)2z}{(1+x^2+y^2+z^2)^2} = \frac{e(1+x^2+y^2+z^2)-(ex+fz)2x}{(1+x^2+y^2+z^2)^2}, \\ \frac{d(1+x^2+y^2+z^2)-(cy+dz)2z}{(1+x^2+y^2+z^2)^2} = \frac{-(cy+dz)2x}{(1+x^2+y^2+z^2)^2} \end{cases}$$

that is

$$\begin{cases} b(1+x^2+y^2+z^2) - (ax+by)2y = -(cy+dz)2x, \\ -(ax+by)2z = e(1+x^2+y^2+z^2) - (ex+fz)2x, \\ d(1+x^2+y^2+z^2) - (cy+dz)2z = -(cy+dz)2x \end{cases}$$

We work on the first equation, the others being similar:

$$b + bx^2 - by^2 + bz^2 - 2(a-c)xy + 2dzx = 0,$$

from which $b = a - c = d = 0$, that is $b = d = 0$ and $a = c$. Similarly, $a = c = f$ and $e = 0$. Thus $\vec{F}$ is irrotational iff

$$\vec{F}(x, y, z) := \left(\frac{ax}{1 + x^2 + y^2 + z^2}, \frac{ay}{1 + x^2 + y^2 + z^2}, \frac{az}{1 + x^2 + y^2 + z^2} \right)$$

ii) To be conservative, $\vec{F}$ must be irrotational, so the values of a, b, c, d, e, f are those found in ii). For such values $\vec{F} = \nabla f$ iff

$$\begin{cases} \partial_x f = \frac{ax}{1+x^2+y^2+z^2}, \\ \partial_y f = \frac{ay}{1+x^2+y^2+z^2}, \\ \partial_z f = \frac{az}{1+x^2+y^2+z^2}. \end{cases}$$

From the first equation we get

$$f(x, y, z) = a \int \frac{x}{1 + x^2 + y^2 + z^2} \, dx + c(y, z) = \frac{a}{2} \log(1 + x^2 + y^2 + z^2) + c(y, z).$$

Imposing the second equation we get

$$\partial_y f = \frac{ay}{1 + x^2 + y^2 + z^2} + \partial_y c(y, z) = \frac{ay}{1 + x^2 + y^2 + z^2}, \quad \iff \quad \partial_y c(y, z) = 0,$$

from which $c(y, z) = c(z)$. Finally, imposing also the third condition we get

$$\partial_z f = \frac{az}{1 + x^2 + y^2 + z^2} + \partial_z c(z) = \frac{az}{1 + x^2 + y^2 + z^2}, \quad \iff \quad \partial_z c(z) = 0,$$

from which $c(y, z) = c(z) = c$. Conclusion: all the potentials of $\vec{F}$ are

$$f(x, y, z) = \frac{a}{2} \log(1 + x^2 + y^2 + z^2) + c. \qquad \square$$

Exercise 43. i) D is closed, being defined by large inequalities involving continuous functions. Let's check that D is bounded (hence compact). Denoting by $\rho = \sqrt{x^2 + y^2} = \|(x, y)\|$ we have

$$(x, y) \in D, \implies \rho^2 \leqslant 2\rho \cos\theta - \rho = \rho(2\cos\theta - 1), \quad \iff \quad \rho \leqslant 2\cos\theta - 1 \leqslant 1.$$

Therefore, D is bounded. In particular, D cannot be be open: only $\emptyset, \mathbb{R}^2$ are both open and closed, and $(0, 0) \in D$ (thus $D \neq \emptyset$), and D is bounded, thus $D \subsetneq \mathbb{R}^2$.

ii) The area of D is

$$\lambda_2(D) = \int_D 1 \, dxdy = \int_{x^2+y^2 \leqslant 2x - \sqrt{x^2+y^2}} 1 \, dxdy \overset{pol\ coords}{=} \int_{\rho \leqslant 2\cos\theta - 1} \rho \, d\rho d\theta.$$

Now, notice that since $\rho \geqslant 0$, this imposes $2\cos\theta - 1 \geqslant 0$, that is $\cos\theta \geqslant \frac{1}{2}$. In one period this means $-\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}$. Thus

$$\begin{aligned} \lambda_2(D) &= \int_{\rho \leqslant 2\cos\theta - 1, \ -\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}} \rho \, d\rho d\theta \overset{RF}{=} \int_{-\pi/3}^{\pi/3} \int_0^{2\cos\theta - 1} \rho \, d\rho \, d\theta = \tfrac{1}{2} \int_{-\pi/3}^{\pi/3} (2\cos\theta - 1)^2 \, d\theta \\ &= \tfrac{1}{2} \left(\tfrac{2\pi}{3} - 4 \int_{-\pi/3}^{\pi/3} \cos\theta \, d\theta + 4 \int_{-\pi/3}^{\pi/3} (\cos\theta)^2 \, d\theta \right) \\ &= \tfrac{\pi}{3} - 2\sqrt{3} + 2 \int_{-\pi/3}^{\pi/3} (\cos\theta)^2 \, d\theta. \end{aligned}$$

About this last integral we have

$$\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta=\int_{-\pi/3}^{\pi/3}(\cos\theta)(\sin\theta)'\,d\theta=[\sin\theta\cos\theta]_{\theta=-\pi/3}^{\theta=\pi/3}+\int_0^{2\pi}(\sin\theta)^2\,d\theta=\frac{\sqrt{3}}{2}-\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta,$$

from which $\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta=\frac{\sqrt{3}}{4}$. We conclude that $\lambda_2(D)=\frac{\pi}{3}-\frac{3\sqrt{3}}{2}$. □

Exercise 44. i) For $z=x+iy\neq 0$ we have

$$f(z)=\frac{(x-iy)^2}{x+iy}\frac{x-iy}{x=iy}=\frac{(x-iy)^3}{x^2+y^2}=\frac{x^3-i3x^2y-3xy^2+iy^3}{x^2+y^2}=\frac{x^3-3xy^2}{x^2+y^2}+i\frac{y^3-3x^2y}{x^2+y^2}.$$

So, $f(x+iy)=u(x,y)+iv(x,y)$ where

$$u(x,y)=\begin{cases}\frac{x^3-3xy^2}{x^2+y^2}, & (x,y)\neq(0,0),\\ \\ 0, & (x,y)=(0,0),\end{cases}\qquad v(x,y)=\begin{cases}\frac{y^3-3x^2y}{x^2+y^2}, & (x,y)\neq(0,0),\\ \\ 0, & (x,y)=(0,0),\end{cases}$$

Let's check that u,v verify the CR equations at $(0,0)$. We need the partial derivatives of u,v at $(0,0)$. To compute these we apply the definition of partial derivative:

$$\partial_x u(0,0)=\lim_{t\to0}\frac{u(t,0)-u(0,0)}{t}=\lim_{t\to0}\frac{t-0}{t}=1,\quad \partial_y u(0,0)=\lim_{t\to0}\frac{u(0,t)-u(0,0)}{t}=\lim_{t\to0}\frac{0-0}{t}=0,$$

and

$$\partial_x v(0,0)=\lim_{t\to0}\frac{v(t,0)-v(0,0)}{t}=\lim_{t\to0}\frac{0-0}{t}=0,\quad \partial_y u(0,0)=\lim_{t\to0}\frac{u(0,t)-u(0,0)}{t}=\lim_{t\to0}\frac{t-0}{t}=1,$$

from which we get

$$\begin{cases}\partial_x u(0,0)=1=\partial_y v(0,0),\\ \\ \partial_y u(0,0)=0=-\partial_x v(0,0),\end{cases}$$

that is, CR equations hold.

ii) To check the differentiability of f at $z=0$ we should check if u,v are $\mathbb{R}-$differentiable at $(0,0)$ or, in alternative, if $\exists f'(0)$. We present here both methods (just one of them is sufficient).

