Calculus 2 — Final Exam

February 2026

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Prompt #1

Exercise 1. .

i) What does it mean that $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ is differentiable at point $\vec{x} \in D$?

ii) Let $f = f(x, y)$ be defined as

$$f(x,y) = \begin{cases} \frac{x^3 - yx^2}{\sqrt{x^2+y^2}}, & (x,y) \neq \vec{0}, \\ \\ 0, & (x,y) = \vec{0}. \end{cases}$$

Is f continuous at $(0,0)$? Is f differentiable at $(0,0)$?

Exercise 2. Let

$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1 + xy\right\}.$$

i) Show that $D \neq \emptyset$ is the zero set of a submersion.

ii) Is D compact?

iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 3. Let

$$D := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leqslant z \leqslant 1 - y^2\}.$$

i) Draw $D \cap \{x = 0\}$ and $D \cap \{y = 0\}$. Is D invariant by some rotation? Justify your answer. Draw D as best as you can.

ii) Compute the volume of D.

Exercise 4. Let

$$\vec{F} := \left(\frac{ax^2 + by^2}{(x^2+y^2)^2}, \frac{xy}{(x^2+y^2)^2}\right)$$

on $D = \mathbb{R}^2 \backslash \{(0,0)\}$. Here $a, b \in \mathbb{R}$ are constants.

i) Determine all possible values for a, b in such a way $\vec{F}$ be irrotational on D.

ii) Determine values of a, b, c in such a way $\vec{F}$ be conservative on D, in this case determining also all the possible potentials.

Exercise 5. .

i) What are the Cauchy–Riemann equations (or conditions)?

ii) Let $f = u + iv$ ($u = \text{Re}\, f$ and $v = \text{Im}\, f$) be a $\mathbb{C}$ differentiable function on the entire plane $\mathbb{C}$. Assume that also $\overline{f} = u - iv = u + i(-v)$ is $\mathbb{C}$ differentiable on $\mathbb{C}$. What conclusion can you draw on f?

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PROMPT #2

Exercise 6. Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 3, \ 0 \leqslant y \leqslant 2\}$ and

$$f(x, y) := x^2 - 2xy + 2y.$$

Determine min/max points of f on D (if any). Justify carefully.

Exercise 7. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (x^2 + y^2)^{1/4} \leqslant z \leqslant 2 - x^2 - y^2\right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 8. Consider the differential equation

$$y' = \frac{e^y}{e^y - t}.$$

i) Determine the domain D of local existence and uniqueness.
ii) Determine all the stationary solutions (if any) and plot the regions of D where solutions are increasing/decreasing.

Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $y(2) = 0$.

iii) Show that y is monotone and deduce that $\alpha > 1$.
iv) Determine the concavity of y. Is $\beta = +\infty$? Justify carerully.
v) Plot a qualitative graph of y.

Exercise 9. Let

$$v(x, y) := e^{-y} (y \cos x + x \sin x), \quad (x, y) \in \mathbb{R}^2.$$

i) Determine all possible $u = u(x, y)$ in such a way that $f(x + iy) := u(x, y) + iv(x, y)$ be $\mathbb{C}$−differentiable on $\mathbb{R}^2$.
ii) Express the f found at i) as function of complex number z, that is $f = f(z)$.

Exercise 10. .

i) State Green's formula.
ii) Let $f \in \mathsf{C}^2(\mathbb{R}^2)$. Prove that

$$\oint_{\partial D} f \nabla f = 0.$$

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Prompt #3

Exercise 11. Let
$$f(x,y) := (x^2+y^2)^3 - x^4 + y^4,\ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x,y)$.
ii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find the eventual min/max points of f. What about $f(\mathbb{R}^2)$?

Exercise 12. Consider the equation
$$y' = \frac{e^y}{e^y - t}.$$

i) Determine the domain D where local existence and uniqueness applies.
ii) Determine the constant solutions and regions of D where solutions are increasing/decreasing.
iii) Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solutions of the Cauchy problem $y(2) = 0$. Determine the monotonicity of y. What can you say about α?
iv) Determine the concavity of y. Is $\beta = +\infty$? Justify carefully.
v) Plot a qualitative graph of y.

Exercise 13. Let $D := \{(x,y,z) \in \mathbb{R}^3 \ :\ x^2+2y^2 \leqslant z \leqslant 4 - 3(x^2+2y^2)\}$.

i) Draw the set D. Is D a rotation volume with respect to the $z-$axis"?
ii) Compute the volume of D.

Exercise 14. Let
$$u(x,y) := x^2 + y^2.$$

i) Determine, if any, $v = v(x,y)$ in such a way that $f(x+iy) := u(x,y)+iv(x,y)$ be $\mathbb{C}-$differentiable on $\mathbb{C}$.
ii) For the f you found at i), write $f = f(z)$ as function of $z \in \mathbb{C}$.

Exercise 15. State the Lagrange multipliers theorem. Then, consider a curve $y = f(x)$ defined by a function $f = f(x) : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in \mathscr{C}^1(\mathbb{R})$. Let $P = (a,b)$ a point in the cartesian plane not belonging to the curve $y = f(x)$. Prove that if Q is a point of the curve $y = f(x)$ where the distance to P is minimum, then the segment $P - Q$ is perpendicular to the tangent to f.

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PROMPT #4

Exercise 16. Let
$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1, \ y^2 + z = 1\}.$$

i) Show that $D \neq \varnothing$ is the zero set of a submersion (g_1, g_2).
ii) Is D compact?
iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 17. Consider the vector field
$$\vec{F}(x, y) := \left(\frac{ax + by}{\sqrt{x^2 + y^2}}, \frac{cx + dy}{\sqrt{x^2 + y^2}} \right), \ (x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

i) Find all possible values of $a, b, c, d \in \mathbb{R}$ such that $\vec{F}$ is irrotational.
ii) Find all possible values for a, b, c, d such that $\vec{F}$ is conservative. For such values, determine the potentials of $\vec{F}$.

Exercise 18. Consider the system
$$\begin{cases} x' = y(x^2 - y^2), \\ \\ y' = (x - 1)(x^2 - y^2). \end{cases}$$

i) Find stationary solutions, and dhow that the system has a non constant first integral.
ii) Draw the phase portrait of the system. Are there periodic solutions? Are there non constant and non periodic global solutions?
iii) Find the x solution of the Cauchy problem $x(0) = 2$, $y(0) = 1$.

Exercise 19. Let $v(x, y) := y^3 - 3x^2 y + 4xy - x$, $(x, y) \in \mathbb{R}^2$. Determine all possible $u = u(x, y)$ such that
$$f(x + iy) := u(x, y) + iv(x, y),$$
be holomorphic on $\mathbb{C}$. What is $f(z)$ as a function of z?

Exercise 20. .

i) State precisely the formula of change of variables for multiple integrals.
ii) Let $\vec{a}, \vec{b}, \vec{c}$ three linearly independent vectors of $\mathbb{R}^3$, $\alpha, \beta, \gamma > 0$ three positive constants, and let $E_{\alpha,\beta,\gamma} := \left\{ \vec{x} \in \mathbb{R}^3 \ : \ 0 \leqslant \vec{a} \cdot \vec{x} \leqslant \alpha, \ 0 \leqslant \vec{b} \cdot \vec{x} \leqslant \beta, \ 0 \leqslant \vec{c} \cdot \vec{x} \leqslant \gamma \right\}$. Prove the formula
$$\int_{E_{\alpha,\beta,\gamma}} (\vec{a} \cdot \vec{x})(\vec{b} \cdot \vec{x})(\vec{c} \cdot \vec{x}) \, d\vec{x} = \frac{(\alpha\beta\gamma)^2}{8 \det M}. \quad \left(\text{where } M = \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \right)$$

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PROMPT #5

Exercise 21. Let $\Gamma \subset \mathbb{R}^3$ the set described by equations

$$\Gamma \ : \ \begin{cases} x^2 + y^2 = 1, \\ \\ x^2 + z^2 = xz + 1. \end{cases}$$

i) Show that $\Gamma \neq \emptyset$ is the zero set of a submersion on Γ.
ii) Is Γ compact? Justify your answer.
iii) Determine points of Γ at minimum/maximum distance to $(0, 0, 0)$ (if any).

Exercise 22. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ 1 - (x^2 + y^2) \leqslant z \leqslant \sqrt{1 - (x^2 + y^2)}\}$.

i) Draw $D \cap \{y = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 23. Consider the equation

$$y' = (\log y)(\log(2 - y)).$$

i) Determine the domain D of local existence and uniqueness, constant solutions and regions of D where the solutions are increasing/decreasing.

Let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution with $y(0) = 1/2$.

ii) Discuss monotonicity and concavity of y.
iii) Is $\alpha = -\infty$? Is $\beta = +\infty$? Determine limits of $y(t)$ when $t \longrightarrow \alpha+, \beta-$.
iv) Plot a qualitative graph of y.

Exercise 24. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\sinh \frac{1}{z} = 0.$$

Exercise 25. .

i) What does it mean that a set $C \subset \mathbb{R}^d$ is closed? What is the Cantor characterization of closed sets?
ii) Given a set $S \subset \mathbb{R}^d$, we define the boundary of S the set

$$\partial S := \left\{ \vec{x} \in \mathbb{R}^d \ : \ \forall r > 0, \ B(\vec{x}, r] \cap S \neq \varnothing, \ B(\vec{x}, r] \cap S^c \neq \varnothing \right\}.$$

Is ∂S always closed? Justify your answer providing a proof if yes, a counterexample if no.

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PROMPT #6

Exercise 26. .

i) Provide a precise statement of the differentiability test says? Is this test a sufficient or necessary condition for differentiability? Justify your answer.

ii) Let

$$f(x, y) := x^2 \log(x^2 + y^2),\ (x, y) \neq \vec{0}.$$

Is it possible to extend f at point $(0, 0)$ in such a way that f is continuous at $(0, 0)$? If yes,what about the differentiability at $(0, 0)$ of the extended function?

Exercise 27. Let

$$f(x, y) := 3xy + x^2y + xy^2,\ \ (x, y) \in D := \{(x, y) \in \mathbb{R}^2\ :\ x \geqslant 0,\ 0 \leqslant y \leqslant 1 - x\}.$$

i) Draw D. Is D closed? open? bounded? compact? Justify carefully.

ii) Discuss the problem of determining min/max (if any) of f on D.

Exercise 28. Let $a, b, c, d \in \mathbb{R}$ and

$$\vec{F}(x, y) := \left(\frac{ax + by}{(x^2 + y^2)^2}, \frac{cx + dy}{(x^2 + y^2)^2} \right),\ \ (x, y) \in D := \mathbb{R}^2 \backslash \{(0, 0)\}.$$

i) Determine $a, b, c, d \in \mathbb{R}$ in such a way that $\vec{F}$ be irrotational on D.

ii) Determine a, b, c, d such that $\vec{D}$ be conservative on D. For these values (if any), determine all possible potentials of $\vec{F}$ on D.

iii) Let $\gamma = \gamma(t) \subset D$ be the segment joining $(1, 0)$ to $(0, 2)$. For $(a, b, c, d) = (2, 0, 0, 2)$ compute

$$\int_\gamma \vec{F}.$$

Exercise 29. Let $D := \{(x, y) \in \mathbb{R}^2\ :\ x \geqslant 1,\ x^3 \leqslant y \leqslant 3\}$.

i) Draw D.

ii) By using the change of variables $u = y - x^3$, $v = y + x^3$, compute the integral

$$\int_D x^2(y - x^3)e^{y+x^3}\ dxdy.$$

Exercise 30. Let $f = u + iv$ be holomorphic on $D \subset \mathbb{C}$. Define

$$g(z) := \overline{f(\overline{z})},\ z \in \overline{D} := \{w \in \mathbb{C}\ :\ \overline{w} \in D\}$$

i) Express real and imaginary part of g in terms of real and imaginary parts u and v of f.

ii) Use i) to discuss whether g is holomorphic on $\overline{D}$ or not.

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Prompt #7

Exercise 31. Let $D := \{(x,y,z) \in \mathbb{R}^3 : \quad x^2 = y^2 + z^2, \quad x^2 + y^2 = xy + 1\}$.

i) Show that D is the zero set of a submersion on D itself.
ii) Is D compact? Justify your answer.
iii) Determine, if any, the points of D at the min / max distance to the origin.

Exercise 32. Consider the system

$$\begin{cases} x' = y(x^2 + y^2), \\ \\ y' = x(x^2 + y^2). \end{cases}$$

We accept that local existence and uniqueness holds.

i) Find stationary solutions, and show that the equation admits a first integral.
ii) Plot the phase portrait of the system. Are there periodic solutions? Are there non constant and non periodic global solutions?
iii) Solve the Cauchy problem $x(0) = 1$, $y(0) = 0$.

Exercise 33. Let

$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ 0 \leqslant z \leqslant \frac{1}{\cosh(x^2 + y^2)}\right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce the figure of D. Is D closed? Open? Bounded? Compact? Justify your answer.
ii) Determine the volume of D.
iii) Determine for which values of α the following integral has a finite value:

$$\int_D e^{\alpha(x^2+y^2)} \, dxdydz.$$

Exercise 34. Let

$$u(x,y) := x^3 + axy^2, \quad v(x,y) := bx^2y - y^3, \quad (x,y) \in \mathbb{R}^2.$$

i) Determine $a, b \in \mathbb{R}$ in such a way that $f(x + iy) := u(x,y) + iv(x,y)$ be holomorphic on $\mathbb{C}$.
ii) For values of a, b found at i), express f as a function of the complex variable z.

Exercise 35. Let $\vec{a}_1, \dots, \vec{a}_N \in \mathbb{R}^d$ be N fixed vectors, $\vec{a}_i \neq \vec{a}_j$ for $i \neq j$. Define

$$f(\vec{x}) := \sum_{j=1}^{N} \|\vec{x} - a_j\|^2.$$

Discuss the problem of determining, if any, points of min/max for f on $\mathbb{R}^d$. Justify carefully, state all general facts you use.

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Prompt #8

Exercise 36. Let
$$D := \left\{(x,y,z) \in \mathbb{R}^3 \;:\; x^2+y^2=z^2,\; y^2+(z-2)^2=1\right\}.$$

i) Show that $D \neq \emptyset$ and it is the zero set of a submersion.
ii) Is D compact? Prove or disprove.
iii) Find points of D at min/max distance to $\vec{0}$.

Exercise 37. By using the change of variables $x = u^2$ and $y = v^2$, compute
$$\int_{[0,+\infty[^2} \frac{e^{-(x+y)}}{\sqrt{x^2y+xy^2}}\,dxdy.$$

Exercise 38. Let $f(x+iy) = u(x,y) + iv(x,y)$.

i) Which properties of u, v characterize $\mathbb{C}-$diffentiability of f at $z = x+iy$? Write a precise statement.
ii) Let
$$u(x,y) := ax^2 + bxy + cy^2, \quad v(x,y) := xy, \quad x+iy \in \mathbb{C}.$$
(a, b, c are real constant). Determine all possible values for a, b, c in such a way f be $\mathbb{C}-$differentiable on $\mathbb{C}$.
iii) For the values of a, b, c at ii), determine explicitly $f(z)$ as a function of $z \in \mathbb{C}$.

Exercise 39. Consider the differential equation
$$y' = \frac{1}{y - \log t}.$$

i) Determine the domain D of local existence and uniqueness.
ii) Find stationary solutions (if any) and plot regions of D where solutions are increasing/decreasing.

Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $y(2) = 0$.

iii) Show that y is monotone. What is the $\lim_{t\to\alpha} y'(t)$?
iv) Discuss the concavity of y and say if $\beta = +\infty$ or less justifying carefully your argument.
v) With all the previous informations plot a qualitative graph of y.

Exercise 40. What does it mean that a set $S \subset \mathbb{R}^d$ is open? Let $\vec{F} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a continuous function on $\mathbb{R}^d$. Prove that the following property holds:
$$S \subset \mathbb{R}^m \text{ open}, \implies \vec{F}^{-1}(S) \text{ open}.$$
(recall that $\vec{F}^{-1}(S) = \{\vec{x} \in \mathbb{R}^d \;:\; \vec{F}(\vec{x}) \in S\}$). Hint: suppose that for some S open, $\vec{F}^{-1}(S)$ is not open...

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PROMPT #9

Exercise 41. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = 1, \ x + y + z = 1\}$.

i) Show that D is the zero set of a submersion.
ii) Is D compact?
iii) Determine, if any, min/max points for $f(x, y, z) = x^2 - x + y^2 + yx + yz - y$ on D.

Exercise 42. .

i) What does it mean that a vector field $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ ($d \in \mathbb{N}$, $d \geqslant 2$) is conservative? And what does it mean that it is irrotational?
ii) Consider now the field
$$\vec{F}(x, y, z) := \left(\frac{ax + by}{1 + x^2 + y^2 + z^2}, \frac{cy + dz}{1 + x^2 + y^2 + z^2}, \frac{ex + fz}{1 + x^2 + y^2 + z^2} \right), \ (x, y, z) \in D := \mathbb{R}^3.$$
Determine all the possible values for $a, b, c, d, e, f \in \mathbb{R}$ such that $\vec{F}$ is irrotational on D.
iii) For which values a, b, c, d, e, f is $\vec{F}$ conservative on D? For these values, if any, determine the potentials of $\vec{F}$.

Exercise 43. Let
$$D := \left\{ (x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant 2x - \sqrt{x^2 + y^2} \right\}.$$

i) Is D closed? open? bounded? compact? Justify carefully.
ii) Compute the area of D.