Differentiability of u at $(0,0)$. We already computed $\nabla u(0,0)=(1,0)$. So, in order u be differentiable at $(0,0)$ we need to verify that

$$\lim_{\vec{h}\to\vec{0}}\frac{u(\vec{h})-u(\vec{0})-\nabla u(\vec{0})\vec{h}}{\|\vec{h}\|}=0.$$

Setting $\vec{h}=(a,b)$, this is

$$\lim_{(a,b)\to(0,0)}\frac{u(a,b)-u(0,0)-(1,0)\cdot(a,b)}{\sqrt{a^2+b^2}}=\lim_{(a,b)\to(0,0)}\frac{\frac{a^3-3ab^2}{a^2+b^2}-0-a}{(a^2+b^2)^{1/2}}=\lim_{(a,b)\to(0,0)}\frac{-4ab^2}{(a^2+b^2)^{3/2}}$$

We see that if $a=b$ this reduces to the limit

$$\lim_{a\to0}\frac{-4a^3}{2^{3/2}|a|^3}=\pm1\neq0$$

according to $a \to 0\mp$, so we deduce that u is not differentiable at $(0,0)$, thus f is not $\mathbb{C}-$differentiable at $z = 0$. □

$\mathbb{C}-$differentiability of f at $z = 0$. We should verify that

$$\exists \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)^2.$$

Now, taking $h = x + i0$,

$$\lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)^2 = \lim_{x\to 0} \left(\frac{x}{x}\right)^2 = 1.$$

Taking $h = x + ix$,

$$\lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)^2 = \lim_{x\to 0} \left(\frac{x - ix}{x + ix}\right)^2 = \left(\frac{1-i}{1+i}\right)^2 = \frac{1}{4}(-i2) = -\frac{i}{2},$$

from which we conclude that the limit does not exists. □

Exercise 45. i) See LN for definitions and general properties.
ii) Since

$$\partial_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v},$$

it is clear that, if $\nabla f(\vec{x}) = 0$, then

$$\max_{\vec{v}\neq\vec{0} \,:\, \|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}) = 0.$$

If $\nabla f(\vec{x}) \neq \vec{0}$, by the Cauchy-Schwarz inequality we get

$$\partial_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v} \leqslant \|\nabla f(\vec{x})\|\|\vec{v}\| = \|\nabla f(\vec{x})\|,$$

for $\|\vec{v}\| = 1$. Now, since $\vec{v}_0 = \frac{1}{\|\nabla f(\vec{x})\|}\nabla f(\vec{x})$ has $\|\vec{v}_0\| = 1$, and

$$\partial_{\vec{v}_0} f(\vec{x}) = \nabla f(\vec{x}) \cdot \left(\frac{1}{\|\nabla f(\vec{x})\|}\nabla f(\vec{x})\right) = \frac{1}{\|\nabla f(\vec{x})\|}\nabla f(\vec{x}) \cdot \nabla f(\vec{x}) = \frac{1}{\|\nabla f(\vec{x})\|}\|\nabla f(\vec{x})\|^2 = \|\nabla f(\vec{x})\|,$$

we conclude that

$$\max_{\vec{v}\neq\vec{0} \,:\, \|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}) = \|\nabla f(\vec{x})\|.$$

So, whatever is $\nabla f(\vec{x})$, the conclusion is

$$\max_{\vec{v}\neq\vec{0} \,:\, \|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}) = \|\nabla f(\vec{x})\|. \quad \square$$

Exercise 46. Recalling of the definition of cosh we have

$$\cosh z = -2, \iff \frac{e^z + e^{-z}}{2} = -2, \iff e^z + e^{-z} = -4, \iff e^{2z} + 4e^z + 1 = 0.$$

Setting $w := e^z$, we have

$$w^2 + 4w + 1 = 0, \iff w = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}.$$

Noticed that $-2 \pm \sqrt{3} < 0$ and that $2 > \sqrt{3}$, we have

$$e^z = -2 \pm \sqrt{3}, \quad \iff \quad z = \log(2 \pm \sqrt{3}) + i(\frac{3\pi}{2} + k2\pi),\ k \in \mathbb{Z},$$

which are the solutions of the proposed equation. □

Exercise 47. i) To be irrotational on $\mathbb{R}^2$, $\vec{F}$ must verify on $\mathbb{R}^2$

$$\partial_y(axy + e^{by}) \equiv \partial_x(xe^y + 2y), \quad \iff \quad ax + be^{by} \equiv e^y.\ (\star)$$

Fix y, for example $y = 0$. Then $ax + b \equiv 1$, iff $a = 0$ and $b = 1$. It is clear that for these a, b, the condition $(\star)$ is verified, so $\vec{F}$ is irrotational on $\mathbb{R}^2$.

ii) To be conservative on $\mathbb{R}^2$, $\vec{F}$ must be irrotational on $\mathbb{R}^2$, so $a = 0$ and $b = 1$. It is a general fact that an irrotational field on $\mathbb{R}^2$ is also conservative, so $\vec{F}$ is conservative on $\mathbb{R}^2$ iff $a = 0$, $b = 1$. In this case the potentials are f such that

$$\begin{cases} \partial_x f = e^y, \\ \\ \partial_y f = xe^y + 2y. \end{cases}$$

From the first equation we have

$$f(x, y) = \int e^y\ dx + c(y) = xe^y + c(y),$$

and plugging this into the second equation we get

$$xe^y + c'(y) = xe^y + 2y, \quad \iff \quad c'(y) = 2y, \quad \iff \quad c(y) = y^2 + c,\ c \in \mathbb{R}.$$

We conclude that the potentials of f are $f(x, y) = xe^y + y^2 + c$, $c \in \mathbb{R}$ constant.

iii) Since the values a, b for which $\vec{F}$ is irrotational are the same for which $\vec{F}$ is conservative, we can use the fundamental theorem of integral calculus for fields to conclude that

$$\int_{\vec{\gamma}} \vec{F} = f(\vec{\gamma}(1)) - f(\vec{\gamma}(0)) = f(1, 0) - f(0, 0) = 1 + c - c = 1. \quad \square$$

Exercise 48. i) We have $(x, 0, z) \in D$ iff $z^2 = 1/2$ and $x^2 = z$, from which $(\pm 1/2, 0, 1/2) \in D$. Let $g_1 := xy + z^2 - 1/2$ and $g_2 := x^2 + y^2 - z$. Clearly, g_1, g_2 are differentiable functions and $D = \{g_1 = 0,\ g_2 = 0\}$. The map (g_1, g_2) is a submersion on D iff

$$\operatorname{rank} \begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} y & x & -2z \\ \\ 2x & 2y & -1 \end{bmatrix} = 2,\ \forall (x, y, z) \in D.$$