Exercise 44. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined as
$$f(z) := \begin{cases} \frac{\overline{z}^2}{z}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

i) Show that $u = \text{re} f$ and $v =$ f verify the Cauchy-Riemann equations at $(0, 0)$.
ii) Is f $\mathbb{C}-$differentiable at $z = 0$?

Exercise 45. Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, differentiable at $\vec{x}$.

i) What is the directional derivative $\partial_{\vec{v}} f$ at $\vec{x}$ along the direction $\vec{v}$? What is the relation between this derivative and the gradient vector $\nabla f(\vec{x})$?
ii) Determine
$$\max_{\vec{v} \neq \vec{0} \ : \ \|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}).$$

Exercise 46. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\cosh z + 4 = 0.$$

Exercise 47. Let

$$\vec{F}(x,y) := \left(axy + e^{by}, xe^{y} + 2y\right), \ (x,y) \in \mathbb{R}^2.$$

i) Determine a, b for which $\vec{F}$ is irrotational on $\mathbb{R}^2$
ii) Determine a, b for which $\vec{F}$ is conservative on $\mathbb{R}^2$, determining also its potentials.
iii) For the values of a, b such that $\vec{F}$ is irrotational, compute the path integral of $\vec{F}$ along the curve $\gamma(t) := (t^2, t^2 - t)$, $t \in [0,1]$.

Exercise 48. Let

$$D := \{(x,y,z) \in \mathbb{R}^3 \ : \ xy + z^2 = 1/2, \ x^2 + y^2 = z\}.$$

i) Check that $D \neq \emptyset$ is the zero set of a submersion on D.
ii) Is D compact?
iii) Determine (if any) point(s) of D at minimum and maximum distance to the z axis.

Exercise 49. Let $D := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 \leqslant 1, \ x^2 + y^2 + z^2 \leqslant 9, \ z \geqslant 0\}$.

i) Draw the figure of D in the plane xz (that is, $D \cap \{y = 0\}$), and deduce a figure for D.
ii) Compute the volume of D (hint: use appropriate cylindrical coordinates)

Exercise 50. Let $f(x+iy) = u(x,y) + iv(x,y) : \mathbb{C} \longrightarrow \mathbb{C}$ be a $\mathbb{C}-$differentiable function on the complex plane $\mathbb{C}$.

i) What properties must verify u and v? Provide a precise statement (no proof is required).
ii) Assume that u is constant. Show that f is constant.
iii) Assume that $|f| = \sqrt{u^2 + v^2}$ is constant. What can be drawn on f? (hint: $|f|^2 = u^2 + v^2 \equiv k$, then $\partial_x(u^2 + v^2) = \ldots$)

Exercise 51. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\sinh z + 2i = 0.$$

Exercise 52. Let

$$\vec{F}(x, y) := \left(ye^x + x, axy + e^{bx}\right), \ (x, y) \in \mathbb{R}^2.$$

i) Determine a, b for which $\vec{F}$ is irrotational on $\mathbb{R}^2$
ii) Determine a, b for which $\vec{F}$ is conservative on $\mathbb{R}^2$, determining also its potentials.
iii) For the values of a, b such that $\vec{F}$ is irrotational, compute the path integral of $\vec{F}$ along the curve $\gamma(t) := (t^2 + 1, t^2)$, $t \in [0, 1]$.

Exercise 53. Let

$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ xz + 2y^2 = 1, \ x^2 + z^2 = 2y\}.$$

i) Check that $D \neq \varnothing$ is the zero set of a submersion on D.
ii) Is D compact?
iii) Determine (if any) point(s) of D at minimum and maximum distance to the y axis.

Exercise 54. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + z^2 - y^2 \leqslant 1, \ x^2 + y^2 + z^2 \leqslant 4, \ z \geqslant 0\}$.

i) Draw the figure of D in the plane xy (that is, $D \cap \{z = 0\}$), and deduce a figure for D.
ii) Compute the volume of D (hint: use appropriate cylindrical coordinates)

Exercise 55. Let $f(x + iy) = u(x, y) + iv(x, y) : \mathbb{C} \longrightarrow \mathbb{C}$ be a $\mathbb{C}-$differentiable function on the complex plane $\mathbb{C}$.

i) What properties must verify u and v? Provide a precise statement (no proof is required).
ii) Assume that v is constant. Show that f is constant.
iii) Assume that $|f| = \sqrt{u^2 + v^2}$ is constant. What can be drawn on f? (hint: $|f|^2 = u^2 + v^2 \equiv k$, then $\partial_x(u^2 + v^2) = \ldots$)

Exercise 56. Let

$$u(x,y) := x^5 - 10x^3y^2 + 5xy^4.$$

i) Determine all possible $v = v(x,y)$ in such a way that $f(x+iy) := u(x,y) + iv(x,y)$ be $\mathbb{C}-$differentiable on $\mathbb{C}$.
ii) For the f found at i), determine the analytical expression of $f(z)$ as function of $z \in \mathbb{C}$.

Exercise 57. Let

$$f(x,y) := x^4 + y^4 - 2(x-y)^2, \ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x,y)$.
ii) Determine and classify the stationary points of f.
iii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find (if any) min/max points of f.

Exercise 58. Let

$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ z \geqslant 1 - (x^2+y^2), \ x^2+y^2+z^2 \leqslant \frac{7}{4}\right\}$$

i) Draw $D \cap \{x = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 59. We consider the differential equation

$$y' = \log(1 - ty).$$

i) Determine the domain $D \subset \mathbb{R}^2$ where local existence and uniqueness applies.
ii) Determine any constant solutions and the regions of D where solutions are increasing/decreasing.

Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $y(0) = c$ with $c > 0$.

iii) Determine the monotonicity of y.
iv) Is $\beta = +\infty$? Determine $\lim_{t\to\beta} y(t)$. Is $\alpha = -\infty$? Determine $\lim_{t\to\alpha} y(t)$.
v) Plot a qualitative graph of $y(t)$.

Exercise 60. Let $\vec{a} \in \mathbb{R}^d \backslash \{\vec{0}\}$. Determine

$$\max_{\vec{x}\in\mathbb{R}^d \ : \ \|\vec{x}\|=1} (\vec{a} \cdot \vec{x})^2.$$

(here: $\vec{a} \cdot \vec{x} = \sum_{j=1}^d a_j x_j$, $\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_d^2}$)

Solutions

Exercise 1. i) See LN for the definition.

ii) To check continuity, we have to check if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$$

For $(x,y) = (\rho\cos\theta, \rho\sin\theta) \neq 0$, in polar coordinates we have

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^3\left(\cos^3\theta - \cos^2\theta\sin\theta\right)}{\rho} = \rho^2\left(\cos^3\theta - \cos^2\theta\sin\theta\right),$$

from which

$$|f(x,y) - 0| = |f(x,y)| = \rho^2\left|\cos^3\theta - \cos^2\theta\sin\theta\right| \leqslant 2\rho^2 \longrightarrow 0,\ \rho\to 0, \iff (x,y)\to(0,0),$$

from which $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so f is continuous at $(0,0)$.

For differentiability, we start computing partial derivatives. We proceed by the definition:

$$\partial_x f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{\frac{t^3}{\sqrt{t^2}}}{t} = \lim_{t\to 0} t\,\mathrm{sgn}\, t = 0.$$

Similarly

$$\partial_y f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0.$$

Therefore, f is differentiable at $(0,0)$ iff

$$\lim_{\vec{h}\to\vec{0}}\frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - (0,0)\cdot\vec{h}}{\|\vec{h}\|} = 0,$$

that is

$$\lim_{\vec{h}\to\vec{0}}\frac{f(\vec{h})}{\|\vec{h}\|} = 0.$$

Setting $\vec{h} = (u,v)$, we have

$$\lim_{(u,v)\to(0,0)}\frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)}\frac{u^3 - vu^2}{u^2+v^2}.$$

Setting $(u,v) = (\rho\cos\theta, \rho\sin\theta)$ we have

$$\left|\frac{u^3 - vu^2}{u^2+v^2}\right| = \frac{\rho^3}{\rho^2}\left|\cos^3\theta - \cos^2\theta\sin\theta\right| \leqslant 2\rho \longrightarrow 0,\ \rho\to 0, \iff (u,v)\to(0,0).$$

This shows that f is differentiable at $(0,0)$ and $\nabla f(0,0) = (0,0)$. □

Exercise 2. i) For instance $(0,0,z)\in D$ iff $z^2 = 1$, thus $(0,0,\pm 1)\in D$ and $D \neq \emptyset$. D is also the zero set of $g(x,y,z) := x^2 + y^2 + z^2 - xy - 1$. This is a submersion on D iff

$$\nabla g \neq \vec{0},\ \text{on } D.$$

We have

$$\nabla g = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} 2x - y = 0, \\ 2y - x = 0, \\ 2z = 0, \end{cases} \quad \Longleftrightarrow \quad (x, y, z) = (0, 0, 0) \notin D,$$

from which it follows that g is a submersion on D.

ii) Certainly, $D = \{g = 0\}$ is closed ($g \in \mathscr{C}$). Is it also bounded? We may see this by using spherical coordinates:

$$\begin{cases} x = \rho \cos\theta \sin\varphi, \\ y = \rho \sin\theta \sin\varphi, \\ z = \rho \cos\varphi. \end{cases} \quad \rho^2 = x^2 + y^2 + z^2 = \|(x, y, z)\|^2.$$

Then, if $(x, y, z) \in D$ we have

$$\rho^2 = 1 + \rho^2 \cos\theta \sin\theta (\sin\varphi)^2 = 1 + \frac{1}{2}\rho^2 \sin(2\theta)(\sin\varphi)^2 \leqslant 1 + \frac{\rho^2}{2},$$

from which

$$\frac{\rho^2}{2} \leqslant 1, \quad \Longleftrightarrow \quad \rho^2 = \|(x, y, z)\|^2 \leqslant 2.$$

Thus, D is bounded, hence compact.

iii) We have to minimize/maximize $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ or, which is equivalent (same min/max points), $f(x, y, z) = x^2 + y^2 + z^2$. According to i), we are in condition to apply Lagrange multipliers theorem. According to this result, at min/max points $(x, y, z) \in D$ we have

$$\nabla f = \lambda \nabla g, \quad \Longleftrightarrow \quad \mathrm{rk} \begin{bmatrix} \nabla f(x, y, z) \\ \\ \nabla g(x, y, z) \end{bmatrix} = \mathrm{rk} \begin{bmatrix} 2x & 2y & 2z \\ \\ 2x - y & 2y - x & 2z \end{bmatrix} < 2.$$

This happens iff all 2×2 subdeterminats equal 0:

$$\begin{cases} 2x(2y - x) - 2y(2x - y) = 0, \\ \\ 2x2z - 2z(2x - y) = 0, \\ \\ 2y2z - 2z(2y - x) = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y^2 - x^2 = 0, \\ yz = 0, \\ xz = 0. \end{cases}$$

The first leads to $y = \pm x$, the second $y = 0$ (then $x = 0$) or $z = 0$. That is we have points $(0, 0, z)$ and $(x, \pm x, 0)$. Now

- $(0, 0, z) \in D$ iff $z^2 = 1$, that is $(0, 0, \pm 1)$.
- $(x, \pm x, 0) \in D$ iff $2x^2 = 1 \pm x^2$. If $+$, $2x^2 = 1 + x^2$, we get $x = \pm 1$, that is points $(1, 1, 0)$ and $(-1, -1, 0)$. It $-$, $x^2 = \frac{1}{3}$, thus points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$.

Prom these we see that $(1, 1, 0)$ and $(-1, -1, 0)$ are points at max distance to $\vec{0}$ while $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$ are points of D at min distance to $\vec{0}$. □

Exercise 3. i) Figures are straightforward. D is not invariant by any rotation because one part of the inequality ($z \geqslant x^2 + y^2$) is invariant by rotations around $z-$axis while the second part ($z \leqslant 1 - y^2$) is not.

ii) We have

$$
\begin{aligned}
\lambda_3(D) &= \int_D 1\, dxdydz \overset{RF}{=} \int_{x^2+y^2\leqslant 1-y^2} \int_{x^2+y^2}^{1-y^2} 1\, dz\, dxdy = \int_{x^2+2y^2\leqslant 1} \left(1-y^2-(x^2+y^2)\right)\, dxdy \\
&= \int_{x^2+2y^2\leqslant 1} \left(1-(x^2+2y^2)\right)\, dxdy \\
&\overset{CV\ x=\rho\cos\theta,\ \sqrt{2}y=\rho\sin\theta}{=} \int_{0\leqslant\rho\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi} \left(1-\rho^2\right)\frac{\rho}{\sqrt{2}}\, d\rho\, d\theta \\
&\overset{RF}{=} \frac{2\pi}{\sqrt{2}}\int_0^1 \rho-\rho^3\, d\rho = \sqrt{2}\pi\left(\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} - \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\right) = \frac{\sqrt{2}\pi}{4}. \quad \square
\end{aligned}
$$

Exercise 4. i) $\vec{F}$ is irrotational on D iff

$$\partial_y \frac{ax^2+by^2}{(x^2+y^2)^2} \equiv \partial_x \frac{xy}{(x^2+y^2)^2} \text{ on } D.$$

By computing derivatives, the previous is equivalent to

$$\frac{2by(x^2+y^2)-(ax^2+by^2)4y}{(x^2+y^2)^3} = \frac{y(x^2+y^2)-4x^2y}{(x^2+y^2)^3}$$

that is, iff

$$(2b-4a)yx^2-2by^3 = -3x^2y+y^3, \quad \Longleftrightarrow \quad 2b=-1,\ -1-4a=-3, \quad \Longleftrightarrow \quad b=-\frac{1}{2},\ a=\frac{1}{2}.$$

ii) To be conservative, $\vec{F}$ must be irrotational, hence, necessarily, $a=\frac{1}{2}=-b$. Thus,

$$\vec{F} = \left(\frac{1}{2}\frac{x^2-y^2}{(x^2+y^2)^2}, \frac{xy}{(x^2+y^2)^2}\right) = \nabla f, \quad \Longleftrightarrow \quad \begin{cases} \partial_x f = \frac{1}{2}\frac{x^2-y^2}{(x^2+y^2)^2}, \\ \partial_y f = \frac{xy}{(x^2+y^2)^2}. \end{cases}$$

Looking at the second equation,

$$f(x,y) = \int \frac{xy}{(x^2+y^2)^2}\, dy + c(x) = \frac{x}{2}\int 2y(x^2+y^2)^{-2}\, dy + c(x) = \frac{x}{2}\frac{(x^2+y^2)^{-1}}{-1} + c(x) = -\frac{1}{2(x^2+y^2)} + c(x).$$

Now, by imposing also the first equation we get

$$c'(x)=0, \quad \Longleftrightarrow \quad c(x)\equiv \text{constant}.$$

Thus, all the potentials of $\vec{F}$ are

$$f(x,y) = -\frac{1}{2(x^2+y^2)} + c. \quad \square$$

Exercise 5. About the CR equations see the course notes. Assume that $f = u + iv$ is $\mathbb{C}$ differentiable on $\mathbb{C}$. Then, u, v are $\mathbb{R}$ differentiable and the CR eqns hold,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

If also $\overline{f} = u - iv = u + i(-v)$ is $\mathbb{C}$ differentiable, $u, -v$ fulfill the CR eqns,

$$\begin{cases} \partial_x u = \partial_y(-v) = -\partial_y v, \\ \\ \partial_y u = -\partial_x(-v) = +\partial_x v. \end{cases}$$

But then, combining the two CR eqns, we get

$$\partial_x u = -\partial_y v = -\partial_x u, \implies 2\partial_x u \equiv 0,$$

and, similarly, $\partial_y u \equiv 0$. From this $\nabla u \equiv 0$ hence u is constant. Similar conclusion holds for v. We conclude that both u and v must be constant, hence also f must be constant. □

Exercise 6. The domain D is defined by large inequalities involving continuous functions: by a known fact, D is closed. Moreover, by the constraints, D is manifestly bounded. Therefore D is compact, and since f is continuous on $\mathbb{R}^2$, it is $f \in \mathscr{C}(D)$. So, the Weierstrass theorem applies and this ensures existence on global min/max points for f on D.

To determine these points we argue as follows. If (x, y) is any min/max point then

- either $(x, y) \in \operatorname{Int} D$, so, being f differentiable (polynomial), Fermat theorem applies and $\nabla f(x, y) = 0$. Now,

$$\nabla f(x, y) = (2x - y, -2x + 2) = (0, 0), \iff (x, y) = (1, 2).$$

  Since $(1, 2) \notin \operatorname{Int} D$, we conclude that there aren't stationary points for f in $\operatorname{Int} D$. In particular, any min/max point cannot lie into $\operatorname{Int} D$.
- or $(x, y) \in \partial D$. Now, since D is a rectangle,

$$\begin{aligned} \partial D &= \{(x, 0) \; : \; 0 \leqslant x \leqslant 3\} \cup \{(x, 2) \; : \; 0 \leqslant x \leqslant 3\} \cup \{(0, y) \; : \; 0 \leqslant y \leqslant 2\} \cup \{(3, y) \; : \; 0 \leqslant y \leqslant 2\} \\ &= E_1 \cup E_2 \cup E_3 \cup E_4. \end{aligned}$$

  We have

$$f(x, 0) = x^2,$$

  so, on E_1 the minimum point is for $x = 0$ (point $(0, 0)$) while the maximum point is for $x = 3$ (point $(3, 0)$). Then,

$$f(x, 2) = x^2 - 4x + 4 = (x - 2)^2,$$

  so, on E_2 the min pt is for $x = 2$ (point $(2, 2)$), while max points are $x = 0$ (point $(0, 2)$) and $x = 3$ (point $(3, 2)$).

  On E_3 we have

$$f(0, y) = 2y,$$

so the min point is at $y = 0$ (point $(0,0)$) and the max point is at $y = 2$ (point $(0,2)$). Finally, on E_4,

$$f(3, y) = 9 - 6y + 2y = 9 - 4y,$$

so min point is at $y = 2$ (point $(3,2)$) and max point is at $y = 0$ (point $(3,0)$).

From the previous analysis we conclude that:

- candidates to be min points are $(0,0)$, $(2,2)$, $(3,2)$, and since
$$f(0,0) = 0, \quad f(2,2) = 0, \ f(3,2) = 1,$$
we conclude that $(0,0)$ and $(2,2)$ are both min points for f on D.
- candidates to be max points are $(3,0)$, $(3,2)$, $(0,2)$, and since
$$f(3,0) = 9, \quad f(3,2) = 1, \ f(0,2) = 4,$$
we conclude that $(3,0)$ is the unique max point for f on D. □

Exercise 7. i) $D \cap \{x = 0\} = \{(0, y, z) \ : \ \sqrt{|y|} \leqslant z \leqslant 2 - y^2\}$. Thus, in the plane yz, $D \cap \{x = 0\}$ is the plane region between $z = \sqrt{|y|}$ and the parabola $z = 2 - y^2$ (see figure). Since $(x, y, z) \in D$ depends on (x, y) through $x^2 + y^2$, D is invariant by rotations around the z−axis.

ii) We have

$$\begin{aligned}
\lambda_3(D) &= \int_D 1 \, dxdydz = \int_{\sqrt[4]{x^2+y^2} \leqslant z \leqslant 2-(x^2+y^2)} 1 \, dxdydz \overset{RF}{=} \int_{\sqrt[4]{x^2+y^2} \leqslant 2-(x^2+y^2)} \int_{\sqrt[4]{x^2+y^2}}^{2-(x^2+y^2)} 1 \, dz \, dxdy \\
&= \int_{\sqrt[4]{x^2+y^2} \leqslant 2-(x^2+y^2)} \left(2 - (x^2 + y^2) - \sqrt[4]{x^2 + y^2}\right) \, dxdy \\
&\overset{CV}{=} \int_{\sqrt{\rho} \leqslant 2-\rho^2, \ \theta \in [0,2\pi]} \left(\sqrt{\rho} - \left(2 - \rho^2\right)\right) \rho \, d\rho d\theta.
\end{aligned}$$

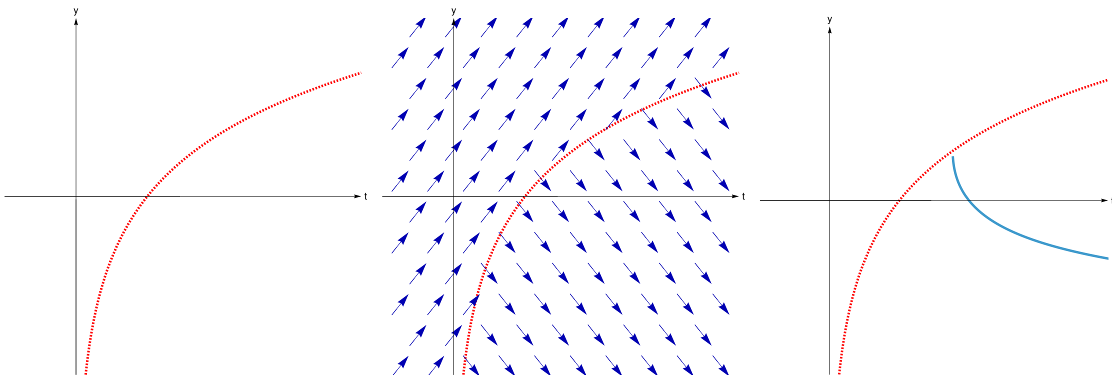
Now, $\sqrt{\rho} \leqslant 2 - \rho^2$ might be hard to solve. However, here $\rho \geqslant 0$; $\sqrt{\rho}$ is increasing while $2 - \rho^2$ decreases. Since at $\rho = 1$ they are equal, we conclude that $\sqrt{\rho} \leqslant 2 - \rho^2$ iff $0 \leqslant \rho \leqslant 1$. We can continue previous chain by the RF:

$$\begin{aligned}
&\overset{RF}{=} \int_0^1 \int_0^{2\pi} \left(2\rho - \rho^3 - \rho^{3/2}\right) \, d\theta \, d\rho = 2\pi \left(-[\rho^2]_{\rho=0}^{\rho=1} - \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1} - \left[\frac{\rho^{5/2}}{5/2}\right]_{\rho=0}^{\rho=1} \right) \\
&= 2\pi \left(1 - \frac{1}{4} - \frac{2}{5}\right) = \frac{7\pi}{10}. \quad \square
\end{aligned}$$

Exercise 8. i) Let $f(t, y) := \frac{e^y}{e^y - t}$. Then, f is defined on

$$D = \{(t, y) \in \mathbb{R}^2 \ : \ e^y - t \neq 0\}.$$

Now, $e^y - t = 0$ iff $e^y = t$, that is $y = \log t$. Therefore $D = \{(t, y) \ : \ y \neq \log t\}$. Clearly, $f, \partial_y f \in \mathscr{C}(D)$, so local existence and uniqueness applies on D.



ii) $y \equiv C$ is a constant solution iff $0 = \frac{e^C}{e^C - t}$ iff $e^C = 0$, which is impossible. Conclusion: no stationary solutions. Furthermore,

$$y \nearrow, \quad \Longleftrightarrow \quad 0 \leqslant y' = \frac{e^y}{e^y - t}, \quad \overset{e^y > 0}{\Longleftrightarrow} \quad e^y - t > 0, \quad \Longleftrightarrow \quad e^y > t.$$

This is always true if $t \leqslant 0$, while, for $t > 0$, it is true iff $e^y > t$, that is $y > \log t$.

iii) Claim: $y \searrow$. This follows once we prove that $y(t) < \log t$ for all t. This is indeed true at $t = 2$: $y(2) = 0 < \log 2$. If the conclusion were false, there would be a $\widehat{t}$ such that $y(\widehat{t}) \geqslant \log \widehat{t}$. But,

- if $y(\widehat{t}) = \log \widehat{t}$, $(\widehat{t}, y(\widehat{t})) \notin D$, which is impossible for a solution.
- if $y(\widehat{t}) > \log \widehat{t}$ then, by continuity, there would be another $\widehat{\widehat{t}}$ such that $y(\widehat{\widehat{t}}) = \log \widehat{\widehat{t}}$, so we fall again into a contradiction.

In any case, we get a contradiction, so $y(t) < \log t$ must be always true and the conclusion follows.

$\alpha > 1$: indeed, if $\alpha \leqslant 1$ we would have $y(\alpha) > y(2) = 0 = \log 1 \leqslant \log \alpha$, contradicting $y(t) < \log t$ always.

iv) For the concavity we notice that

$$y'' = \left(\frac{e^y}{e^y - t} \right)' = \frac{e^y y' (e^y - t) - e^y (e^y y' - 1)}{(e^y - t)^2} = \frac{e^y (-t y' + 1)}{(e^y - t)^2}.$$

Therefore,

$$y \uparrow, \quad \Longleftrightarrow \quad y'' \geqslant 0, \quad \Longleftrightarrow \quad 1 - t y' \geqslant 0.$$

Since $t \in]\alpha, \beta[$, $t > \alpha > 1 > 0$ and $y' \leqslant 0$, we conclude that $1 - t y' \geqslant 1 > 0$ for all t, thus $y'' > 0$ for all t and y is convex.

$\beta = +\infty$? We have two possibilities: $\beta < +\infty$, $\beta = +\infty$. If $\beta < +\infty$ then, defined $y(\beta) = \lim_{t \to \beta} y(t) \geqslant -\infty$ (the limit exists being y decreasing), we would have

- either $y(\beta) > -\infty$: in this case, the solution would remain into the compact box $[2, \beta] \times [y(\beta), 0]$ for all future times, violating the exit from compact sets, impossible;
- or $y(\beta) = -\infty$: also this is impossible because, by convexity, y would be above each of its tangents, in particular to that one for $t = 2$:

$$y(t) \geqslant y(2) + y'(2)(t - 2) \quad \Longrightarrow \quad y(\beta) \geqslant y(2) + y'(2)(\beta - 2) > -\infty,$$

  which is impossible being $y(\beta) = -\infty$.

In any case, we got a contradiction. This means $\beta = +\infty$. □

Exercise 9. i) $f = u + iv$ is $\mathbb{C}-$differentiable on $\mathbb{C}$ iff u, v are $\mathbb{R}-$differentiable on $\mathbb{R}^2$ and u, v fulfill the CR conditions. Clearly v is differentiable. Thus we have to look at $u = u(x,y)$ $\mathbb{R}-$differentiable such that

$$\begin{cases} \partial_x u &= \partial_y v = -e^{-y}(y\cos x + x\sin x) + e^{-y}\cos x, \\ \partial_y u &= -\partial_x v = -e^{-y}\left(-y\sin x + \sin x + x\cos x\right). \end{cases}$$

From the first equation,

$$u(x,y) = \int \partial_x u(x,y)\, dx + c(y) = -e^{-y}\left(y\sin x - x\cos x\right) + c(y).$$

We have

$$\partial_y u = e^{-y}\left(y\sin x - x\cos x\right) - e^{-y}\sin x + c'(y) = e^{-y}\left(y\sin x - x\cos x + \sin x\right) + c'(y)$$

thus $\partial_y u = -\partial_x v$ iff $c'(y) = 0$, that is $c(y)$ is constant. We conclude that

$$u(x,y) = -e^{-y}\left(y\sin x - x\cos x\right) + c + e^{-y}\left(y\cos x + x\sin x\right).$$

ii) We have

$$\begin{aligned} f &= u + iv = -e^{-y}\left(y\sin x - x\cos x\right) + ie^{-y}\left(y\cos x + x\sin x\right) \\ &= e^{-y}\left(y\left(-\sin x + i\cos x\right) + x\left(\cos x + i\sin x\right)\right) \\ &= e^{-y}\left(iye^{ix} + xe^{ix}\right) \\ &= e^{ix-y}(iy + x) = e^{i(x+iy)}(x+iy) = e^{iz}z. \quad \square \end{aligned}$$

Exercise 10. Let $\vec{F} := f\nabla f = (f\partial_x f, f\partial_y f) =: (F_1, F_2)$. According to Green formula,

$$\oint_{\partial D} f\nabla f = \oint_{\partial D} \vec{F} = \int_D \left(\partial_y F_1 - \partial_x F_2\right)\, dxdy.$$

Now, since

$$\partial_y F_1 = \partial_y(f\partial_x f) = \partial_y f\partial_x f + f\partial_{yx} f, \ \ \partial_x F_2 = \partial_x(f\partial_y f) = \partial_x f\partial_y f + f\partial_{xy} f$$

we easily deduce that $\partial_y F_1 - \partial_x F_2 \equiv 0$ being $f \in \mathscr{C}^2(\mathbb{R}^2)$. □

Exercise 11. i) Clearly $f(x,0) = x^6 - x^4 \longrightarrow +\infty$ for $|x| \longrightarrow +\infty$. So, if a limit exists it must be $= +\infty$. We check this changing coordinates and using polar coords:

$$f(x,y) = \rho^6 - (\rho\cos\theta)^4 + (\rho\sin\theta)^4 \geqslant \rho^6 - 2\rho^4 \longrightarrow +\infty, \text{ if } \rho = \|(x,y)\| \longrightarrow +\infty.$$

ii) By i) and a consequence of Weierstrass theorem, f has global minimum on $\mathbb{R}^2$ but not any global maximum. Since every point of $\mathbb{R}^2$ lies in its interior, according to Fermat theorem (clearly

$\partial_x f = 6x(x^2+y^2)^2 - 4x^3$ and $\partial_y f = 6y(x^2+y^2)2 + 4y^3$ are both continuous on $\mathbb{R}^2$, hence f is differentiable on $\mathbb{R}^2$ according to the differentiability test), at min we have $\nabla f = \vec{0}$. Now,

$$\nabla f = \vec{0}, \iff \begin{cases} 6x(x^2+y^2)^2 - 4x^3 = 0, \\ 6y(x^2+y^2)^2 + 4y^3 = 0 \end{cases} \iff \begin{cases} x\left(6(x^2+y^2)^2 - 4x^2\right) = 0, \\ y\left(6(x^2+y^2)^2 + 4y^2\right) = 0, \end{cases}$$

Now, looking at second equation, we see that either $y = 0$ or $6(x^2+y^2)^2 + 4y^2 = 0$. In the second case we obtain trivially $x = 0$ and $y = 0$, thus the point $(0,0)$. Plugging $y = 0$ into the first equation we get

$$x\left(6x^4 - 4x^2\right) = 0, \iff x^3(3x^2-2) = 0, \iff x = 0, \vee x = \pm\sqrt{\frac{2}{3}}.$$

Thus we have again $(0,0)$ and two more points $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$. Since $f(0,0) = 0$ while

$$f\left(\pm\sqrt{\frac{2}{3}}, 0\right) = \frac{8}{27} - \frac{4}{9} = -\frac{28}{27} < f(0,0) = 0,$$

we conclude that $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$ are global minimums. Finally, since $\mathbb{R}^2$ is connected,

$$f(\mathbb{R}^2) = \left[-\frac{28}{27}, +\infty\right[. \quad \square$$

Exercise 12.

Exercise 13. ii) We have,

$$\begin{aligned} \lambda_3(D) &= \int_{x^2+2y^2 \leqslant z \leqslant 4-3(x^2+2y^2)} 1 \; dxdydz \\ &\overset{RF}{=} \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} \int_{x^2+2y^2}^{4-3(x^2+2y^2)} 1 \; dz \; dxdy \\ &= \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} 4\left(1-(x^2+2y^2)\right) \; dxdy. \end{aligned}$$

Noticed that $x^2 + 2y^2 \leqslant 4 - 3(x^2+2y^2)$ iff $x^2 + 2y^2 \leqslant 1$, we have

$$\lambda_3(D) = \int_{x^2+2y^2 \leqslant 1} 4\left(1-(x^2+2y^2)\right) \; dxdy.$$

Changing variables to adapted polar coordinates

$$x = \rho\cos\theta, \quad \sqrt{2}y = \rho\sin\theta,$$

we have

$$\lambda_3(D) = \int_{0\leqslant\rho\leqslant 1, \; 0\leqslant\theta\leqslant 2\pi} 4\left(1-\rho^2\right)\frac{\rho}{\sqrt{2}} \; d\rho d\theta \overset{RF}{=} \frac{8\pi}{\sqrt{2}}\int_0^1 (\rho - \rho^3) \; d\rho = \frac{8\pi}{\sqrt{2}}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{4\pi}{\sqrt{2}}. \quad \square$$

Exercise 14. i) Let $u = x^2 + y^2$. Clearly, u is $\mathbb{R}$−differentiable on $\mathbb{R}^2$. Therefore, $f = u + iv$ is $\mathbb{C}$−differentiable iff v is $\mathbb{R}$−differentiable and CR equations hold,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases} \iff \begin{cases} \partial_x v = -\partial_y u = -2y, \\ \\ \partial_y v = \partial_x u = 2x. \end{cases}$$

From the first equation $v(x, y) = -\int 2y \, dx + c(y) = -2xy + c(y)$. Plugging this into the second equation we have $\partial_y v = -2x + c'(y) = 2x$, that is $c'(y) = 4x$, which is impossible since c does not depend on y. We conclude that such v does not exist.

ii) Since there is no v such that $f = u + iv$ is $\mathbb{C}$−differentiable, there is no f to be found. □

Exercise 15. See notes for the statement. We may formally set the optimization problem in the following way. The set $y = f(x)$ is also $f(x) - y = 0$. Setting $g(x, y) := f(x) - y$ we see that

$$\{(x, y) \ : \ y = f(x)\} = \{(x, y) \ : \ g(x, y) = 0\}.$$

Let's check that g is a submersion on $\{g = 0\}$. Indeed $\nabla g = (\partial_x g, \partial_y g) = (f'(x), -1) \neq 0$, whatever is x. Let now

$$d(x, y) := (x - a)^2 + (y - b)^2,$$

the square of distance from (a, b) to (x, y). At minimum (x, y) on the curve, that is $y = f(x)$, according to Lagrange theorem we have

$$\nabla d = \lambda \nabla g = \lambda(f'(x), -1).$$

Since

$$\nabla d = (2(x - a), 2(y - b)) = 2(x - a, y - b) = 2(Q - P),$$

we have

$$Q - P = \frac{\lambda}{2}(f'(x), -1).$$

Now, since the tangent direction to $y = f(x)$ at point $(x, f(x))$ is $(1, f'(x))$, and clearly $(f'(x), -1) \perp (1, f'(x))$, we have that

$$Q - P \parallel (f'(x), -1) \perp (1, f'(x)) \parallel \text{ tangent to } f,$$

from which the conclusion follows. □

Exercise 16. i) Point $(0, y, 0) \in D$ iff $y^2 = 1$ and $y^2 = 1$, that is $y = \pm 1$, so $(0, \pm 1, 0) \in D$. D is the zero set of $(g_1, g_2) = (x^2 + y^2 - z^2 - 1, y^2 + z - 1)$. According to the Definition,

$$(g_1, g_2) \text{ is a submersion on } D \iff \text{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} = 2 \text{ on } D.$$

Since this is a 2×3 matrix, its rank is < 2 iff all 2×2 sub determinant equal 0, or

$$\begin{cases} 4xy = 0, \\ 2x = 0, \\ 2y(-1 + 2z) = 0, \end{cases} \iff \begin{cases} x = 0, \\ \\ y(1 + 2z) = 0. \end{cases} \iff \begin{array}{l} \begin{cases} x = 0, \\ y = 0, \end{cases} \iff (0, 0, z), \\ \\ \begin{cases} x = 0, \\ \\ z = -\frac{1}{2}, \end{cases} \iff (0, y, -\frac{1}{2}). \end{array}$$

Now,

- $(0,0,z) \in D$ iff $-z^2 = 1$ and $z = 1$, impossible;
- $(0, y, -\frac{1}{2}) \in D$ iff $y^2 = \frac{5}{4}$ and $y^2 = \frac{3}{2}$, impossible.