This is false at (x, y, z) iff all 2×2 subdeterminants of the previous matrix vanish, that is iff

$$\begin{cases} 2y^2 - 2x^2 = 0, \\ -y + 4xz = 0, \\ -x + 4zy = 0. \end{cases}$$

The first equation is equivalent to $(y-x)(y+x)=0$, that yields the alternative $y=x$ or $y=-x$. In the first case we have

$$\begin{cases} y=x, \\ x(1-4z)=0, \end{cases} \iff x=0, \ \vee \ z=\frac{1}{4},$$

from which we find points $(0,0,z)$ $(z\in\mathbb{R})$ and $(x,x,\frac{1}{4})$ $(x\in\mathbb{R})$. Similarly, for $y=-x$ we get

$$\begin{cases} y=-x, \\ x(1+4z)=0, \end{cases} \iff x=0, \ \vee \ z=-\frac{1}{4},$$

from which we find points $(0,0,z)$ $(z\in\mathbb{R})$ and $(x,-x,-\frac{1}{4})$ $(x\in\mathbb{R})$. Now, let's check if these bad points belong to D:

- $(0,0,z)\in D$ iff $z^2=1/2$ and $z=0$, impossible.
- $(x,x,\frac{1}{4})\in D$ iff $x^2+\frac{1}{16}=\frac{1}{2}$ and $2x^2=\frac{1}{4}$, that is $x^2=\frac{7}{16}$ and $x^2=\frac{1}{8}$: impossible.
- $(x,-x,-\frac{1}{4})\in D$ iff $-x^2+\frac{1}{16}=\frac{1}{2}$ and $2x^2=-\frac{1}{4}$, impossible.

We conclude that the rank of the jacobian matrix of (g_1,g_2) is $=2$ on D, that is (g_1,g_2) is a submersion on D.

ii) D is defined by equalities on continuous functions, therefore it is closed. Let's check it is also bounded: we have

$$(x,y,z)\in D, \implies x^2+y^2=z, \implies (x^2+y^2)^2=z^2=\frac{1}{2}-xy$$

so, in particular,

$$(x^2+y^2)^2=\frac{1}{2}-xy.$$

Now, if $(x,y)=(\rho\cos\theta,\rho\sin\theta)$ this says that

$$\rho^4=\frac{1}{2}-\rho^2\cos\theta\sin\theta=\frac{1}{2}\left(1-\rho^2\sin(2\theta)\right)\leqslant\frac{1}{2}(1+\rho^2), \iff \frac{\rho^4}{1+\rho^2}\leqslant\frac{1}{2}.$$

From this we deduce that ρ must be bounded: if unbounded, $\rho\to+\infty$, we would have $\frac{\rho^4}{1+\rho^2}\to+\infty$ and the previous bound would be impossible. Thus, $x^2+y^2=\rho^2\leqslant M$ for some M, and since $z=x^2+y^2$ we get $0\leqslant z\leqslant M$, so D is bounded. Conclusion: D is compact.

Alternative solution: from the remarkable inequality $|xy|\leqslant\frac{x^2+y^2}{2}$, we get

$$z^2=\frac{1}{2}-xy\leqslant\frac{1}{2}+\frac{x^2+y^2}{2}=\frac{1}{2}+\frac{z}{2},$$

from which

$$2z^2-z-1\leqslant 0, \iff -\frac{1}{2}\leqslant z\leqslant 1.$$

Therefore, $x^2+y^2=z\leqslant 1$ and also $|x|,|y|\leqslant 1$, that is D is bounded.

iii) Let $f(x,y,z):=x^2+y^2$ be the square of the distance to the z axis. We have to determine min/max of f on D. Since D is compact and f is continuous, the Weierstrass theorem ensures the existence. To

determine these points, we apply the Lagrange multipliers theorem. By i), we see that the assumptions of the theorem are verified. Then, if $(x, y, z) \in D$ is a min/max point for f we have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rank} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 2x & 2y & 0 \\ y & x & -2z \\ 2x & 2y & -1 \end{bmatrix} = 2.$$

This condition is equivalent to the determinant of the previous matrix = 0,

$$-1(2x^2 - 2y^2) = 0, \iff (y - x)(y + x) = 0,$$

from which we get points of type (x, x, z) and $(x, -x, z)$. Now,

$$(x, x, z) \in D, \iff \begin{cases} x^2 + z^2 = \frac{1}{2}, \\ 2x^2 = z, \end{cases} \iff \begin{cases} z^2 + \frac{z}{2} = \frac{1}{2}, \\ x^2 = \frac{z}{2}. \end{cases} \iff \begin{cases} z = -1, \frac{1}{2}, \\ x^2 = \frac{z}{2}. \end{cases}$$

It is clear that $z = -1$ yields no solutions while $z = \frac{1}{2}$ yields $x^2 = \frac{1}{4}$ from which $x = \pm\frac{1}{2}$. We get points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2})$ (two points).

Similarly,

$$(x, -x, z) \in D, \iff \begin{cases} -x^2 + z^2 = \frac{1}{2}, \\ 2x^2 = z, \end{cases} \iff \begin{cases} z^2 - \frac{z}{2} = \frac{1}{2}, \\ x^2 = \frac{z}{2}. \end{cases} \iff \begin{cases} z = -\frac{1}{2}, 1, \\ x^2 = \frac{z}{2}. \end{cases}$$

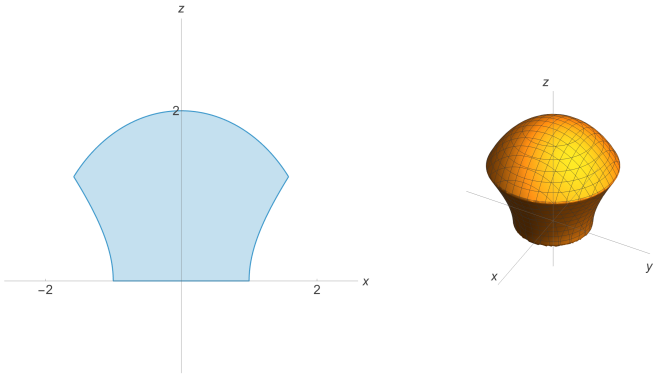
It is clear that $z = -1/2$ yields no solutions while $z = 1$ yields $x^2 = \frac{1}{2}$ from which $x = \pm\frac{1}{\sqrt{2}}$. We get points $(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 2)$ (two points).

Conclusion: since $f(\pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ and $f(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 2) = 1$, the former are max points, the latter min points. □

Exercise 49. i) We have

$$D \cap \{y = 0\} = \{(x, 0, z) \ : \ x^2 - z^2 \leqslant 1, \ x^2 + z^2 \leqslant 4, \ z \geqslant 0\} = \{x^2 - z^2 \leqslant 1\} \cap \{x^2 + z^2 \leqslant 4\} \cap \{z \geqslant 0\}.$$

Now, in the upper half-plane of xz plane with $z \geqslant 0$, $\{x^2 + z^2 \leqslant 4\}$ is a disk centered at the origin with radius 2, while $\{x^2 - z^2 \leqslant 1\}$ is the plane region delimited by the hyperbola $x^2 - z^2 = 1$.



ii) To compute the volume we use the formula

$$\lambda_3(D) = \int_D 1\ dxdydz = \int_{x^2+y^2-z^2\leqslant 1,\ x^2+y^2+z^2\leqslant 4,\ z\geqslant 0} 1\ dxdydz.$$

Using cylindrical coordinates, we get

$$\lambda_3(D) = \int_{\rho^2-z^2\leqslant 1,\ \rho^2+z^2\leqslant 4,\ z\geqslant 0,\ 0\leqslant\theta\leqslant 2\pi} \rho\ d\rho d\theta\ dz \overset{RF}{=} 2\pi \int_{\rho^2-z^2\leqslant 1,\ \rho^2+z^2\leqslant 4,\ z\geqslant 0} \rho\ d\rho\ dz$$