Conclusion: at no point of D the rank of the matrix $\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix}$ is less than 2, thus (g_1, g_2) is a submersion on D.

ii) D is certainly closed being defined by equations involving continuous functions. Is it also bounded? From the second equation $y^2 = 1 - z$, thus $y = \pm\sqrt{1-z}$ for $z \leqslant 1$. Plugging this into the first equation

$$x^2 = z^2 - (1-z) + 1 = z^2 + z = z(z+1), \implies x = \pm\sqrt{z^2+z} \text{ for } z \leqslant 0 \ \vee \ z \geqslant 1.$$

In particular, for $z \leqslant 0$ points

$$(\pm\sqrt{z^2+z}, \pm\sqrt{1-z}, z) \in D, \ \forall z \leqslant 0.$$

These points are unbounded because

$$\|(\pm\sqrt{z^2+z}, \pm\sqrt{1-z}, z)\|^2 = z^2 + z + (1-z) + z^2 = 2z^2 + 1 \longrightarrow +\infty, \ z \longrightarrow -\infty.$$

We conclude that D is unbounded.

iii) By ii) D is closed and unbounded. We have to min/max $\sqrt{x^2+y^2+z^2}$ or, equivalently, $f := x^2 + y^2 + z^2$, which is continuous on D and such that $\lim_{\infty_3} f = +\infty$. We conclude f has no max point on D while it has min points. By i) and according to the Lagrange multipliers theorem, at min point we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \text{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} < 3.$$

This happens iff the determinant of the previous jacobian matrix equals 0, that is

$$8xy(x+z) = 0, \quad \iff \quad x = 0, \ \vee \ y = 0, \ \vee \ z = -x.$$

This leads to points $(0, y, z)$, $(x, 0, z)$ and $(x, y, -x)$. Now,

- $(0, y, z) \in D$ iff $y^2 - z^2 = 1$ and $y^2 + z = 1$. From these, $z^2 + z = 0$ that is, $z = 0$ or $z = -1$, thus we have points $(0, \pm 1, 0)$ and $(0, \pm\sqrt{2}, -1)$;
- $(x, 0, z) \in D$ iff $x^2 - z^2 = 1$ and $z = 1$, that is $(\pm\sqrt{2}, 0, 1)$.
- $(x, y, -x) \in D$ iff $x^2 + y^2 - x^2 = 1$ and $y^2 - x = 1$, that is $y^2 = 1$ and $x = 0$, from which we have points $(0, \pm 1, 0)$.

Conclusion: min points are among $(0, \pm 1, 0)$, $(0, \pm\sqrt{2}, -1)$, $(\pm\sqrt{2}, 0, 1)$, and clearly thos at min distance to $\vec{0}$ are $(0, \pm 1, 0)$. □

Exercise 17. i) To be irrotational, the field must verify

$$\partial_y \frac{ax+by}{\sqrt{x^2+y^2}} \equiv \partial_x \frac{cx+dy}{\sqrt{x^2+y^2}}, \ \forall (x,y) \in D = \mathbb{R}^2 \backslash \{\vec{0}\}.$$

We have

$$\partial_y \frac{ax+by}{\sqrt{x^2+y^2}} = \frac{b\sqrt{x^2+y^2} - (ax+by)\frac{2y}{2\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{b(x^2+y^2) - y(ax+by)}{(x^2+y^2)^{3/2}} = \frac{bx^2 - axy}{(x^2+y^2)^{3/2}},$$

and, similarly

$$\partial_x \frac{cx+dy}{\sqrt{x^2+y^2}} = \frac{cy^2 - dxy}{(x^2+y^2)^{3/2}}.$$

Thus, the field is irrotational iff

$$\frac{bx^2 - axy}{(x^2+y^2)^{3/2}} \equiv \frac{cy^2 - dxy}{(x^2+y^2)^{3/2}}, \quad \Longleftrightarrow \quad bx^2 - axy = cy^2 - dxy, \ \ \forall (x,y) \in \mathbb{R}^2 \backslash \{\vec{0}\}.$$

Since the identity is trivally verified at $(x,y) = \vec{0}$, we may say that the field is irrotational iff

$$bx^2 - axy \equiv cy^2 - dxy, \quad \Longleftrightarrow \quad b = c = 0, \ a = d.$$

ii) By i), to be conservative $\vec{F}$ must have the form

$$\vec{F} = \left(\frac{ax}{\sqrt{x^2+y^2}}, \frac{ay}{\sqrt{x^2+y^2}} \right)$$

Now, such a $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = \frac{ax}{\sqrt{x^2+y^2}}, \\ \\ \partial_y f = \frac{ay}{\sqrt{x^2+y^2}}. \end{cases}$$

From first equation,

$$f(x,y) = \int \frac{ax}{\sqrt{x^2+y^2}} \, dx + k(y) = \frac{a}{2} \int (x^2+y^2)^{-1/2} (2x) \, dx + k(y) = a(x^2+y^2)^{1/2} + k(y).$$

Plugging this into the second equation we have

$$\partial_y f = a\frac{1}{2}(x^2+y^2)^{-1/2} 2y + k'(y) = \frac{ay}{\sqrt{x^2+y^2}}, \quad \Longleftrightarrow \quad k'(y) = 0.$$

Thus, we deduce that

$$f(x,y) = a\sqrt{x^2+y^2} + k, \ \ k \in \mathbb{R},$$

are all the potentials for $\vec{F}$. □

Exercise 18.

Exercise 19. In order $f = u + iv$ be holomorphic, we need that u, v are both $\mathbb{R}-$differentiable (and certainly v it is), and they verify the CR equations,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

Thus we have to look for an $\mathbb{R}-$differentiable u such that

$$\begin{cases} \partial_x u = 3y^2 - 3x^2 + 4x, \\ \partial_y u = -(-6xy + 4y - 1). \end{cases}$$

From the first equation we get,

$$u(x, y) = \int (3y^2 - 3x^2 + 4x)\, dx + k(y) = 3y^2 x - x^3 + 2x^2 + k(y).$$

Plugging this into the second equation we have

$$6xy + k'(y) = 6xy - 4y + 1, \quad \Longleftrightarrow \quad k'(y) = -4y + 1, \quad \Longleftrightarrow \quad k(y) = -2y^2 + y + k,\ k \in \mathbb{R}.$$

Thus, all the possible u that verify the CR eqns together with v are

$$u(x, y) = 3y^2 x - x^3 + 2x^2 - 2y^2 + y + k.$$

Since such u are clearly $\mathbb{R}-$differentiable, $f = u + iv$ is $\mathbb{C}-$differentiable (holomorphic) on $\mathbb{R}^2$.

To determine the analytical expression for f as a function of complex variable $z = x + iy$, we may notice that

$$\begin{aligned} f &= u + iv = 3y^2 x - x^3 + 2x^2 - 2y^2 + y + k + i\left(y^3 - 3x^2 y + 4xy - x\right) \\ &= -i\underbrace{(x + iy)}_{z} + 2\underbrace{(x^2 - y^2 + i2xy)}_{z^2} - \underbrace{\left(x^3 - iy^3 - 3y^2 x + i3x^2 y\right)}_{z^3} + k \\ &= -z^3 + 2z^2 - iz + k. \quad \square \end{aligned}$$

Exercise 20. i) See LN.
ii) We introduce the change of variable

$$(u, v, w) := (\vec{a} \cdot \vec{x}, \vec{b} \cdot \vec{x}, \vec{c} \cdot \vec{x}) = \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \vec{x} =: M\vec{x} =: \Psi(\vec{x})$$

where M is the 3×3 matrix made by $\vec{a}, \vec{b}, \vec{c}$ as lines. Since $\vec{a}, \vec{b}, \vec{c}$ are linearly independent, M has rank 3, thus it is invertible and the previous is a true change of variable. Moreover, $\Phi'(\vec{x}) = M$ so

$$|\det(\Psi^{-1})'| = \frac{1}{|\det \Psi'|} = \frac{1}{|\det M|}.$$

therefore, according to the change of variable formula

$$\int_{E_{\alpha,\beta,\gamma}} (\vec{a}\cdot\vec{x})(\vec{b}\cdot\vec{x})(\vec{c}\cdot\vec{x})\,d\vec{x} = \int_{0\leqslant u\leqslant\alpha,\ 0\leqslant v\leqslant\beta,\ 0\leqslant w\leqslant\gamma} uvw\frac{1}{|\det M|}\,dudvdw$$

$$\stackrel{RF}{=} \frac{1}{|\det M|}\left(\int_0^{\alpha} u\,du\right)\left(\int_0^{\beta} v\,dv\right)\left(\int_0^{\gamma} w\,dw\right)$$

$$= \frac{1}{|\det M|}\frac{\alpha^2}{2}\frac{\beta^2}{2}\frac{\gamma^2}{2} = \frac{(\alpha\beta\gamma)^2}{8|\det M|}.$$

(clearly, there is a typo in the text of the exercise, $\det M$ must be replaced by $|\det M|$). □

Exercise 21. i) We have $(x,y,0)\in\Gamma$ iff $x^2+y^2=1$ and $x^2=1$, thus $x=\pm1$ and $y^2=0$, hence $(\pm1,0,0)\in\Gamma$. Now, $\Gamma=\{g_1=0,\ g_2=0\}$, where $g_1=x^2+y^2-1$, and $g_2=x^2+z^2-xz-1$. Clearly $g_1,g_2\in\mathscr{C}^1$ and (g_1,g_2) is a submersion on Γ iff

$$\operatorname{rank}\begin{bmatrix}\nabla g_1\\ \\ \nabla g_2\end{bmatrix} = \operatorname{rank}\begin{bmatrix}2x & 2y & 0\\ \\ 2x-z & 0 & 2z-x\end{bmatrix} = 2,\ \forall(x,y,z)\in\Gamma.$$

This is false iff all 2×2 submatrices have determinant =0, that is

$$\begin{cases}2y(2x-z)=0,\\ 2x(2z-x)=0,\\ 2y(2z-x)=0.\end{cases}$$

Working on the first equation, we have the alternatives $y=0$ or $2x-z=0$. In the first case, the system reduces to $x(2z-x)=0$ that is $x=0$ (points $(0,0,z)$) or $x=2z$ (points $(2z,0,z)$). In the second case, the system reduces to

$$\begin{cases}z=2x,\\ 3x^2=0,\\ 3yx=0,\end{cases} \iff (0,y,0).$$

Thus, rank is less than 2 at points $(0,0,z)$, $(2z,0,z)$ and $(0,y,0)$. Now:

- $(0,0,z)\in\Gamma$ iff $0=1$ (first condition), impossible;
- $(2z,0,z)\in\Gamma$ iff $4z^2=1$ and $5z^2=2z^2+1$, that is $z^2=\frac14$ and $z^2=\frac13$ which are impossible together.
- $(0,y,0)\in\Gamma$ iff $y^2=1$ and $0=1$, which is, again, impossible.

Conclusion: none of points where rank is ¡2 belong to Γ, this meaning that rank =2 on Γ, hence (g_1,g_2) is a submersion on Γ.

ii) Clearly Γ is closed because defined by equations involving continuous functions. Boundedness: from first equation we deduce $x^2, y^2\leqslant 1$. From second equation, recalling that $ab\leqslant\frac{a^2+b^2}{2}$ we have

$$x^2+z^2=xz+1\leqslant\frac{x^2+z^2}{2}+1, \implies \frac{x^2+z^2}{2}\leqslant 1,$$

from which, in particular, $z^2 \leqslant 2$. Therefore $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2} \leqslant \sqrt{1+1+2} = \sqrt{4} = 2$, for every $(x,y,z) \in \Gamma$. Conclusion: Γ is bounded, hence compact.

iii) We have to minimize/maximize $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ or, equivalently, $f(x,y,z) = x^2+y^2+z^2$. By ii), Γ is compact and obviously $f \in \mathscr{C}$, thus existence of min and max for f is ensured by Weierstrass' theorem. To determine min/max points we apply Lagrange's thm. According to i), this thm can be applied on Γ. We deduce that, at min/max points $(x,y,z) \in \Gamma$,

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \text{rank} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 2x-z & 0 & 2z-x \end{bmatrix} = 2,$$

or, equivalently, the determinant of this last matrix equals 0. We obtain

$$2z \cdot (-2y(2x-z)) = 0, \iff yz(2x-z) = 0, \iff y = 0, \vee z = 0, \vee z = 2x.$$

Thus possible min/max points are among points $(x,0,z)$, $(x,y,0)$ and $(x,y,2x)$. Now,

- $(x,0,z) \in \Gamma$ iff $x^2 = 1$ and $x^2+z^2 = xz+1$, or, equivalently, $x^2 = 1$ and $z^2 = xz+1$. For $x = 1$ we get $z^2 = z+1$, that is $z = \frac{1\pm\sqrt{5}}{2}$, namely points $(1, 0, \frac{1\pm\sqrt{5}}{2})$. For $x = -1$ we get $z^2 = -z+1$, that is $z = \frac{-1\pm\sqrt{5}}{2}$, namely points $(-1, 0, \frac{-1\pm\sqrt{5}}{2})$.
- $(x,y,0) \in \Gamma$ iff $x^2+y^2 = 1$ and $x^2 = 1$, that is $x = \pm 1$ and $y^2 = 0$, namely points $(\pm 1, 0, 0)$.
- $(x,y,2x) \in \Gamma$ iff $x^2+y^2 = 1$ and $x^2+4x^2 = 2x^2+1$, from which $x^2 = \frac{1}{3}$, $x = \pm\frac{1}{\sqrt{3}}$ and $y^2 = \frac{2}{3}$, $y = \pm\sqrt{\frac{2}{3}}$, thus we get points $\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right)$ (4 points).

We have

- $f(1, 0, \frac{1\pm\sqrt{5}}{2}) = 1 + \left(\frac{1\pm\sqrt{5}}{2}\right)^2 = \frac{10\pm2\sqrt{5}}{4}$, $f(-1, 0, \frac{-1\pm\sqrt{5}}{2}) = 1 + \left(\frac{-1\pm\sqrt{5}}{2}\right)^2 = \frac{10\pm2\sqrt{5}}{4}$;
- $f(\pm 1, 0, 0) = 1$;
- $f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = \frac{1}{3}+\frac{2}{3}+\frac{4}{3} = \frac{7}{3}$ and $f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{1}{3}+\frac{2}{3}+\frac{4}{3} = \frac{7}{3}$.

From this we see that $(1, 0, \frac{1+\sqrt{5}}{2})$ and $(-1, 0, \frac{-1-\sqrt{5}}{2})$ are maximum points while $(\pm 1, 0, 0)$ are min points. □

Exercise 22. ii) D is closed (because defined by large inequalities involving continuous functions) and bounded (the root imposes $x^2+y^2 \leqslant 1$ and, consequently, $0 \leqslant 1-(x^2+y^2) \leqslant z \leqslant \sqrt{1-(x^2+y^2)} \leqslant \sqrt{1}$, that is $0 \leqslant z \leqslant 1$). Thus D is compact, hence 1_D is integrable on D. Furthermore, noticed that, calling $\rho^2 = x^2+y^2$,

$$1-\rho^2 \leqslant \sqrt{1-\rho^2}, \iff \sqrt{1-\rho^2} \leqslant 1,$$

which is always true, thus $1-(x^2+y^2)\leqslant\sqrt{1-(x^2+y^2)}$ always when defined. Then

$$\begin{aligned}\text{Vol}\, D &= \int_D 1\, dxdydz \stackrel{RF}{=} \int_{x^2+y^2\leqslant 1}\int_{1-(x^2+y^2)}^{\sqrt{1-(x^2+y^2)}} 1\, dz\, dxdy\\ &= \int_{x^2+y^2\leqslant 1}\left(\sqrt{1-(x^2+y^2)}-(1-(x^2+y^2))\right)\, dxdy\\ &\stackrel{pol.\ coords}{=} \int_{0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\rho\leqslant 1}\left(\sqrt{1-\rho^2}-1+\rho^2\right)\rho\, d\rho d\theta\\ &\stackrel{RF}{=} 2\pi\int_0^1 \rho(1-\rho^2)^{1/2}-\rho+\rho^3\, d\rho = 2\pi\left[\left[-\tfrac{1}{3}(1-\rho^2)^{3/2}\right]_{\rho=0}^{\rho=1}-\left[\tfrac{\rho^2}{2}\right]_{\rho=0}^{\rho=1}+\left[\tfrac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\right]\\ &= 2\pi\left[+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{4}\right]=\tfrac{\pi}{6}. \quad \square\end{aligned}$$

Exercise 23.

Exercise 24. Let $z\neq 0$. Setting $w=\frac{1}{z}$, we have to solve

$$\sinh w = 0, \iff \frac{e^w-e^{-w}}{2}=0, \iff e^{2w}=1, \iff 2w=\log|1|+i\,(0+k2\pi)=ik2\pi,\ k\in\mathbb{Z}.$$

Thus

$$\frac{1}{z}=w=ik\pi, \iff z=\frac{1}{ik\pi}=\frac{-i}{k\pi}=\frac{i}{k\pi},\ k\in\mathbb{Z}\backslash\{0\}. \quad \square$$

Exercise 25. i) See notes for definitions and characterizations.

ii) If $\partial S=\varnothing$, ∂S it is closed. Let $\partial S\neq\varnothing$. To verify that ∂S is closed, we use the Cantor characterization. Let $(\vec{x}_n)\subset\partial S$ be such that $\vec{x}_n\longrightarrow\vec{x}\in\mathbb{R}^d$. We prove that $\vec{x}\in\partial S$. Fix $r>0$. Since $\vec{x}_n\longrightarrow\vec{x}$, we have that for $n\geqslant N$ $\|\vec{x}_n-\vec{x}\|\leqslant\frac{r}{2}$. Now, since $\vec{x}_n\in\partial S$,

$$B(\vec{x}_n,r/2]\cap S\neq\varnothing,\ \wedge\ B(\vec{x}_n,r/2]\cap S^c\neq\varnothing.$$

Since $\|\vec{x}_n-\vec{x}\|\leqslant\frac{r}{2}$, we have that

$$B(\vec{x}_n,r/2]\subset B(\vec{x},r],$$

therefore

$$B(\vec{x},r]\cap S\supset B(\vec{x}_n,r/2]\cap S\neq\varnothing,$$

and, similarly, $B(\vec{x},r]\cap S^c\neq\varnothing$. We conclude that $\vec{x}\in\partial S$, thus ∂S is closed. $\square$

Exercise 26. i) See LN for statements.

ii) To extend f at $(0,0)$ as a continuous function, we need to compute the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)} x^2\log(x^2+y^2).$$

In polar coordinates

$$f(x,y)=\rho^2\cos^2\theta\log\rho^2,\implies |f(x,y)|\leqslant 2\rho^2|\log\rho|\longrightarrow 0,\ \rho\to 0+.$$

We conclude that

$$\exists \lim_{(x,y)\to(0,0)} f(x,y)=0.$$

Thus, setting

$$f(x,y) := \begin{cases} x^2 \log(x^2+y^2), & (x,y) \neq (0,0), \\ \\ 0, & (x,y) = (0,0), \end{cases}$$

we get a continuous function at $(0,0)$. To check differentiability we notice that, for $(x,y) \neq (0,0)$,

$$\partial_x f(x,y) = 2x\log(x^2+y^2) + x^2 \frac{2x}{x^2+y^2}, \quad \partial_y f(x,y) = x^2 \frac{2y}{x^2+y^2}.$$

Clearly $\partial_x f, \partial_y f \in \mathscr{C}(\mathbb{R}^2 \backslash \{(0,0)\})$ so, according to the differentiability test, f is differentiable on $\mathbb{R}^2 \backslash \{(0,0)\}$. Let's discuss differentiability at $(0,0)$. Previous formulas do not allow to compute $\partial_x f(0,0)$ and $\partial_y f(0,0)$. However, we notice that

$$\partial_x f(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{t^2 \log t^2}{t} = \lim_{t\to 0} t \log t^2 = 0,$$

and,

$$\partial_y f(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

We can now easily check that $\partial_x f$ and $\partial_y f$ are continuous at $(0,0)$ too. Indeed, for example

$$|\partial_x f(x,y)| = \left| 2\rho\cos\theta \log\rho^2 + 2\frac{\rho^3 \cos^3\theta}{\rho^2} \right| \leqslant 2\rho |\log \rho^2| + 2\rho \longrightarrow 0, \ \rho \to 0+.$$

Therefore

$$\lim_{(x,y)\to(0,0)} \partial_x f(x,y) = 0 = \partial_x f(0,0),$$

thus $\partial_x f$ is continuous at $(0,0)$. Similarly, $\partial_y f$ is continuous at $(0,0)$. We conclude that f is differentiable at $(0,0)$. □

Exercise 27. i) D is closed being defined by large inequalities involving continuous functions of (x,y). It is not open since $D \neq \emptyset, \mathbb{R}^2$. It is bounded because $x \geqslant 0$ and from $0 \leqslant y \leqslant 1-x$, in particular $1-x \geqslant 0$, that is $x \leqslant 1$, so $0 \leqslant x \leqslant 1$ and, at same time, $0 \leqslant y \leqslant 1-x \leqslant 1$. Thus $0 \leqslant x, y \leqslant 1$ and this implies that D is bounded. Since D is closed and bounded it is also compact.

ii) Since f is clearly continuous on D and D is compact, f admits both global min/max on D. To determine min/max points, we may argue in the following way. If $(x,y) \in D$ is a min/max point for f then

- either $(x,y) \in \operatorname{Int} D$
- or $(x,y) \in D \backslash \operatorname{Int} D = \partial D$.

In the first case, since

$$\partial_x f = 3y + 2xy + y^2, \quad \partial_y f = 3x + x^2 + 2xy$$

so $\partial_x f, \partial_y f \in \mathscr{C}(D)$, f is then differentiable on D, according to Fermat theorem, at min/max points

$$\nabla f(x,y) = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} 3y + 2xy + y^2 = 0, \\ \\ 3x + x^2 + 2xy = 0. \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y(3+2x+y) = 0, \\ \\ x(3+2y+x) = 0. \end{cases}$$

The first equation leads to the alternative $y = 0$ or $3 + 2x + y = 0$. In the first case, the second equation becomes $x(3 + x) = 0$. whose solutions are $x = 0$ and $x = -3$. This produces points $(0, 0)$ and $(-3, 0)$. In any case these do not belong to Int D. In the second case, $y = -2x - 3$, from the second equation we obtain $x(-3 - 3x) = 0$, that is $x = 0$ or $x = -1$. This yields points $(0, -3), (-1, -1) \notin D$. In conclusion, no stationary point for f is in the interior of D.

Thus, min/max points for f are on $\partial D = A \cup B \cup C$ where $A = \{(0, y) \ : \ 0 \leqslant y \leqslant 1\}$, $B = \{(x, 0) \ : \ 0 \leqslant x \leqslant 1\}$ and, finally, $C = \{(x, 1 - x) \ : \ 0 \leqslant x \leqslant 1\}$. On A we have

$$f(0, y) \equiv 0,$$

thus every point is min/max point for f on A. On B, similarly, we have $f(x, 0) \equiv 0$, thus every point of B is at same time min/max for f on B. Finally, on C

$$f(x, 1-x) = 3x(1-x) + x^2(1-x) + x(1-x)^2 = 3x - 3x^2 + x^2 - x^3 + x - 2x^2 + x^3 = -4x^2 + 4x =: g(x).$$

Let's determine min/max points for g with $x \in [0, 1]$. We have $g'(x) = -8x + 4 \geqslant 0$ iff $x \leqslant \frac{1}{2}$. Thus $x = \frac{1}{2}$ is max point for g and $x = 0, 1$ are min points for g. This means that

- $\left(\frac{1}{2}, \frac{1}{2}\right)$ is max point for f on C
- $(0, 1)$, $(1, 0)$ are min points for f on C.

We can now draw the conclusion:

- for minimum, candidates are points $(x, 0)$, $(0, y)$ with $0 \leqslant x, y \leqslant 1$ where $f = 0$. All these are min points for f on D;
- for maximum, candidates are points $\left(\frac{1}{2}, \frac{1}{2}\right)$ (where $f = 1$) and $(x, 0)$ and $(0, y)$ with $0 \leqslant x, y \leqslant 1$ (where $f = 0$). Thus, the max point is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

□

Exercise 28. i) Let $\vec{F} = (\phi, \psi)$. In order $\vec{F}$ be irrotational on D we need

$$\partial_y \phi \equiv \partial_x \psi, \text{ on } D.$$

We have

$$\partial_y \phi = \frac{b(x^2+y^2)^2 - (ax+by)2(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{b(x^2+y^2) - 4y(ax+by)}{(x^2+y^2)^2} = \frac{bx^2 - 4axy - 3by^2}{(x^2+y^2)^2},$$

$$\partial_x \psi = \frac{c(x^2+y^2)^2 - (cx+dy)2(x^2+y^2)2x}{(x^2+y^2)^4} = \frac{c(x^2+y^2) - 4x(cx+dy)}{(x^2+y^2)^2} = \frac{-3cx^2 - 4dxy + cy^2}{(x^2+y^2)^2}.$$

Hence,

$$\partial_y \phi \equiv \partial_x \psi, \quad \Longleftrightarrow \quad bx^2 - 4axy - 3by^2 \equiv -3cx^2 - 4dxy + cy^2, \quad \Longleftrightarrow \quad \begin{cases} b = -3c, \\ a = d, \\ -3b = c \end{cases}$$

from which $b = c = 0$ and $a = d \in \mathbb{R}$. Thus

$$\vec{F} = \left(\frac{ax}{(x^2+y^2)^2}, \frac{ay}{(x^2+y^2)^2} \right), \ \forall (x, y) \in D.$$

ii) Necessary condition to be conservative is that $\vec{F}$ be irrotational, thus $\vec{F}$ is given as at the end of i). Now, such $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = \frac{ax}{(x^2+y^2)^2}, \\ \partial_y f = \frac{ay}{(x^2+y^2)^2}. \end{cases}$$

From the first equation

$$f(x,y) = \int \frac{ax}{(x^2+y^2)^2}\,dx + k(y) = \frac{a}{2}\int \partial_x - (x^2+y^2)^{-1}\,dx + k(y) = -\frac{a}{2}(x^2+y^2)^{-1} + k(y).$$

Plugging this into the second equation we have

$$\partial_y f = \frac{ay}{(x^2+y^2)^2}, \iff \frac{ay}{(x^2+y^2)^2} + k'(y) = \frac{ay}{(x^2+y^2)^2}, \iff k'(y) = 0, \iff . .k(y) = k \in \mathbb{R}.$$

Thus, $\vec{F}$ is conservative with potentials

$$f(x,y) = -\frac{a}{2}(x^2+y^2)^{-1} + k,\ k \in \mathbb{R}.$$

iii) By previous discussion, when $(a,b,c,d) = (2,0,0,2)$, field $\vec{F}$ is conservative. Thus

$$\int_\gamma \vec{F} = f(0,2) - f(1,0) = -\frac{1}{4} - (-1) = \frac{3}{4}. \quad \square$$

Exercise 29. ii) The change or variable is given in the form $(u,v) = \Phi(x,y) = (y - x^3, y + x^3)$. According to the change of variable formula,

$$\int_D f(x,y)\,dxdy = \int_{\Phi(D)} f(\Phi^{-1}(u,v))|\det(\Phi^{-1})'(u,v)|\,dudv.$$

We need to determine Φ^{-1}. Notice that

$$\begin{cases} u = y - x^3, \\ v = y + x^3, \end{cases} \iff \begin{cases} u + v = 2y, \\ v - u = 2x^3, \end{cases} \iff \begin{cases} y = \frac{u+v}{2}, \\ x^3 = \frac{v-u}{2}, \end{cases} \iff \begin{cases} y = \frac{u+v}{2}, \\ x = \left(\frac{v-u}{2}\right)^{1/3}, \end{cases}$$

Therefore

$$\Phi^{-1}(u,v) = \left(\left(\frac{v-u}{2}\right)^{1/3}, \frac{u+v}{2}\right).$$

Moreover,

$$(x,y) \in D, \iff \begin{cases} x \geqslant 1, \\ x^3 \leqslant y \leqslant 3, \end{cases} \iff \begin{cases} \left(\frac{v-u}{2}\right)^{1/3} \geqslant 1, \\ \frac{v-u}{2} \leqslant \frac{u+v}{2} \leqslant 3 \end{cases} \iff \begin{cases} v - u \geqslant 2, \\ v - u \leqslant v + u \leqslant 6 \end{cases}$$

that is

$$\Phi(D) = \{(u,v)\ :\ 2 \leqslant v - u \leqslant v + u \leqslant 6\}.$$

Now, to be $v - u \leqslant v + u$ it must be $u \geqslant 0$, and from $2 \leqslant v - u \leqslant v + u \leqslant 6$ we get $2 + u \leqslant v \leqslant 6 - u$ provided $2 + u \leqslant 6 - u$, that is $u \leqslant 2$. In conclusion

$$\Phi(D) = \{(u, v) \ : \ 0 \leqslant u \leqslant 2, \ 2 + u \leqslant v \leqslant 6 - u\}.$$

About f, in coordinates (u, v) we have

$$f(\Phi^{-1}(u, v)) = \left(\frac{v - u}{2}\right)^{2/3} ue^v,$$

while

$$\det(\Phi^{-1})' = \det \begin{bmatrix} \frac{1}{3}\left(\frac{v-u}{2}\right)^{-2/3}\left(-\frac{1}{2}\right) & \frac{1}{3}\left(\frac{v-u}{2}\right)^{-2/3}\left(+\frac{1}{2}\right) \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = -\frac{1}{6}\left(\frac{v - u}{2}\right)^{-2/3}.$$

In conclusion

$$\begin{aligned}
\int_D f \ dxdy &= \int_{0\leqslant u\leqslant 2, \ 2+u\leqslant v\leqslant 6-u} \left(\tfrac{v-u}{2}\right)^{2/3} ue^v \tfrac{1}{6}\left(\tfrac{v-u}{2}\right)^{-2/3} \ dudv = \tfrac{1}{6}\int_{0\leqslant u\leqslant 2, \ 2+u\leqslant v\leqslant 6-u} ue^v \ dudv \\
&\overset{RF}{=} \tfrac{1}{6}\int_0^2 \int_{2+u}^{6-u} ue^v \ dv \ du = \tfrac{1}{6}\int_0^2 u \int_{2+u}^{6-u} e^v \ dv \ du = \tfrac{1}{6}\int_0^2 u \, [e^v]_{v=2+u}^{v=6-u} \ du \\
&= \tfrac{1}{6}\int_0^2 u\left(e^{6-u} - e^{2+u}\right) \ du = \tfrac{1}{6}\left(e^6 \int_0^2 ue^{-u} \ du - e^2 \int_0^2 ue^u \ du\right) \\
&= \tfrac{1}{6}\left(e^6\left([-ue^{-u}]_{u=0}^{u=2} + \int_0^2 e^{-u} \ du\right) - e^2\left([ue^u]_{u=0}^{u=2} - \int_0^2 e^u \ du\right)\right) \\
&= \tfrac{1}{6}\left(e^6\left(-2e^{-2} - (e^{-2} - 1)\right) - e^2\left(2e^2 - (e^2 - 1)\right)\right) \\
&= \tfrac{e^2}{6}\left(-2e^2 + e^4 - 1\right). \quad \square
\end{aligned}$$

Exercise 30. i) If $u(x, y) = \mathrm{Re}\, f(x + iy)$ and $v(x, y) = \mathrm{Im}\, f(x + iy)$, then

$$g(x + iy) = \overline{f(x - iy)} = \overline{u(x, -y) + iv(x, -y)} = u(x, -y) - iv(x, -y),$$

from which we see that

$$U(x, y) = \mathrm{Re}\, g(x + iy) = u(x, -y), \ \ V(x, y) = \mathrm{Im}\, g(x + iy) = -v(x, -y).$$

ii) g is holomorphic iff U, V are $\mathbb{R}-$differentiable and they verify CR equations. Clearly, sunce f is holomorphic, u, v are $\mathbb{R}-$differentiable, hence also U, V are $\mathbb{R}-$differentiable. Therefore, we have to verify if U, V fulfil also the CR equations, that is

$$\begin{cases} \partial_x U \equiv \partial_y V, \\ \partial_y U \equiv -\partial_x V. \end{cases}$$

We have,

$$\partial_x U = \partial_x(u(x, -y)) = \partial_x u(x, -y), \ \ \partial_y V = \partial_y\left(-v(x, -y)\right) = -\partial_y v(x, -y)(-1) = \partial_y v(x, -y).$$

And since $\partial_x u \equiv \partial_y v$ we deduce that also $\partial_x U = \partial_y V$. Similarly, $\partial_y U = -\partial_x V$ and the check is completed. □

Exercise 31. i) Let $g_1 := x^2 - y^2 - z^2$ and $g_2 := x^2 + y^2 - xy - 1$. Then, $\vec{g} = (g_1, g_2)$ is a submersion on D iff $\mathrm{rk}\vec{g}'(x, y, z) = 2$ for all $(x, y, z) \in D$. Now,

$$\mathrm{rk}\,\vec{g}'(x,y,z) = \mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & -2y & -2z \\ \\ 2x-y & 2y-x & 0 \end{bmatrix} < 2, \iff \begin{cases} 2x(2y-x)+2y(2x-y)=0, \\ 2z(2x-y)=0, \\ 2z(2y-x)=0. \end{cases}$$

Simplifying, we get the system

$$\begin{cases} x^2+y^2-4xy=0, \\ z(2x-y)=0, \\ z(2y-x)=0. \end{cases}$$

Choosing the second equation, we have the alternative $z = 0$ or $2x - y = 0$. In the first case the system reduces to

$$\begin{cases} z=0, \\ \\ x^2+y^2-4xy=0. \end{cases}$$

These points belong to D iff

$$\begin{cases} x^2=y^2, \\ \\ 4xy=xy+1, \end{cases} \iff \begin{cases} y=\pm x, \\ \\ 3xy=1. \end{cases}$$

However, since $x^2 + y^2 = 4xy$ implies that, for $y = \pm x$, that $x = 0 = y$, it is impossible that $3xy = 1$, thus no solutions are in D.