We do the double integration beginning with ρ and finishing with z. We notice that

$$\rho^2 - z^2 \leqslant 1,\ \rho^2 + z^2 \leqslant 4,\ \implies\ \rho^2 \leqslant 1 + z^2, 4 - z^2.$$

Therefore,

$$\rho^2 \leqslant \min(1+z^2, 4-z^2) = \begin{cases} 1+z^2, & \text{if } 1+z^2 \leqslant 4-z^2, \iff z^2 \leqslant \frac{3}{2}, \overset{z\geqslant 0}{\iff} 0 \leqslant z \leqslant \sqrt{\frac{3}{2}}, \\ 4-z^2, & \text{if } z \geqslant \sqrt{\frac{3}{2}}. \end{cases}$$

Notice also that, since $\rho^2 \geqslant 0$, $\rho^2 = 4 - z^2 \geqslant 0$ iff $0 \leqslant z \leqslant 2$. Therefore,

$$\rho^2 \leqslant \min(1+z^2, 4-z^2) = \begin{cases} 1+z^2, & 0 \leqslant z \leqslant \sqrt{\frac{3}{2}}, \\ 4-z^2, & \sqrt{\frac{3}{2}} \leqslant z \leqslant 2. \end{cases}$$

Therefore

$$\begin{aligned} \lambda_3(D) &= 2\pi \left(\int_0^{\sqrt{3/2}} \int_0^{\sqrt{1+z^2}} \rho\ d\rho\ dz + \int_{\sqrt{3/2}}^2 \int_0^{\sqrt{4-z^2}} \rho\ d\rho\ dz \right) \\ &= 2\pi \left(\int_0^{\sqrt{3/2}} \left[\frac{\rho^2}{2} \right]_{\rho=0}^{\rho=\sqrt{1+z^2}} dz + \int_{\sqrt{3/2}}^2 \left[\frac{\rho^2}{2} \right]_{\rho=0}^{\rho=\sqrt{4-z^2}} dz \right) \\ &= \pi \left(\int_0^{\sqrt{3/2}} (1+z^2)\ dz + \int_{\sqrt{3/2}}^2 4 - z^2\ dz \right) \\ &= \pi \left(\sqrt{\frac{3}{2}} + 4\left(2 - \sqrt{\frac{3}{2}}\right) + \left[\frac{z^3}{3} \right]_{z=0}^{z=\sqrt{3/2}} - \left[\frac{z^3}{3} \right]_{z=\sqrt{3/2}}^{z=2} \right) \\ &= \pi \left(8 - 3\sqrt{\frac{3}{2}} + \frac{1}{3}\frac{3}{2}\sqrt{\frac{3}{2}} - \frac{1}{3}\left(8 - \frac{3}{2}\sqrt{\frac{3}{2}}\right) \right) = \pi \left(\frac{16}{3} - 3\sqrt{\frac{3}{2}} \right). \quad \square \end{aligned}$$

Exercise 50. i) u and v must be $\mathbb{R}-$differentiable and verify the Cauchy-Riemann equations at each point $(x, y) \in \mathbb{R}^2$.

ii) If u is constant, then by the CR equations we get

$$\begin{cases} \partial_x v = -\partial_y u = 0, \\ \\ \partial_y v = \partial_x u = 0, \end{cases}$$

From the first equation $v(x, y) = \int 0 \, dx + c(y) = c(y)$, and plugging this into the second equation we get $c'(y) = 0$, that is c is constant. Therefore $v(x, y) \equiv c$ so $f = 0 + ic = ic$ is constant.

iii) Following the hint

$$0 = \partial_x(u^2 + v^2) = 2u\partial_x u + 2v\partial_x v = 2(u, v) \cdot (\partial_x u, \partial_x v).$$

Similarly,

$$0 = \partial_y(u^2 + v^2) = 2u\partial_y u + 2v\partial_y v = 2(u, v) \cdot (\partial_y u, \partial_y v).$$

These relations say that $(\partial_x u, \partial_x v)$ and $(\partial_y u, \partial_y v)$ are both perpendicular to (u, v). Assume that $(u, v) \neq (0, 0)$. Since the plane vectors $(\partial_x u, \partial_x v)$ and $(\partial_y u, \partial_y v)$ are both perpendicular to $(u, v) \neq \vec{0}$, they must be parallel, so

$$\exists \alpha \ : \ (\partial_x u, \partial_x v) = \alpha(\partial_y u, \partial_y v), \quad \Longleftrightarrow \quad \begin{cases} \partial_x u = \alpha \partial_y u, \\ \\ \partial_x v = \alpha \partial_y v, \end{cases}$$

Now, by the CR equations we have

$$\partial_x u = \alpha \partial_y u \overset{CR}{=} \alpha(-\partial_x v) = -\alpha(\alpha \partial_y v) \overset{CR}{=} -\alpha^2 \partial_x u$$

that is

$$(1 + \alpha^2)\partial_x u = 0, \implies \partial_x u = 0.$$

In the same way we deduce $\partial_y u = 0$, and from this u is constant. Similarly for v, we conclude that $f = u + iv$ is a constant function. □

Exercise 56. i) From the general theory, $u+iv$ is $\mathbb{C}-$differentiable on $\mathbb{C}$ iff both u, v are $\mathbb{R}-$differentiable and they fulfill the Cauchy-Riemann equations. Since u is a polynomial, it is $\mathbb{R}-$differentiable. So, we look for v $\mathbb{R}-$differentiable in such a way that

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \partial_x v = -\partial_y u = -(-20x^3 y + 20xy^3) = 20(x^3 y - xy^3), \\ \\ \partial_y v = \partial_x u = 5x^4 - 30x^2 y^2 + 5y^4, \end{cases}$$

From the first equation,

$$v(x, y) = \int 20(x^3 y - xy^3) \, dx + c(y) = 5x^4 y - 10x^2 y^3 + c(y).$$

Imposing the second equation we get,

$$5x^4 - 30x^2 y^2 + c'(y) = 5x^4 - 30x^2 y^2 + 5y^4, \quad \Longleftrightarrow \quad c'(y) = 5y^4, \quad \Longleftrightarrow \quad c(y) = y^5 + c,$$

where $c \in \mathbb{R}$. Conclusion: the unique possible v such that $u + iv$ be $\mathbb{C}-$differentiable on $\mathbb{C}$ is

$$v(x, y) = 5x^4 y - 10x^2 y^3 + y^5 + c.$$

ii) We have

$$\begin{aligned} f(x+iy) &= u(x,y)+iv(x,y) = x^5 - 10x^3y^2 + 5xy^4 + i(5x^4y - 10x^2y^3 + y^5 + c) \\ &= x^5 + i5x^4y - 10x^3y2 + i10x^2y^3 + 5xy^4 + iy^5 + ic \\ &= (x+iy)^5 + ic, \end{aligned}$$

from which $f(z) = z^5 + ic$. □

Exercise 57. We have $f(x,x) = 2x^4 \longrightarrow +\infty$ when $(x,x) \to \infty_2$. So, if a limit exists it must be $= +\infty$. Using polar coordinates

$$f(x,y) = \rho^4\left(\cos^4\theta + \sin^4\theta\right) - 2\rho^2(\cos\theta - \sin\theta)^2 \geqslant \rho^4\left(\cos^4\theta + \sin^4\theta\right) - 8\rho^2.$$