In the second case, namely, $z \neq 0$ and $2x - y = 0$ or $y = 2x$, condition $\mathrm{rk}\vec{g}'(x, y, z) < 2$ reduces to

$$\begin{cases} y-2x, \\ x(2y-x)=0, \\ 2y-x=0, \end{cases}$$

we easily get $x = y = 0$, that is a point of type $(0, 0, z)$. Now,

$$(0,0,z) \in D, \quad \iff \quad \begin{cases} z=0, \\ \\ 0=1, \end{cases}$$

clearly impossible. Conclusion: rank of $\vec{g}'(x, y, z)$ is never less than 2 on D, that is $\vec{g}$ is a submersion on D.

ii) D is clearly closed being defined by equalities involving continuous functions. To determine whether D is bounded or less, we look first at constraint $x^2 + y^2 = xy + 1$. Writing $x = \rho\cos\theta$ and $y = \rho\sin\theta$, this reads as

$$\rho^2 = \rho^2\cos\theta\sin\theta + 1 = \frac{\rho^2}{2}\sin(2\theta) + 1, \leqslant \frac{\rho^2}{2} + 1, \implies \frac{\rho^2}{2} \leqslant 1, \implies x^2+y^2 \leqslant 2,\ \forall (x,y,z) \in D.$$

But then, by the first equation,

$$z^2 = x^2 - y^2 \leqslant x^2 \leqslant x^2 + y^2 \leqslant 2, \implies x^2 + y^2 + z^2 \leqslant 4, \implies \|(x, y, z)\| \leqslant 2, \ \forall (x, y, z) \in D.$$

This means that D is bounded, hence compact.

iii) We have to miunimize/maximize $f(x, y, z) = \|(x, y, z)\|$ or, which is the same, $f(x, y, z) = \|(x, y, z)\|^2 = x^2 + y^2 + z^2$. The existence of min and max is ensured by the Weierstrass theorem being D compact by ii).

To determine min/max points, we apply Lagrange multipliers theorem. By i), assumptions of this theorem are verified. Thus, at min/max point $(x, y, z) \in D$ we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \iff \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 0.$$

Now,

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & -2y & -2z \\ 2x - y & 2y - x & 0 \end{bmatrix} = -(2y - z)(-8xz) = 8xz(2y - z),$$

iff $x = 0$, or $z = 0$ or $2y - z = 0$. Thus, we have points $(0, y, z)$, $(x, y, 0)$ and $(x, y, 2y)$. Now:

- $(0, y, z) \in D$ iff $0 = y^2 + z^2$ and $y^2 = 1$, and of course this is impossible.
- $(x, y, 0) \in D$ iff $x^2 = y^2$ and $x^2 + y^2 = xy + 1$. From the first we have $y = \pm x$. For $y = x$, second condition becomes $2x^2 = x^2 = 1$, thus $x^2 = 1$, so $x = \pm 1$ and we have points $(\pm 1, \pm 1, 0)$ (same sign). For $y = -x$, second condition becomes $2x^2 = -x^2 + 1$, that is $x^2 = \frac{1}{3}$, that is $x = \pm \frac{1}{\sqrt{3}}$, from which we have points $\left(\pm \frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}, 0 \right)$ (opposite sign).
- $(x, y, 2y) \in D$ iff $x^2 = y^2 + 4y^2 = 5y^2$ and $x^2 + y^2 = xy + 1$. From first equation we get $x = \pm\sqrt{5}y$. In the case $x = \sqrt{5}y$, from second eqn we have $5y^2 + y^2 = \sqrt{5}y^2 + 1$, that is $(6 - \sqrt{5})y^2 = 1$, that is $y = \pm \frac{1}{\sqrt{6 - \sqrt{5}}}$, this yielding to points $\left(\pm \frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm \frac{1}{\sqrt{6-\sqrt{5}}}, 0 \right)$ (same sign). In the case $x = -\sqrt{5}y$, second condition yields to $5y^2 + y^2 = -\sqrt{5}y^1$, that is $y^2 = \frac{1}{5+\sqrt{5}}$, or $y = \pm \frac{1}{\sqrt{5+\sqrt{5}}}$, from which we get points $\left(\mp \frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm \frac{1}{\sqrt{5+\sqrt{5}}}, 0 \right)$ (opposite sign).

Previous analysis figured out possible min/max points. To decide which are min and which max it suffices to compute f at these points. We have:

- $f(\pm 1, \pm 1, 0) = 2$;
- $f\left(\pm \frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}, 0 \right) = \frac{2}{3} = 0,\bar{6}$;
- $f\left(\pm \frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm \frac{1}{\sqrt{6-\sqrt{5}}}, 0 \right) = \frac{6}{6-\sqrt{5}} \approx 1,59\ldots$
- $f\left(\mp \frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm \frac{1}{\sqrt{5+\sqrt{5}}}, 0 \right) = \frac{6}{5+\sqrt{5}} \approx 0,83\ldots$

From this it is clear that $(\pm 1, \pm 1, 0)$ are points of D at max distance to $\vec{0}$, while $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, 0\right)$ are poitns of D at min distance to $\vec{0}$. □

Exercise 32.

Exercise 33. i) D is closed because it is defined by large inequalities. It is not open because $D \neq \emptyset, \mathbb{R}^3$. It is unbounded since $(n, n, \frac{1}{\cosh(2n^2)}) \in D$ for every $n \in \mathbb{N}$, therefore it is not compact.

ii) We have

$$\lambda_3(D) = \int_D 1\, dxdydz \stackrel{RF}{=} \int_{\mathbb{R}^2} \left(\int_0^{1/\cosh(x^2+y^2)} dz \right) dxdy = \int_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)}\, dxdy.$$

By introducing polar coordinates,

$$\lambda_3(D) = \int_{\rho \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi} \frac{1}{\cosh \rho^2} \rho\, d\rho d\theta = 2\pi \int_0^{+\infty} \frac{\rho}{\cosh \rho^2}\, d\rho.$$

Notice that

$$\frac{\rho}{\cosh \rho^2} = \frac{2\rho}{e^{\rho^2} + e^{-\rho^2}} = \frac{2\rho e^{\rho^2}}{1 + e^{2\rho^2}} = \partial_\rho \arctan(e^{\rho^2}),$$

thus

$$\lambda_3(D) = 2\pi \left[\arctan(e^{\rho^2}) \right]_{\rho=0}^{\rho=+\infty} = 2\pi \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi^2}{2}.$$

iii) Proceeding as in ii), we have

$$I_\alpha := \int_D e^{\alpha(x^2+y^2)}\, dxdydz \stackrel{RF}{=} \int_{\mathbb{R}^2} \left(\int_0^{1/\cosh(x^2+y^2)} e^{\alpha(x^2+y^2)}\, dz \right) dxdy = \int_{\mathbb{R}^2} \frac{e^{\alpha(x^2+y^2)}}{\cosh(x^2+y^2)}\, dxdy.$$

Changing vars to polar coords,

$$I_\alpha = \int_{\rho \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi} \frac{e^{\alpha\rho^2}}{\cosh \rho^2} \rho\, d\rho d\theta = 2\pi \int_0^{+\infty} \frac{2\rho e^{(\alpha+1)\rho^2}}{1 + e^{2\rho^2}}\, d\rho.$$

Notice that

$$\frac{2\rho e^{(\alpha+1)\rho^2}}{1 + e^{2\rho^2}} \sim_{+\infty} 2\rho \frac{e^{(\alpha+1)\rho^2}}{e^{2\rho^2}} = 2\rho e^{(\alpha-1)\rho^2}$$

and

$$\exists \int_0^{+\infty} \rho e^{(\alpha-1)\rho^2}\, d\rho \iff \alpha - 1 < 0, \iff \alpha < 1. \quad \square$$

Exercise 34. i) In order $f = u + iv$ be $\mathbb{C}-$differentiable on $\mathbb{C}$ we need 1. that u, v are $\mathbb{R}$ differentiable on $\mathbb{R}^2$ (which is true, being u, v polynomials) and 2. u, v fulfil the CR equations, namely

$$\begin{cases} \partial_x u \equiv \partial_y v, \\ \\ \partial_y u \equiv -\partial_x v, \end{cases} \iff \begin{cases} 3x^2 + ay^2 \equiv bx^2 - 3y^2, \\ \\ 2axy \equiv -2bxy, \end{cases} \iff b = 3,\ a = -3.$$

ii) We have

$$f = (x^3 - 3xy^2) + i(3x^2y - y^3) = (x + iy)^3 = z^3. \quad \square$$

Exercise 35. Clearly $f \in \mathscr{C}(\mathbb{R}^d)$ and moreover $f \geqslant 0$ (trivial) and

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty.$$

Just notice that $f(\vec{x}) \geqslant \|\vec{x}-\vec{a}_1\|^2 \longrightarrow +\infty$ when $\vec{x} \longrightarrow \infty_d$. Thus f cannot have a maximum but it has a minimum according to Weierstrass' thm. Now, f is differentiable on $\mathbb{R}^d$,

$$\nabla f = \sum_{j=1}^{N} \nabla \|\vec{x}-\vec{a}_j\|^2$$

and

$$\nabla \|\vec{x}-\vec{a}_j\|^2 = \left(\partial_1 \|\vec{x}-\vec{a}_j\|^2, \dots, \partial_d \|\vec{x}-\vec{a}_j\|^2\right),$$

so, writing

$$\|\vec{x}-\vec{a}_j\|^2 = \sum_{k=1}^{d} (x_k - a_{j,k})^2, \implies \partial_i \|\vec{x}-\vec{a}_j\|^2 = \partial_i \sum_{k=1}^{d} (x_k - a_{j,k})^2 = 2(x_i - a_{j,i}),$$

we deduce

$$\nabla \|\vec{x}-\vec{a}_j\|^2 = \big(2(x_1 - a_{j,1}), 2(x_2 - a_{j,2}, \dots, 2(x_d - a_{j,d})\big) = 2(\vec{x}-\vec{a}_j).$$

Therefore, $\nabla f \in \mathscr{C}$ and f is differentiable. According to Fermat thm, at min point we must have

$$\nabla f = \vec{0}, \iff \sum_{j=1}^{N} 2(\vec{x}-\vec{a}_j) = 0, \iff N\vec{x} - \sum_{j=1}^{N} \vec{a}_j = \vec{0}, \iff \vec{x} = \frac{1}{N}\sum_{j=1}^{N} \vec{a}_j. \quad \square$$

Exercise 36. i) For $D \neq \emptyset$ we consider a point of type $(x, y, 2)$. Then $(x, y, 2) \in D$ iff $x^2 + y^2 = 4$ and $y^2 = 1$, thus $y = \pm 1$ and $x^2 = 3$, that is $x = \pm\sqrt{3}$. We conclude that points $(\pm\sqrt{3}, \pm 1, 2)$ (four points, all possible combinations of sign) belong to D.

We have that $D = \{g_1 = 0,\ g_2 = 0\}$ where $g_1 = x^2 + y^2 - z^2$, and $g_2 = y^2 + (z-2)^2 - 1$. Clearly, both g_1 and g_2 are differentiable functions (they are polynomials). In order $\vec{g} = (g_1, g_2)$ be a submersion on D we need to verify that

$$\operatorname{rk} \vec{g}' = \operatorname{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \operatorname{rk}\begin{bmatrix} 2x & 2y & -2z \\ \\ 0 & 2y & 2(z-2) \end{bmatrix} = 2,\ \forall (x,y,z) \in D.$$

Now, this is false iff all 2×2 sub-determinants of the Jacobian matrix $\vec{g}'$ vanish, that is iff

$$\begin{cases} 4xy = 0, \\ 4x(z-2) = 0, \\ 8y(z-1) = 0. \end{cases} \iff \begin{cases} x = 0, \\ \\ y(z-1) = 0, \end{cases} \vee \begin{cases} y = 0, \\ \\ x(z-2) = 0, \end{cases}$$

The first subsystem has solutions $(0,0,z)$ and $(0,y,1)$ $(x, y \in \mathbb{R})$; the second, $(0,0,z)$ and $(x,0,2)$, $(x, z \in \mathbb{R})$. Now:

- $(0,0,z) \in D$ iff $z^2 = 0$ and $(z-2)^2 = 1$, impossible;
- $(0,y,1) \in D$ iff $y^2 = 1$ and $y^2 + 1 = 1$, again impossible;
- $(x,0,2) \in D$ iff $x^2 = 4$ and $0 = 1$, impossible.

Cocnlusion: there is no point on D at which rank of $\vec{g}'$ is less than 2, therefore rank of $\vec{g}'(x, y, z)$ is 2 for every $(x, y, z) \in D$, that is $\vec{g}$ is a submersion on D.

ii) D is defined by equalities involving continuous functions, it is therefore closed. From the second equation

$$y^2 + (z-2)^2 = 1, \implies y^2 \leqslant 1, \ (z-2)^2 \leqslant 1.$$

In particular, $-1 \leqslant z - 2 \leqslant 1$, that is $1 \leqslant z \leqslant 3$, thus $z^2 \leqslant 9$. Plugging this into the first equation,

$$x^2 + y^2 = z^2, \ \leqslant \ x^2 + y^2 \leqslant 9, \implies x^2 \leqslant 9.$$

In conclusion $x^2 + y^2 + z^2 \leqslant 9 + 1 + 9 = 19$, for every $(x, y, z) \in D$, from which we see that D is bounded. We conclude that D is compact.

iii) Points at min/max distance to $\vec{0}$ minimize/maximize the function $f = x^2 + y^2 + z^2$. Since f is continuous and D is compact, according to the Weierstrass theorem, f has both min and max on D.

To determine these points, we apply the Lagrange multipliers' theorem. By i), hypotheses of the theorem are fulfilled. Thus, at every $(x, y, z) \in D$ min/max point for f in D we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \iff \det\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det\begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 2(z-2) \end{bmatrix} = 0.$$

By computing the determinant we get

$$0 = 2x \cdot 4y(z - 2 + z) - 2x \cdot 4y(z - 2 - z) = 16xyz,$$

whose solutions are points $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$. Now,

- $(0, y, z) \in D$ iff $y^2 = z^2$ and $y^2 + (z-2)^2 = 1$, from which $z^2 + (z-2)^2 = 1$, or $2z^2 - 2z + 3 = 0$, and since $\Delta < 0$ there are no solutions to this equation;
- $(x, 0, z) \in D$ iff $x^2 = z^2$ and $(z-2)^2 = 1$, from which $z = 1, 3$ and $x^2 = 1$ (that is $x = \pm 1$), or $x^2 = 9$ (that is $x = \pm 3$). We obtain points $(\pm 1, 0, 1)$ and $(\pm 3, 0, 3)$;
- $(x, y, 0) \in D$ iff $x^2 + y^2 = 0$, $y^2 + 4 = 1$ which is impossible.

Since $f(\pm 1, 0, 1) = 2$ and $f(\pm 3, 0, 3) = 18$ we deduce that $(\pm 1, 0, 1)$ are points of D at min distance to $\vec{0}$, $(\pm 3, 0, 3)$ are points of D at max distance to $\vec{0}$. □

Exercise 37. Following the hint, we set $(x, y) = (u^2, v^2)$ in such a way that, with notations of the change of variables formula, $(x, y) = \Phi^{-1}(u, v)$. To have a true change of variable, Φ, Φ^{-1} must be bijections, that is injective and surjective. Now, as defined, Φ^{-1} is not injective on $\mathbb{R}^2$. However, since

$$(x, y) \in [0, +\infty[^2, \ \begin{cases} x = u^2, \\ \\ y = v^2, \end{cases} \iff \begin{cases} u = \pm\sqrt{x}, \\ \\ v = \pm\sqrt{y} \end{cases}$$

we see that Φ^{-1} becomes injective (and bijective) if also $(u, v) \in [0, +\infty[^2$. Therefore, Φ, Φ^{-1} are a bijection from $[0, +\infty[^2$ into itself:

$$(x, y) \in [0, +\infty[^2, \ \Phi^{-1} \begin{cases} x = u^2, \\ \\ y = v^2, \end{cases} \iff \Phi \begin{cases} u = \sqrt{x}, \\ \\ v = \sqrt{y} \end{cases}$$

According to the change of variables formula

$$\int_D f(x,y)\, dxdy = \int_{\Phi(D)} f(\Phi^{-1}(u,v))|\det(\Phi^{-1})'(u,v)|\, dudv$$

Here $D = [0,+\infty[^2$, and $\Phi(D) = [0,+\infty[^2$.

$$f(\Phi^{-1}(u,v)) = f(u^2,v^2) = \frac{e^{-(u^2+v^2)}}{\sqrt{u^4v^2+u^2v^4}} = \frac{e^{-(u^2+y^2)}}{\sqrt{u^2v^2(u^2+v^2)}}.$$

and since

$$\det(\Phi^{-1})'(u,v) = \det\begin{bmatrix} 2u & 0 \\ 0 & 2v \end{bmatrix} = 4uv,$$

we have

$$\begin{aligned}
\int_{[0,+\infty[^2} \tfrac{e^{-(x+y)}}{\sqrt{x^2y+xy^2}}\, dxdy &= \int_{[0,+\infty[^2} \tfrac{e^{-(u^2+y^2)}}{\sqrt{u^2v^2(u^2+v^2)}}|4uv|\, dudv = \\
&= 4\int_{[0,+\infty[^2} \tfrac{e^{-(u^2+v^2)}}{\sqrt{u^2+v^2}}\, dudv \\
&= 4\int_{\rho\geqslant 0,\ 0\leqslant\theta\leqslant\frac{\pi}{2}} \tfrac{e^{-\rho^2}}{\rho}\rho\, d\rho d\theta \quad (u=\rho\cos\theta,\ v=\rho\sin\theta) \\
&\overset{RF}{=} 2\pi\int_0^{+\infty} e^{-\rho^2}\, d\rho = 2\pi\tfrac{\sqrt{\pi}}{2} = \pi\sqrt{\pi}. \quad \square
\end{aligned}$$

Exercise 38. i) In order $f = u + iv$ is holomorphic on $\mathbb{C}$ we need that $u, v \in \mathscr{C}^1$ (true, u and v are polynomials) and they fulfill the CR equations:

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v, \end{cases} \iff \begin{cases} 2ax+by = x, \\ \\ bx+2cy = -y, \end{cases} \forall (x,y)\in\mathbb{R}^2., \iff \begin{cases} 2a=1,\ b=0, \\ \\ b=0,\ 2c=-1. \end{cases}$$

Thus,

$$u = \frac{1}{2}x^2 - \frac{1}{2}y^2,\quad v = xy,$$

and $f = u + iv$ is holomorphic on $\mathbb{C}$.

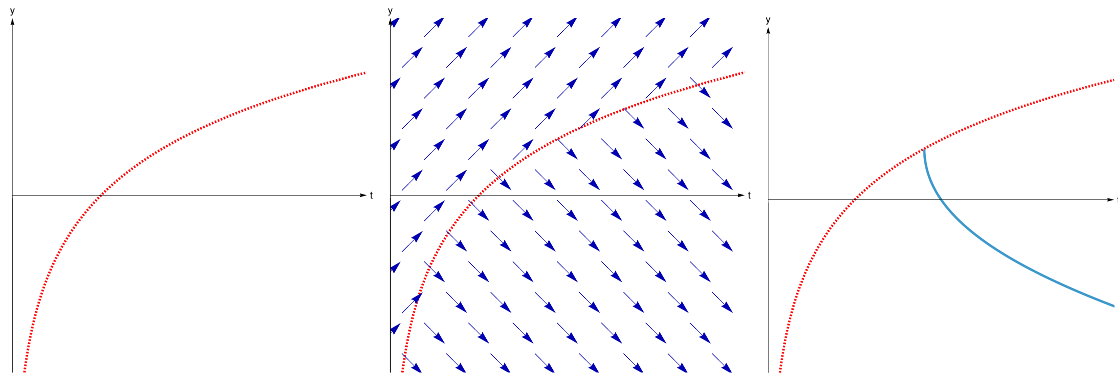
ii) Notice that

$$f = u+iv = \frac{1}{2}x^2 - \frac{1}{2}y^2 + ixy = \frac{1}{2}\left(x^2 - y^2 + i2xy\right) = \frac{1}{2}(x+iy)^2 \equiv \frac{z^2}{2},\ z\in\mathbb{C}. \quad \square$$

Exercise 39. i) Let $f(t,y) := \frac{1}{y-\log t}$. Clearly, f is well defined on

$$D := \{(t,y)\in\mathbb{R}^2\ :\ y-\log t \neq 0\} = \{(t,y)\ :\ t>0,\ y\neq\log t\}.$$

Clearly $f, \partial_y f \in \mathscr{C}(D)$, so local existence and uniqueness applies on D.



ii) $y \equiv C$ is a constant solution iff $0 = \frac{1}{C-\log t}$, which is impossible. Conclusion: no stationary solutions. Furthermore,

$$y \nearrow, \iff 0 \leqslant y' = \frac{1}{y-\log t}, \iff y - \log t > 0, \iff y > \log t.$$

iii) Claim: $y \searrow$. This follows once we prove that $y(t) < \log t$ for all t. This is indeed true at $t = 2$: $y(2) = 0 < \log 2$. If the conclusion were false, there would be a $\widehat{t}$ such that $y(\widehat{t}) \geqslant \log \widehat{t}$. But,

- if $y(\widehat{t}) = \log \widehat{t}$, $(\widehat{t}, y(\widehat{t})) \notin D$, which is impossible for a solution.
- if $y(\widehat{t}) > \log \widehat{t}$ then, by continuity, there would be another $\widehat{\widehat{t}}$ such that $y(\widehat{\widehat{t}}) = \log \widehat{\widehat{t}}$, so we fall again into a contradiction.

In any case, we get a contradiction, so $y(t) < \log t$ must be always true and the conclusion follows.

We have

$$\lim_{t \to \alpha} y'(t) = \lim_{t \to \alpha} \frac{1}{y(t) - \log t}.$$

Claim: $y(t) \longrightarrow \log \alpha$. Indeed, we know $y(t) < \log t$ for every t, so letting $t \to \alpha$ we have $y(\alpha) \leqslant \log \alpha$. If $y(\alpha) < \log \alpha$ then

$$\forall t \in]\alpha, 2], \ (t, y(t)) \in [\alpha, 2] \times [0, y(\alpha)] =: K \text{ compact } \subset D,$$

so the solution never leaves the compact K in the past. This is in conflict with the exit from compact sets, so $y(\alpha) = \log \alpha$, from which $y(t) - \log t \longrightarrow 0$. Since $y(t) < \log t$, this implies $y(t) - \log t \longrightarrow 0-$, so

$$\lim_{t \to \alpha} y'(t) = \frac{1}{0-} = -\infty.$$

iv) For the concavity we notice that

$$y'' = \left(\frac{1}{y - \log t}\right)' = -\frac{y' - \frac{1}{t}}{(y - \log t)^2}.$$

Therefore,

$$y \uparrow, \iff y'' \geqslant 0, \iff y' - \frac{1}{t} \leqslant 0.$$

Since $t \in]\alpha, \beta[$, $t > \alpha > 0$ and $y' \leqslant 0$, we conclude that $y' - \frac{1}{t} \leqslant 0$ for all $t > 0$, thus $y'' > 0$ for all t and y is convex.

$\beta = +\infty$? We have two possibilities: $\beta < +\infty, \beta = +\infty$. If $\beta < +\infty$ then, defined $y(\beta) = \lim_{t\to\beta} y(t) \geqslant -\infty$ (the limit exists being y decreasing), we would have

- either $y(\beta) > -\infty$: in this case, the solution would remain into the compact box $[2, \beta] \times [y(\beta), 0]$ for all future times, violating the exit from compact sets, impossible;
- or $y(\beta) = -\infty$: also this is impossible because, by convexity, y would be above each of its tangents, in particular to that one for $t = 2$:

$$y(t) \geqslant y(2) + y'(2)(t-2) \implies y(\beta) \geqslant y(2) + y'(2)(\beta - 2) > -\infty,$$

  which is impossible being $y(\beta) = -\infty$.

In any case, we got a contradiction. This means $\beta = +\infty$. □

Exercise 40. See notes for definitions. We aim to prove that $\vec{F}^{-1}(S)$ is open if S it is. Suppose this is false. There exists then a point $\vec{x} \in \vec{F}^{-1}(S)$ for which

$$\nexists B(\vec{x}, r] \subset \vec{F}^{-1}(S).$$

This means that:

$$\forall r > 0,\ B(\vec{x}, r] \cap \vec{F}^{-1}(S)^c \neq \varnothing.$$

Taking $r = \frac{1}{n}$

$$\forall n \in \mathbb{N},\ n \geqslant 1,\ \exists \vec{x}_n \in B(\vec{x}, 1/n] \cap \vec{F}^{-1}(S)^c.$$

This means that $\|\vec{x}_n - \vec{x}\| \leqslant \frac{1}{n}$, thus $\vec{x}_n \longrightarrow \vec{x}$. By continuity, $\vec{F}(\vec{x}_n) \longrightarrow \vec{F}(\vec{x})$. Furthermore, by construction of $(\vec{x}_n)$, we have that $\vec{x}_n \notin \vec{F}^{-1}(S)$, that is $\vec{F}(\vec{x}_n) \notin S$ for every n. Since $\vec{F}(\vec{x}) \in S$ (recall that $\vec{x} \in \vec{F}^{-1}(S)$), and S is supposed to be open,

$$\exists B(\vec{F}(\vec{x}), \rho] \subset S.$$

And since $\vec{F}(\vec{x}_n) \longrightarrow \vec{F}(\vec{x})$, we have that

$$\exists N\ :\ \vec{F}(\vec{x}_n) \in B(\vec{F}(\vec{x}), \rho] \subset S,\ \forall n \geqslant N,$$

which is in contradiction with the construction of $(\vec{x}_n)$. We deduce that the initial assumption must be false, that is $\vec{F}^{-1}(S)$ is open. □

Exercise 41. i) Let $(g_1, g_2) := (x^2 + y^2 - 1, x + y + z - 1)$ in such a way $D = \{g_1 = 0,\ g_2 = 0\}$. To check that (g_1, g_2) is a submersion on D we have to verify that

$$\mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & 2y & 0 \\ \\ 1 & 1 & 1 \end{bmatrix} = 2,\ \forall (x, y, z) \in D.$$

Now, rank is < 2 iff the two gradients are linearly dependent. This is manifestly impossible because of their third component.

ii) D is closed being defined by equalities involving continuous functions. D is also bounded: indeed, by first equation we have $x^2, y^2 \leqslant 1$, thus $-1 \leqslant x, y \leqslant 1$, and by the secon

$$-1 \geqslant z = 1 - (x + y) \leqslant 3,$$

thus $z^2 \leqslant 9$ and $x^2 + y^2 + z^2 \leqslant 11$.

iii) Function f is continuous on D compact: existence of min/max is ensured by Weierstrass thm. To determine these points, we apply Lagrange multipliers thm. By i), D fulfils the assumption of the thm. Thus, at (x, y, z) min/max point for f on D we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \Longleftrightarrow \quad \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \mathrm{rk} \begin{bmatrix} 2x + y - 1 & 2y + x + z - 1 & y \\ 2x & 2y & 0 \\ 1 & 1 & 1 \end{bmatrix} < 3,$$

that is iff the determinant of previous matrix vanishes. We get the condition

$$2y(x - y) + 2(y(2x + y - 1) - x(2y + x + z - 1)) = 0,$$

from which, simplifying,

$$y(y - x) + (y^2 - y - x^2 + x - xz) = 0.$$

Since we are looking for solutions $(x, y, z) \in D$, we must have $z = 1 - x - y$, and plugging this into previous equation yields,

$$y\,(2y - 1) = 0, \quad \Longleftrightarrow \quad y = 0, \ \vee\ y = \frac{1}{2}.$$

Thus we get points $(x, 0, 1-x)$ and $(x, \frac{1}{2}, \frac{1}{2}-x)$, to which we have still to impose the condition $x^2+y^2 = 1$. In the first case $x^2 + 0^2 = 1$, thus $x = \pm 1$, that is points $(\pm 1, 0, \mp 1)$ (two points). In the second case, $x^2 + \frac{1}{4} = 1$, thus $x^2 = \frac{3}{4}$ and $x = \pm\frac{\sqrt{3}}{2}$, that is points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1+\sqrt{3}}{2}\right)$. We have

- $f(1, 0, -1) = 0$
- $f(-1, 0, 1) = 2$
- $f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} - 2)$
- $f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1+\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} + 2)$

From this we see that $(-1, 0, 1)$ is max point, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right)$ is min point. □

Exercise 42. i) See LN for the definitions and general properties.

ii) To be irrotational, $\vec{F}$ must verify

$$\begin{cases} \partial_y \frac{ax+by}{1+x^2+y^2+z^2} = \partial_x \frac{cy+dz}{1+x^2+y^2+z^2}, \\ \partial_z \frac{ax+by}{1+x^2+y^2+z^2} = \partial_x \frac{ex+fz}{1+x^2+y^2+z^2}, \\ \partial_z \frac{cy+dz}{1+x^2+y^2+z^2} = \partial_y \frac{ex+fz}{1+x^2+y^2+z^2}, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \frac{b(1+x^2+y^2+z^2)-(ax+by)2y}{(1+x^2+y^2+z^2)^2} = \frac{-(cy+dz)2x}{(1+x^2+y^2+z^2)^2}, \\ \frac{-(ax+by)2z}{(1+x^2+y^2+z^2)^2} = \frac{e(1+x^2+y^2+z^2)-(ex+fz)2x}{(1+x^2+y^2+z^2)^2}, \\ \frac{d(1+x^2+y^2+z^2)-(cy+dz)2z}{(1+x^2+y^2+z^2)^2} = \frac{-(cy+dz)2x}{(1+x^2+y^2+z^2)^2} \end{cases}$$

that is

$$\begin{cases} b(1 + x^2 + y^2 + z^2) - (ax + by)2y = -(cy + dz)2x, \\ -(ax + by)2z = e(1 + x^2 + y^2 + z^2) - (ex + fz)2x, \\ d(1 + x^2 + y^2 + z^2) - (cy + dz)2z = -(cy + dz)2x \end{cases}$$

We work on the first equation, the others being similar:

$$b + bx^2 - by^2 + bz^2 - 2(a - c)xy + 2dzx = 0,$$

from which $b = a - c = d = 0$, that is $b = d = 0$ and $a = c$. Similarly, $a = c = f$ and $e = 0$. Thus $\vec{F}$ is irrotational iff

$$\vec{F}(x,y,z) := \left(\frac{ax}{1+x^2+y^2+z^2}, \frac{ay}{1+x^2+y^2+z^2}, \frac{az}{1+x^2+y^2+z^2}\right)$$

ii) To be conservative, $\vec{F}$ must be irrotational, so the values of a, b, c, d, e, f are those found in ii). For such values $\vec{F} = \nabla f$ iff

$$\begin{cases} \partial_x f = \frac{ax}{1+x^2+y^2+z^2}, \\ \partial_y f = \frac{ay}{1+x^2+y^2+z^2}, \\ \partial_z f = \frac{az}{1+x^2+y^2+z^2}. \end{cases}$$

From the first equation we get

$$f(x,y,z) = a\int \frac{x}{1+x^2+y^2+z^2}\,dx + c(y,z) = \frac{a}{2}\log(1+x^2+y^2+z^2) + c(y,z).$$

Imposing the second equation we get

$$\partial_y f = \frac{ay}{1+x^2+y^2+z^2} + \partial_y c(y,z) = \frac{ay}{1+x^2+y^2+z^2}, \quad \Longleftrightarrow \quad \partial_y c(y,z) = 0,$$

from which $c(y,z) = c(z)$. Finally, imposing also the third condition we get

$$\partial_z f = \frac{az}{1+x^2+y^2+z^2} + \partial_z c(z) = \frac{az}{1+x^2+y^2+z^2}, \quad \Longleftrightarrow \quad \partial_z c(z) = 0,$$

from which $c(y,z) = c(z) = c$. Conclusion: all the potentials of $\vec{F}$ are

$$f(x,y,z) = \frac{a}{2}\log(1+x^2+y^2+z^2) + c. \quad \square$$

Exercise 43. i) D is closed, being defined by large inequalities involving continuous functions. Let's check that D is bounded (hence compact). Denoting by $\rho = \sqrt{x^2+y^2} = \|(x,y)\|$ we have

$$(x,y) \in D, \implies \rho^2 \leqslant 2\rho\cos\theta - \rho = \rho(2\cos\theta - 1), \iff \rho \leqslant 2\cos\theta - 1 \leqslant 1.$$

Therefore, D is bounded. In particular, D cannot be be open: only $\emptyset, \mathbb{R}^2$ are both open and closed, and $(0,0) \in D$ (thus $D \neq \emptyset$), and D is bounded, thus $D \subsetneq \mathbb{R}^2$.

ii) The area of D is

$$\lambda_2(D) = \int_D 1\,dxdy = \int_{x^2+y^2 \leqslant 2x - \sqrt{x^2+y^2}} 1\,dxdy \overset{pol\ coords}{=} \int_{\rho \leqslant 2\cos\theta - 1} \rho\,d\rho d\theta.$$

Now, notice that since $\rho \geqslant 0$, this imposes $2\cos\theta - 1 \geqslant 0$, that is $\cos\theta \geqslant \frac{1}{2}$. In one period this means $-\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}$. Thus

$$\begin{aligned} \lambda_2(D) &= \int_{\rho \leqslant 2\cos\theta - 1,\ -\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}} \rho\,d\rho d\theta \overset{RF}{=} \int_{-\pi/3}^{\pi/3}\int_0^{2\cos\theta - 1} \rho\,d\rho\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta - 1)^2\,d\theta \\ &= \frac{1}{2}\left(\frac{2\pi}{3} - 4\int_{-\pi/3}^{\pi/3}\cos\theta\,d\theta + 4\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta\right) \\ &= \frac{\pi}{3} - 2\sqrt{3} + 2\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta. \end{aligned}$$

About this last integral we have

$$\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta = \int_{-\pi/3}^{\pi/3}(\cos\theta)(\sin\theta)'\,d\theta = [\sin\theta\cos\theta]_{\theta=-\pi/3}^{\theta=\pi/3}+\int_0^{2\pi}(\sin\theta)^2\,d\theta = \frac{\sqrt{3}}{2}-\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta,$$

from which $\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\,d\theta = \frac{\sqrt{3}}{4}$. We conclude that $\lambda_2(D) = \frac{\pi}{3} - \frac{3\sqrt{3}}{2}$. □

Exercise 44. i) For $z = x + iy \neq 0$ we have

$$f(z) = \frac{(x-iy)^2}{x+iy}\frac{x-iy}{x=iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - i3x^2y - 3xy^2 + iy^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\frac{y^3-3x^2y}{x^2+y^2}.$$

So, $f(x+iy) = u(x,y) + iv(x,y)$ where

$$u(x,y) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ \\ 0, & (x,y) = (0,0), \end{cases} \qquad v(x,y) = \begin{cases} \frac{y^3-3x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ \\ 0, & (x,y) = (0,0), \end{cases}$$

Let's check that u, v verify the CR equations at $(0,0)$. We need the partial derivatives of u, v at $(0,0)$. To compute these we apply the definition of partial derivative:

$$\partial_x u(0,0) = \lim_{t\to 0}\frac{u(t,0)-u(0,0)}{t} = \lim_{t\to 0}\frac{t-0}{t} = 1, \quad \partial_y u(0,0) = \lim_{t\to 0}\frac{u(0,t)-u(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0,$$

and

$$\partial_x v(0,0) = \lim_{t\to 0}\frac{v(t,0)-v(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0, \quad \partial_y u(0,0) = \lim_{t\to 0}\frac{u(0,t)-u(0,0)}{t} = \lim_{t\to 0}\frac{t-0}{t} = 1,$$

from which we get

$$\begin{cases} \partial_x u(0,0) = 1 = \partial_y v(0,0), \\ \\ \partial_y u(0,0) = 0 = -\partial_x v(0,0), \end{cases}$$

that is, CR equations hold.

ii) To check the differentiability of f at $z = 0$ we should check if u, v are $\mathbb{R}-$differentiable at $(0,0)$ or, in alternative, if $\exists f'(0)$. We present here both methods (just one of them is sufficient).