Now, let $g(\theta) := \cos^4\theta + \sin^4\theta$. g is positive valued and continuous on $[0, 2\pi]$. By the Weierstrass theorem there exists a global minimum, that is $\exists\theta_{min}$ such that $g(\theta) \geqslant g(\theta_{min})\ \forall\theta \in [0,2\pi]$. Let $C := g(\theta_{min}) \geqslant 0$. If $C = 0$, $\cos^4\theta_{min} + \sin^4\theta_{min} = 0$, that is $\cos\theta_{min} = \sin\theta_{min} = 0$, which is impossible. So,

$$f(x,y) \geqslant C\rho^4 - 8\rho^2 \longrightarrow +\infty, \text{ when } \rho = \|(x,y)\| \longrightarrow +\infty.$$

We conclude that $\exists \lim_{(x,y)\to\infty_2} f(x,y) = +\infty$.

ii) f is a polynomial, hence it is differentiable on $\mathbb{R}^2$ and

$$\nabla f(x,y) = \left(4x^3 - 4(x-y), 4y^3 + 4(x-y)\right).$$

A point (x,y) is a stationary point iff

$$\nabla f(x,y) = \vec{0}, \iff \begin{cases} 4x^3 - 4(x-y) = 0, \\ \\ 4y^3 + 4(x-y) = 0, \end{cases} \iff \begin{cases} x - y = x^3, \\ \\ y^3 = -x^3. \end{cases}$$

From the second equation we get $y = -x$, and plugging into the first one we get

$$x^3 = 2x, \iff x(x^2-2) = 0, \iff x = 0, \pm\sqrt{2}.$$

The stationary points are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. To classify them we compute the Hessian matrix:

$$\nabla^2 f(x,y) = \begin{bmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{bmatrix}$$

From this we see that

$$\nabla^2 f(0,0) = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}, \quad \nabla^2 f(\pm\sqrt{2}, \mp\sqrt{2}) = \begin{bmatrix} 20 & 4 \\ 4 & 20 \end{bmatrix}$$

The diagonal sub-determinants of $\nabla^2 f(0,0)$ are $-4, 0$ so $\nabla^2 f(0,0) \leqslant 0$ but not < 0. It is not possible to conclude anything from the hessian matrix. However, we may notice that $f(x,x) = 2x^4$, so $(0,0)$ is a local min point along $y = x$, while $f(x,0) = x^4 - 2x^2 \sim_{x\to 0} -2x^2$, from which we see that $(0,0)$ is a local max point along $x = 0$. This shows that $(0,0)$ is a saddle point.

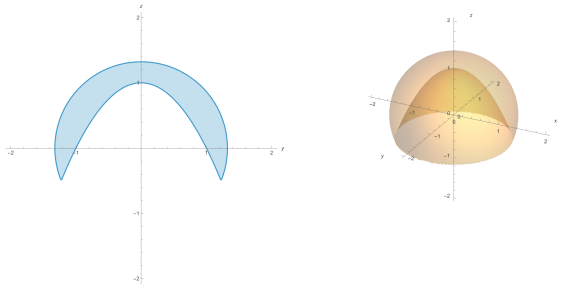
Abot $\nabla^2 f(\pm\sqrt{2}, \mp\sqrt{2})$ the diagonal sub-determinants are $20, 384$, both positive, hence $\nabla^2 f(\pm\sqrt{2}, \mp\sqrt{2}) > 0$, both points are local min. points.

iii) By i), $\nexists \max_{\mathbb{R}^2} f$ while $\exists \min_{\mathbb{R}^2} f$. Let (x, y) be any min point. Since $\mathbb{R}^2$ is open, $(x, y) \in \mathrm{Int}(\mathbb{R}^2)$. According to the Fermat theorem, $\nabla f(x, y) = \vec{0}$, so (x, y) is one of $(0, 0)$, $(\pm\sqrt{2}, \mp\sqrt{2})$. By ii) the unique possibilities are $(\pm\sqrt{2}, \mp\sqrt{2})$, and since $f(\pm\sqrt{2}, \mp\sqrt{2}) = 8$, we conclude that both points are global min points for f. □

Exercise 58. i) We have

$$D \cap \{x = 0\} = \{(0, y, z) \in \mathbb{R}^3 \ : \ z \geqslant 1 - y^2, \ y^2 + z^2 \leqslant \tfrac{7}{4}\}.$$

In the plane yz, this is the part of plane inside the disk $y^2 + z^2 \leqslant \frac{7}{4}$ and above the parabola $z = 1 - y^2$.



D is then the rotation solid obtained by rotation of $D \cap \{x = 0\}$ around the $z-$axis.

ii) We have

$$\lambda_3(D) = \int_D 1 \ dxdydz.$$

Using standard cylindrical coordinates we get

$$(x, y, z) \in D, \quad \Longleftrightarrow \quad z \geqslant 1 - \rho^2, \ \rho^2 + z^2 \leqslant \tfrac{7}{4}, \ \theta \in [0, 2\pi].$$

Therefore

$$\lambda_3(D) = \int_{z \geqslant 1-\rho^2, \ \rho^2+z^2 \leqslant \frac{7}{4}, \ 0 \leqslant \theta \leqslant 2\pi} \rho \ d\rho d\theta dz \overset{RF}{=} 2\pi \int_{z \geqslant 1-\rho^2, \ \rho^2+z^2 \leqslant \frac{7}{4}} \rho \ d\rho dz.$$

Here we have two possible ways to compute the integral, by integrating first in ρ and then in z or vice versa. Here, we choose to do the first integration in ρ. We notice that

$$z \geqslant 1 - \rho^2, \ \rho^2 + z^2 \leqslant \tfrac{7}{4}, \quad \Longleftrightarrow \quad 1 - z \leqslant \rho^2 \leqslant \tfrac{7}{4} - z^2.$$

Notice that $1 - z \leqslant \frac{7}{4} - z^2$ iff $z^2 - z - \frac{3}{4} \leqslant 0$ or $4z^2 - 4z - 3 \leqslant 0$. This happens iff $-\frac{1}{2} \leqslant z \leqslant \frac{3}{2}$. Actually, by the sphere constraint, $z^2 \leqslant \frac{7}{4}$, that is $|z| \leqslant \frac{\sqrt{7}}{2} < \frac{3}{2}$, so $-\frac{1}{2} \leqslant z \leqslant \frac{\sqrt{7}}{2}$. For such z, $1 - z < 0$ for $z > 1$. Therefore,

$$1 - z \leqslant \rho^2 \leqslant \frac{7}{4} - z^2, \quad \Longleftrightarrow \quad \begin{cases} -\frac{1}{2} \leqslant z \leqslant 1, \quad \sqrt{1 - z} \leqslant \rho \leqslant \sqrt{\frac{7}{4} - z^2}, \\ \\ 1 \leqslant z \leqslant \frac{\sqrt{7}}{2}, \quad 0 \leqslant \rho \leqslant \sqrt{\frac{7}{4} - z^2}. \end{cases}$$

Consequently,

$$\begin{aligned}
\int_{z\geqslant 1-\rho^2,\ \rho^2+z^2\leqslant\frac{7}{4}} \rho\ d\rho dz &= \int_{-1/2}^{1}\int_{\sqrt{1-z}}^{\sqrt{\frac{7}{4}-z^2}} \rho\ d\rho\ dz + \int_{1}^{\sqrt{7}/2}\int_{0}^{\sqrt{\frac{7}{4}-z^2}} \rho\ d\rho\ dz \\
&= \int_{-1/2}^{1}\left[\frac{\rho^2}{2}\right]_{\rho=\sqrt{1-z}}^{\rho=\sqrt{\frac{7}{4}-z^2}} dz + \int_{1}^{\sqrt{7}/2}\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sqrt{\frac{7}{4}-z^2}} dz \\
&= \frac{1}{2}\left(\int_{-1/2}^{1}\left(\tfrac{7}{4}-z^2-(1-z)\right)\ dz + \int_{1}^{\sqrt{7}/2}\tfrac{7}{4}-z^2\ dz\right) \\
&= \frac{1}{2}\left(\int_{-1/2}^{\sqrt{7}/2}\tfrac{7}{4}-z^2\ dz - \int_{-1/2}^{1}(1-z)\ dz\right) \\
&= \tfrac{1}{2}\left(\tfrac{7}{4}\tfrac{\sqrt{7}+1}{2} - \tfrac{1}{3}\left[z^3\right]_{z=-1/2}^{z=\sqrt{7}/2} - \left(\tfrac{3}{2} - \tfrac{1}{2}[z^2]_{z=-1/2}^{z=1}\right)\right) \\
&= \tfrac{7}{48}\left(2\sqrt{7}-1\right)
\end{aligned}$$

from which $\lambda_3(D) = \frac{7}{24}\left(2\sqrt{7}-1\right)\pi$. □

Exercise 59. i) The equation has form $y' = f(t,y)$ where

$$f(t,y) := \log(1-ty).$$

The domain of definition for f is

$$D := \{(t,y)\in\mathbb{R}^2\ :\ 1-ty>0\}.$$

Now,

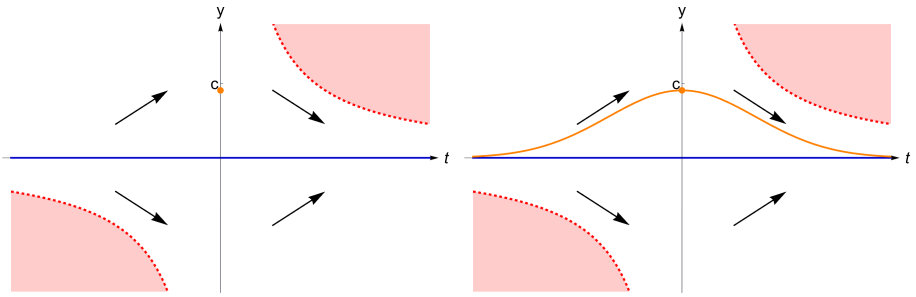
$$1-ty>0, \quad\Longleftrightarrow\quad ty<1, \quad\Longleftrightarrow\quad \begin{cases} t>0,\ y<\frac{1}{t}, \\ t=0,\ \forall y, \\ t<0,\ y>\frac{1}{t}. \end{cases}$$

On its natural domain f is continuous and also $\partial_y f = \frac{-t}{1-ty}$ is continuos on D. We conclude that local existence and uniqueness holds on D.

ii) $y\equiv C$ is a solution iff $0=\log(1-tC)$, that is $1-tC\equiv 1$, or $tC\equiv 0$. Since C is a constant and t is variable, the previous holds iff $C=0$, so $y\equiv 0$ is the unique constant solution. Moreover

$$0\leqslant y' = \log(1-ty), \quad\Longleftrightarrow\quad 1-ty\geqslant 1, \quad\Longleftrightarrow\quad ty\leqslant 0, \quad\Longleftrightarrow\quad t\leqslant 0,\ y\geqslant 0, \ \text{ or } \ t<0,\ y\geqslant 0.$$

We get the following picture.



iii) We show that $y(t) > 0$ for every t. Indeed, if there exists a t^* such that $y(t^*) \leqslant 0$ then

- if $y(t^*) = 0$, y would cross the constant solution. By uniqueness, $y(t) \equiv 0$ so also $y(0) = 0$, but $y(0) = c > 0$, so we get a contradiction.
- if $y(t^*) < 0$ then, by continuity, there would be a t^{**} such that $y(t^{**}) = 0$, so we fall back to the previous contradiction.

Since $y(t) > 0$ for every t, by ii) we deduce that $y \nearrow$ for $t < 0$ and $y \searrow$ for $t > 0$.

iv) Since $y \searrow$ for $t > 0$, there exists $\ell := \lim_{t\to\beta} y(t)$. Since $y > 0$ it follows that $\ell \geqslant 0$. We check that $\beta = +\infty$. If $\beta < +\infty$, we claim that $\ell < \frac{1}{\beta}$. Indeed: $1 - ty(t) > 0$ so, letting $t \to \beta$, $1 - \beta\ell \geqslant 0$. If $1 - \beta\ell = 0$, then

$$y'(t) \longrightarrow \log(1 - \beta\ell) = \log 0+ = -\infty,$$

but then the solution would have a vertical asymptote at β. This is impossible being $y(t) > 0$ for every t. Thus $1 - \beta\ell > 0$, that is the final point $(\beta, \ell) \in D$, and this is not possible by the argument of exit from compact sets. Contradiction.

Since all this follows by the assumption $\beta < +\infty$ we conclude that $\beta = +\infty$. We claim that $\ell = 0$. If $\ell > 0$, then $1 - ty(t) \to 1 - (+\infty)\ell = -\infty$, so for t large $1 - ty(t) < 0$, impossible for a solution.

Similarly, we can prove that $\alpha = -\infty$ and $\lim_{t\to\alpha} y(t) = 0$.

v) Figure: see above. □

Exercise 60. We can set the problem as a constrained optimization problem. Let $f(\vec{x}) := (\vec{a} \cdot \vec{x})^2 = \left(\sum_{j=1}^d a_j x_j\right)^2$. We see that f is polynomial, hence it is continuous. The optimization domain

$$D := \{\vec{x} \in \mathbb{R}^d \ : \ \|\vec{x}\| = 1\} = \left\{\vec{x} \in \mathbb{R}^d \ : \ \sqrt{\sum_{j=1}^d x_j^2} = 1\right\}$$

is closed (defined by an equation on a continuous function) and bounded (obviously because $\|\vec{x}\| = 1$ on D). Thus, D is compact, so f attains both min and max on D because of the Weierstrass theorem. To determine these points we apply the Lagrange multipliers theorem. Before we start, to simplify calculations it is convenient to notice that

$$D = \{\vec{x} \in \mathbb{R}^d \ : \ x_1^2 + \cdots + x_d^2 = 1\}.$$

We notice that $D = \{g = 0\}$ where $g(x_1, \ldots, x_d) = x_1^2 + \cdots + x_d^2 - 1$, and g is a submersion on D. This because g is not a submersion iff $\nabla g = 0$. Since $\nabla g(\vec{x}) = 2\vec{x}$, we have

$$\nabla g(\vec{x}) = \vec{0}, \quad \Longleftrightarrow \quad 2\vec{x} = \vec{0}, \quad \Longleftrightarrow \quad \vec{x} = \vec{0} \notin D.$$

Now, at any min/max point for f on D we have

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x}).$$

We have

$$\partial_k f(\vec{x}) = \partial_k (\vec{a} \cdot \vec{x})^2 = 2(\vec{a} \cdot \vec{x}) \partial_k (\vec{a} \cdot \vec{x}),$$

and since

$$\partial_k \vec{a} \cdot \vec{x} = \partial_k \sum_{j=1}^{d} a_j x_j = a_k,$$

we conclude that

$$\nabla f(\vec{x}) = 2(\vec{a} \cdot \vec{x}) \vec{a}.$$

So,

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x}), \quad \Longleftrightarrow \quad 2(\vec{a} \cdot \vec{x}) \vec{a} = 2\lambda \vec{x}, \quad \Longleftrightarrow \quad \lambda \vec{x} = (\vec{a} \cdot \vec{x}) \vec{a}.$$

This equation for $\vec{x}$ yields the following alternatives:

- $\lambda = 0$: in this case $(\vec{a} \cdot \vec{x}) \vec{a} = \vec{0}$, and since $\vec{a} \neq \vec{0}$, this is possible iff $\vec{a} \cdot \vec{x} = 0$, that is $\vec{x} \perp \vec{a}$. For these $\vec{x}$ we have $f(\vec{x}) = (\vec{a} \cdot \vec{x})^2 = 0$.
- $\lambda \neq 0$: in this case $\vec{x} = \mu \vec{a}$ (where we set $\mu := \frac{\vec{a} \cdot \vec{x}}{\lambda} \in \mathbb{R}$). Imposing $\vec{x} \in D$ we have $\|\mu \vec{a}\| = 1$ that is $|\mu| \|\vec{a}\| = 1$, from which

  $$\mu = \pm \frac{1}{\|\vec{a}\|}, \implies \vec{x} = \pm \frac{\vec{a}}{\|\vec{a}\|}.$$

  In this case

  $$f(\vec{x}) = \left(\vec{a} \cdot \pm \frac{\vec{a}}{\|\vec{a}\|} \right)^2 = \left(\pm \frac{\|\vec{a}\|^2}{\|\vec{a}\|} \right)^2 = \|\vec{a}\|^2 > 0.$$

Conclusion: the maximum is attained at $\vec{x} = \pm \frac{\vec{a}}{\|\vec{a}\|}$ with value $\|\vec{a}\|^2$.

Alternative solution. This is based on the Cauchy-Schwarz inequality and does not require any use of differential calculus. We recall that

$$|\vec{x} \cdot \vec{y}| \leqslant \|\vec{x}\| \|\vec{y}\|, \ \forall \vec{x}, \vec{y} \in \mathbb{R}^d.$$

So, if $\|\vec{x}\| = 1$,

$$(\star) \ |\vec{a} \cdot \vec{x}| \leqslant \|\vec{a}\| \|\vec{x}\| \leqslant \|\vec{a}\|, \implies (\vec{a} \cdot \vec{x})^2 = |\vec{a} \cdot \vec{x}|^2 \leqslant \|\vec{a}\|^2, \ \forall \vec{x} \ : \ \|\vec{x}\| = 1.$$

If we take $\vec{x}_\pm := \pm \frac{\vec{a}}{\|\vec{a}\|}$. These $\vec{x}_\pm$ are well defined because $\vec{a} \neq \vec{0}$ and

$$\|\vec{x}_\pm\| = \left\| \pm \frac{\vec{a}}{\|\vec{a}\|} \right\| = \frac{1}{\|\vec{a}\|} \|\vec{a}\| = 1.$$

Moreover,

$$(\star\star) \ (\vec{a} \cdot \vec{x}_\pm)^2 = \left(\vec{a} \cdot \frac{\vec{a}}{\|\vec{a}\|} \right)^2 = \frac{1}{\|\vec{a}\|^2} (\vec{a} \cdot \vec{a})^2 = \frac{1}{\|\vec{a}\|^2} \|\vec{a}\|^4 = \|\vec{a}\|^2.$$

Combining $(\star)$ and $(\star\star)$ we get that

$$\max_{\|\vec{x}\|=1} (\vec{a} \cdot \vec{x})^2 = \|\vec{a}\|^2,$$

and the maximum is achieved at $\vec{x}_\pm = \pm\frac{\vec{a}}{\|\vec{a}\|}$. Are there other max points? If $\vec{x}^*$ is any max point s.t. $\|\vec{x}^*\| = 1$, then

$$(\vec{a}\cdot\vec{x}^*)^2 = \|\vec{a}\|^2.$$

If this happens, it means that the CS is actually an equality, and this happens iff $\vec{x}^* = \lambda\vec{a}$. Imposing $\|\vec{x}^*\| = 1$ we get $|\lambda|\|\vec{a}\| = 1$, that is $\lambda = \pm\frac{1}{\|\vec{a}\|}$, that is $\vec{x}^*$ is either $\vec{x}_+$ or $\vec{x}_-$. □

Exercise 61. Using Euler's formulas,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

we have

$$\frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iz} + e^{-iz}}{2} = 2,$$

that is, after straightforward manipulations,

$$(1-i)w^2 - 4w + (1+i) = 0, \quad w = e^{iz}.$$

This is a second degree equation whose solutions are

$$w_{1,2} = \frac{4 \pm \sqrt{8}}{2(1-i)} = \frac{2 \pm \sqrt{2}}{1-i} = (1+i)\left(1 \pm \frac{1}{\sqrt{2}}\right).$$

Both have argument $\pi/4$ and

$$|w_1| = \sqrt{2} + 1, \qquad |w_2| = \sqrt{2} - 1 = \frac{1}{\sqrt{2}+1}.$$

Thus

$$iz = \log\left((\sqrt{2}\pm 1)e^{i\pi/4}\right) = \log(\sqrt{2}\pm 1) + i\left(\frac{\pi}{4} + k2\pi\right), \ k \in \mathbb{Z},$$

from which

$$z = \frac{\pi}{4} + 2k\pi - i\log(\sqrt{2}\pm 1), \qquad k \in \mathbb{Z}.$$

Exercise 62. i) The inequalities defining D are

$$x \geqslant 0, \qquad 0 \leqslant x + y \leqslant 1, \quad \Longleftrightarrow \quad x \geqslant 0, \qquad -x \leqslant y \leqslant 1 - x.$$

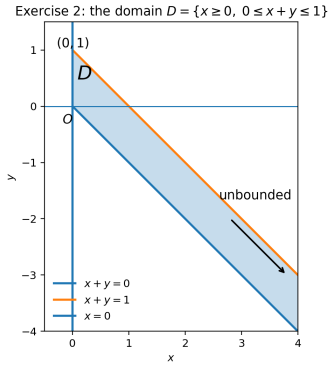
Therefore D is the strip between the two parallel lines

$$x + y = 0$$

and

$$x + y = 1,$$

restricted to the half-plane $x \geqslant 0$.



Clearly, D is unbounded, so it cannot be compact. The set D is closed being defined by large inequalities with continuous functions. It is not open, because the unique both open and closed sets are $\varnothing$ and $\mathbb{R}^2$, and this is not the case.

ii) We have

$$f(x, y) = xy(x + y) - xy = xy(x + y - 1).$$

So, on $y = -x$ we have

$$f(x, -x) = x^2 \to +\infty,\ x \to +\infty.$$

Therefore, f is unbounded on D, $\sup_D f = +\infty$ and $\nexists \max_D f$. About min, we notice that for $x \geqslant 0$, $y \leqslant 0$, being $x + y \leqslant 1$ on D, that is $x + y - 1 \leqslant 0$, we have

$$f(x, y) \geqslant 0.$$

On the triangle

$$T = D \cap \{y \geqslant 0\},\ \implies\ f \leqslant 0.$$

Thus

$$\inf_D f = \inf_T f = \min_T f.$$

Since T closed and bounded it is compact. The min point of f exists (Weierstrass theorem). We have the following alternative:

- either $(x, y) \in \mathrm{Int}(T)$, and in this case $\nabla f(x, y) = \vec{0}$ according to the Fermat theorem. We have

$$\nabla f(x, y) = (y^2 + 2xy - y, 2xy + x^2 - x) = \vec{0},\ \iff\ \begin{cases} y(y + 2x - 1) = 0, \\ x(x + 2y - 1) = 0. \end{cases}$$

  Since here $(x, y) \in \mathrm{Int}(T)$, we cannot have $x = 0$ or $y = 0$, thus

$$\nabla f(x, y) = \vec{0},\ \iff\ \begin{cases} y + 2x - 1 = 0, \\ x + 2y - 1 = 0, \end{cases}\ \iff\ \begin{cases} x = \frac{1}{3}, \\ y = \frac{1}{3}, \end{cases}$$

  and since $(\frac{1}{3}, \frac{1}{3}) \in \mathrm{Int}(T)$, this is a stationary point for f on T.
- or $(x, y) \in \partial D$, thus $(x, y) = (x, 0)$ with $x \in [0, 1]$, or $(x, y) = (0, y)$ with $y \in [0, 1]$, or again $(x, y) = (x, 1 - x)$ with $x \in [0, 1]$. We have

$$f(x, 0) = f(0, y) = f(x, 1 - x) \equiv 0.$$

Since $f(\frac{1}{3},\frac{1}{3}) = \frac{1}{9}\left(\frac{2}{3}-1\right) = -\frac{1}{27}$ we conclude that this is the unique min point for f on D. Thus

$$\boxed{\min_D f = -\frac{1}{27}, \qquad \operatorname*{argmin}_D f = \left\{\left(\frac{1}{3},\frac{1}{3}\right)\right\},}$$

and

$$\boxed{f \text{ has no maximum on } D.}$$

Exercise 63. Set

$$u = x^2, \qquad v = y^2, \quad (u,v) = \Phi(x,y) : [0,+\infty[^2 \to [0,+\infty[^2.$$

Φ is a bijection with

$$\Phi^{-1} : [0,+\infty[^2 \to [0,+\infty[^2, \;\; \Phi^{-1}(u,v) = \left(\sqrt{u},\sqrt{v}\right).$$

Recalling the change of variables formula

$$\int_D f(x,y)\, dxdy = \int_{\Phi(D)} f\left(\Phi^{-1}(u,v)\right) |\det(\Phi^{-1})'(u,v)|\, dudv,$$

we have

$$\int_{[0,+\infty[^2} xye^{-(x^4+y^4)}\, dxdy = \int_{[0,+\infty[^2} \sqrt{uv}e^{-(u^2+v^2)} |\det(\Phi^{-1})'(u,v)|\, dudv.$$

To compute the determinant we notice that

$$\Phi'(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}, \implies |\det(\Phi^{-1})(u,v)| = \frac{1}{|\det \Phi(\Phi^{-1}(u,v))|} = \frac{1}{|4xy|} = \frac{1}{4\sqrt{uv}},$$

so,

$$\int_{[0,+\infty[^2} xye^{-(x^4+y^4)}\, dxdy = \int_{[0,+\infty[^2} \sqrt{uv}e^{-(u^2+v^2)} \frac{1}{4\sqrt{uv}}\, dudv = \frac{1}{4}\int_{[0,+\infty[^2} e^{-(u^2+v^2)}\, dudv.$$

Here we may remember of the Gaussian integral, or we can compute it directly. Introducing polar coordinates in the (u,v)-plane:

$$u = r\cos\theta, \qquad v = r\sin\theta.$$

Since $u, v \geqslant 0$, the region corresponds to

$$r \in [0,+\infty[, \qquad \theta \in \left[0,\frac{\pi}{2}\right].$$

The Jacobian is

$$du\, dv = r\, dr\, d\theta.$$

Therefore

$$\int_{[0,+\infty[^2} xye^{-(x^4+y^4)}\, dxdy = \frac{1}{4}\int_0^{\pi/2}\int_0^{+\infty} e^{-r^2} r\, dr\, d\theta.$$

Now

$$\int_0^{+\infty} e^{-r^2} r\, dr = \left[\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=+\infty} = \frac{1}{2}.$$

Therefore

$$I = \frac{1}{4} \cdot \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{16}.$$

Thus

$$\boxed{\int_{[0,+\infty[^2} xye^{-x^4-y^4}\, dx\, dy = \frac{\pi}{16}.}$$

Exercise 64. i) The field $\vec{F} = (F_1(x, y), F_2(x, y))$ is irrotational on $\mathbb{R}^2$ if and only if

$$\partial_y F_1 = \partial_x F_2.$$

We have

$$\partial_y F_1 = ax^2 + 3by^2, \qquad \partial_x F_2 = 3cx^2 + dy^2.$$

Thus we require

$$ax^2 + 3by^2 = 3cx^2 + dy^2$$

for every $(x, y) \in \mathbb{R}^2$. Therefore

$$a = 3c, \qquad 3b = d.$$

Hence

$$\boxed{\vec{F} \text{ is irrotational on } \mathbb{R}^2 \iff a = 3c,\ d = 3b.}$$

ii) Since $\mathbb{R}^2$ is simply connected, every C^1 irrotational vector field on $\mathbb{R}^2$ is conservative. Thus $\vec{F}$ is conservative exactly when

$$a = 3c, \qquad d = 3b.$$

We determine a potential f. We need

$$\nabla f = \vec{F}.$$

Thus

$$\partial_x f = F_1 = ax^2y + by^3 + x.$$

Integrating with respect to x, we obtain

$$f(x, y) = \frac{a}{3}x^3y + bxy^3 + \frac{x^2}{2} + h(y),$$

where h depends only on y. Now differentiate with respect to y:

$$\partial_y f(x, y) = \frac{a}{3}x^3 + 3bxy^2 + h'(y).$$

We require

$$\partial f_y = F_2 = cx^3 + dxy^2 + y.$$

Using

$$a = 3c, \qquad d = 3b,$$

we get

$$h'(y) = y.$$

Therefore

$$h(y) = \frac{y^2}{2} + C, \qquad C \in \mathbb{R}.$$

Hence all potentials are

$$\boxed{f(x,y) = \frac{a}{3}x^3y + bxy^3 + \frac{x^2+y^2}{2} + C, \qquad C \in \mathbb{R}.}$$

iii) Let γ be the positively oriented boundary of

$$D := \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ x^2 \leqslant y \leqslant x\}.$$

For the values of a, b, c, d found above, the vector field is conservative. The line integral of a conservative vector field along a closed curve is zero. Since γ is a closed curve,

$$\oint_\gamma \vec{F} = 0.$$

Exercise 65. i) A set

$$K \subset \mathbb{R}^d$$

is compact if it is closed and bounded.

ii) Let

$$S := \{x \in \mathbb{R}^d : f(x) \leqslant M\}.$$

We prove that S is compact.

First, S is closed. Indeed, S is defined by a large inequality involving a continuous function (by assumption f is continuous).

Now we prove that S is bounded. Suppose, by contradiction, that S is unbounded. Then, there exists a sequence $(\vec{x}_n) \subset S$ such that

$$\|\vec{x}_n\| \to +\infty.$$

Since

$$\lim_{\|x\|\to+\infty} f(x) = +\infty,$$

we have

$$f(\vec{x}_n) \to +\infty.$$

But $(\vec{x}_n) \subset S$, so $f(\vec{x}_n) \leqslant M$ for every n. This contradicts $f(\vec{x}_n) \to +\infty$. We got a contradiction. Thus S must be bounded.

We have shown that S is closed and bounded. Thus, S is compact.