Differentiability of u at $(0,0)$. We already computed $\nabla u(0,0) = (1,0)$. So, in order u be differentiable at $(0,0)$ we need to verify that

$$\lim_{\vec{h}\to\vec{0}}\frac{u(\vec{h}) - u(\vec{0}) - \nabla u(\vec{0})\vec{h}}{\|\vec{h}\|} = 0.$$

Setting $\vec{h} = (a,b)$, this is

$$\lim_{(a,b)\to(0,0)}\frac{u(a,b)-u(0,0)-(1,0)\cdot(a,b)}{\sqrt{a^2+b^2}} = \lim_{(a,b)\to(0,0)}\frac{\frac{a^3-3ab^2}{a^2+b^2} - 0 - a}{(a^2+b^2)^{1/2}} = \lim_{(a,b)\to(0,0)}\frac{-4ab^2}{(a^2+b^2)^{3/2}}$$

We see that if $a = b$ this reduces to the limit

$$\lim_{a\to 0}\frac{-4a^3}{2^{3/2}|a|^3} = \pm 1 \neq 0$$

according to $a \to 0\mp$, so we deduce that u is not differentiable at $(0,0)$, thus f is not $\mathbb{C}-$differentiable at $z = 0$. $\square$

$\mathbb{C}-$differentiability of f at $z = 0$. We should verify that

$$\exists \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)^2.$$

Now, taking $h = x + i0$,

$$\lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)^2 = \lim_{x\to 0} \left(\frac{x}{x}\right)^2 = 1.$$

Taking $h = x + ix$,

$$\lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)^2 = \lim_{x\to 0} \left(\frac{x - ix}{x + ix}\right)^2 = \left(\frac{1-i}{1+i}\right)^2 = \frac{1}{4}(-i2) = -\frac{i}{2},$$

from which we conclude that the limit does not exists. $\square$

Exercise 45. i) See LN for definitions and general properties.

ii) Since

$$\partial_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v},$$

it is clear that, if $\nabla f(\vec{x}) = 0$, then

$$\max_{\vec{v}\neq\vec{0}\,:\,\|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}) = 0.$$

If $\nabla f(\vec{x}) \neq \vec{0}$, by the Cauchy-Schwarz inequality we get

$$\partial_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v} \leqslant \|\nabla f(\vec{x})\|\|\vec{v}\| = \|\nabla f(\vec{x})\|,$$

for $\|\vec{v}\| = 1$. Now, since $\vec{v}_0 = \frac{1}{\|\nabla f(\vec{x})\|}\nabla f(\vec{x})$ has $\|\vec{v}_0\| = 1$, and

$$\partial_{\vec{v}_0} f(\vec{x}) = \nabla f(\vec{x}) \cdot \left(\frac{1}{\|\nabla f(\vec{x})\|}\nabla f(\vec{x})\right) = \frac{1}{\|\nabla f(\vec{x})\|}\nabla f(\vec{x}) \cdot \nabla f(\vec{x}) = \frac{1}{\|\nabla f(\vec{x})\|}\|\nabla f(\vec{x})\|^2 = \|\nabla f(\vec{x})\|,$$

we conclude that

$$\max_{\vec{v}\neq\vec{0}\,:\,\|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}) = \|\nabla f(\vec{x})\|.$$

So, whatever is $\nabla f(\vec{x})$, the conclusion is

$$\max_{\vec{v}\neq\vec{0}\,:\,\|\vec{v}\|=1} \partial_{\vec{v}} f(\vec{x}) = \|\nabla f(\vec{x})\|. \quad \square$$

Exercise 46. Recalling of the definition of cosh we have

$$\cosh z = -2, \iff \frac{e^z + e^{-z}}{2} = -2, \iff e^z + e^{-z} = -4, \iff e^{2z} + 4e^z + 1 = 0.$$

Setting $w := e^z$, we have

$$w^2 + 4w + 1 = 0, \iff w = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}.$$

Noticed that $-2 \pm \sqrt{3} < 0$ and that $2 > \sqrt{3}$, we have

$$e^z = -2 \pm \sqrt{3}, \quad \Longleftrightarrow \quad z = \log(2 \pm \sqrt{3}) + i(\frac{3\pi}{2} + k2\pi),\ k \in \mathbb{Z},$$

which are the solutions of the proposed equation. □

Exercise 47. i) To be irrotational on $\mathbb{R}^2$, $\vec{F}$ must verify on $\mathbb{R}^2$

$$\partial_y(axy + e^{by}) \equiv \partial_x(xe^y + 2y), \quad \Longleftrightarrow \quad ax + be^{by} \equiv e^y.\ (\star)$$

Fix y, for example $y = 0$. Then $ax + b \equiv 1$, iff $a = 0$ and $b = 1$. It is clear that for these a, b, the condition $(\star)$ is verified, so $\vec{F}$ is irrotational on $\mathbb{R}^2$.

ii) To be conservative on $\mathbb{R}^2$, $\vec{F}$ must be irrotational on $\mathbb{R}^2$, so $a = 0$ and $b = 1$. It is a general fact that an irrotational field on $\mathbb{R}^2$ is also conservative, so $\vec{F}$ is conservative on $\mathbb{R}^2$ iff $a = 0$, $b = 1$. In this case the potentials are f such that

$$\begin{cases} \partial_x f = e^y, \\ \\ \partial_y f = xe^y + 2y. \end{cases}$$

From the first equation we have

$$f(x, y) = \int e^y\ dx + c(y) = xe^y + c(y),$$

and plugging this into the second equation we get

$$xe^y + c'(y) = xe^y + 2y, \quad \Longleftrightarrow \quad c'(y) = 2y, \quad \Longleftrightarrow \quad c(y) = y^2 + c,\ c \in \mathbb{R}.$$

We conclude that the potentials of f are $f(x, y) = xe^y + y^2 + c$, $c \in \mathbb{R}$ constant.

iii) Since the values a, b for which $\vec{F}$ is irrotational are the same for which $\vec{F}$ is conservative, we can use the fundamental theorem of integral calculus for fields to conclude that

$$\int_{\vec{\gamma}} \vec{F} = f(\vec{\gamma}(1)) - f(\vec{\gamma}(0)) = f(1, 0) - f(0, 0) = 1 + c - c = 1. \quad \square$$

Exercise 48. i) We have $(x, 0, z) \in D$ iff $z^2 = 1/2$ and $x^2 = z$, from which $(\pm 1/2, 0, 1/2) \in D$. Let $g_1 := xy + z^2 - 1/2$ and $g_2 := x^2 + y^2 - z$. Clearly, g_1, g_2 are differentiable functions and $D = \{g_1 = 0,\ g_2 = 0\}$. The map (g_1, g_2) is a submersion on D iff

$$\text{rank}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \text{rank}\begin{bmatrix} y & x & -2z \\ \\ 2x & 2y & -1 \end{bmatrix} = 2,\ \forall (x, y, z) \in D.$$

This is false at (x, y, z) iff all 2×2 subdeterminants of the previous matrix vanish, that is iff

$$\begin{cases} 2y^2 - 2x^2 = 0, \\ -y + 4xz = 0, \\ -x + 4zy = 0. \end{cases}$$

The first equation is equivalent to $(y-x)(y+x)=0$, that yields the alternative $y=x$ or $y=-x$. In the first case we have

$$\begin{cases} y=x, \\ x(1-4z)=0, \end{cases} \iff x=0, \ \vee\ z=\frac{1}{4},$$

from which we find points $(0,0,z)$ $(z\in\mathbb{R})$ and $(x,x,\frac{1}{4})$ $(x\in\mathbb{R})$. Similarly, for $y=-x$ we get

$$\begin{cases} y=-x, \\ x(1+4z)=0, \end{cases} \iff x=0, \ \vee\ z=-\frac{1}{4},$$

from which we find points $(0,0,z)$ $(z\in\mathbb{R})$ and $(x,-x,-\frac{1}{4})$ $(x\in\mathbb{R})$. Now, let's check if these bad points belong to D:

- $(0,0,z)\in D$ iff $z^2=1/2$ and $z=0$, impossible.
- $(x,x,\frac{1}{4})\in D$ iff $x^2+\frac{1}{16}=\frac{1}{2}$ and $2x^2=\frac{1}{4}$, that is $x^2=\frac{7}{16}$ and $x^2=\frac{1}{8}$: impossible.
- $(x,-x,-\frac{1}{4})\in D$ iff $-x^2+\frac{1}{16}=\frac{1}{2}$ and $2x^2=-\frac{1}{4}$, impossible.

We conclude that the rank of the jacobian matrix of (g_1,g_2) is $=2$ on D, that is (g_1,g_2) is a submersion on D.

ii) D is defined by equalities on continuous functions, therefore it is closed. Let's check it is also bounded: we have

$$(x,y,z)\in D, \implies x^2+y^2=z, \implies (x^2+y^2)^2=z^2=\frac{1}{2}-xy$$

so, in particular,

$$(x^2+y^2)^2=\frac{1}{2}-xy.$$

Now, if $(x,y)=(\rho\cos\theta,\rho\sin\theta)$ this says that

$$\rho^4=\frac{1}{2}-\rho^2\cos\theta\sin\theta=\frac{1}{2}\Big(1-\rho^2\sin(2\theta)\Big)\leqslant\frac{1}{2}(1+\rho^2), \iff \frac{\rho^4}{1+\rho^2}\leqslant\frac{1}{2}.$$

From this we deduce that ρ must be bounded: if unbounded, $\rho\to+\infty$, we would have $\frac{\rho^4}{1+\rho^2}\to+\infty$ and the previous bound would be impossible. Thus, $x^2+y^2=\rho^2\leqslant M$ for some M, and since $z=x^2+y^2$ we get $0\leqslant z\leqslant M$, so D is bounded. Conclusion: D is compact.

Alternative solution: from the remarkable inequality $|xy|\leqslant\frac{x^2+y^2}{2}$, we get

$$z^2=\frac{1}{2}-xy\leqslant\frac{1}{2}+\frac{x^2+y^2}{2}=\frac{1}{2}+\frac{z}{2},$$

from which

$$2z^2-z-1\leqslant 0, \iff -\frac{1}{2}\leqslant z\leqslant 1.$$

Therefore, $x^2+y^2=z\leqslant 1$ and also $|x|,|y|\leqslant 1$, that is D is bounded.

iii) Let $f(x,y,z):=x^2+y^2$ be the square of the distance to the z axis. We have to determine min/max of f on D. Since D is compact and f is continuous, the Weierstrass theorem ensures the existence. To determine these points, we apply the Lagrange multipliers theorem. By i), we see that the assumptions of the theorem are verified. Then, if $(x, y, z) \in D$ is a min/max point for f we have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \Longleftrightarrow \quad \operatorname{rank}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 2x & 2y & 0 \\ y & x & -2z \\ 2x & 2y & -1 \end{bmatrix} = 2.$$

This condition is equivalent to the determinant of the previous matrix = 0,

$$-1(2x^2 - 2y^2) = 0, \quad \Longleftrightarrow \quad (y - x)(y + x) = 0,$$

from which we get points of type (x, x, z) and $(x, -x, z)$. Now,

$$(x, x, z) \in D, \quad \Longleftrightarrow \quad \begin{cases} x^2 + z^2 = \frac{1}{2}, \\ 2x^2 = z, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} z^2 + \frac{z}{2} = \frac{1}{2}, \\ x^2 = \frac{z}{2}. \end{cases} \quad \Longleftrightarrow \quad \begin{cases} z = -1, \frac{1}{2}, \\ x^2 = \frac{z}{2}. \end{cases}$$

It is clear that $z = -1$ yields no solutions while $z = \frac{1}{2}$ yields $x^2 = \frac{1}{4}$ from which $x = \pm\frac{1}{2}$. We get points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2})$ (two points).

Similarly,

$$(x, -x, z) \in D, \quad \Longleftrightarrow \quad \begin{cases} -x^2 + z^2 = \frac{1}{2}, \\ 2x^2 = z, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} z^2 - \frac{z}{2} = \frac{1}{2}, \\ x^2 = \frac{z}{2}. \end{cases} \quad \Longleftrightarrow \quad \begin{cases} z = -\frac{1}{2}, 1, \\ x^2 = \frac{z}{2}. \end{cases}$$

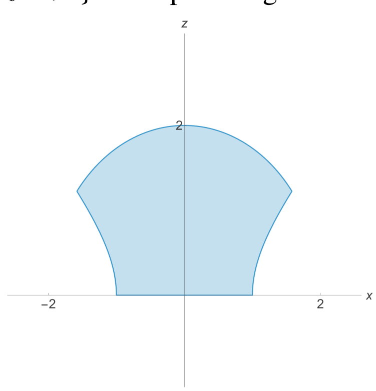
It is clear that $z = -1/2$ yields no solutions while $z = 1$ yields $x^2 = \frac{1}{2}$ from which $x = \pm\frac{1}{\sqrt{2}}$. We get points $(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 2)$ (two points).

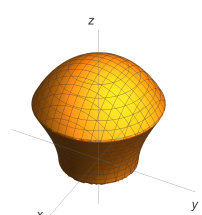
Conclusion: since $f(\pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ and $f(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 2) = 1$, the former are max points, the latter min points. □

Exercise 49. i) We have

$$D \cap \{y = 0\} = \{(x, 0, z) \ : \ x^2 - z^2 \leqslant 1, \ x^2 + z^2 \leqslant 4, \ z \geqslant 0\} = \{x^2 - z^2 \leqslant 1\} \cap \{x^2 + z^2 \leqslant 4\} \cap \{z \geqslant 0\}.$$

Now, in the upper half-plane of xz plane with $z \geqslant 0$, $\{x^2 + z^2 \leqslant 4\}$ is a disk centered at the origin with radius 2, while $\{x^2 - z^2 \leqslant 1\}$ is the plane region delimited by the hyperbola $x^2 - z^2 = 1$.





ii) To compute the volume we use the formula

$$\lambda_3(D) = \int_D 1\ dxdydz = \int_{x^2+y^2-z^2\leqslant 1,\ x^2+y^2+z^2\leqslant 4,\ z\geqslant 0} 1\ dxdydz.$$

Using cylindrical coordinates, we get

$$\lambda_3(D) = \int_{\rho^2-z^2\leqslant 1,\ \rho^2+z^2\leqslant 4,\ z\geqslant 0,\ 0\leqslant\theta\leqslant 2\pi} \rho\ d\rho d\theta\ dz \overset{RF}{=} 2\pi \int_{\rho^2-z^2\leqslant 1,\ \rho^2+z^2\leqslant 4,\ z\geqslant 0} \rho\ d\rho\ dz$$

We do the double integration beginning with ρ and finishing with z. We notice that

$$\rho^2 - z^2 \leqslant 1,\ \rho^2 + z^2 \leqslant 4, \implies \rho^2 \leqslant 1 + z^2, 4 - z^2.$$

Therefore,

$$\rho^2 \leqslant \min(1+z^2, 4-z^2) = \begin{cases} 1+z^2, & \text{if } 1+z^2 \leqslant 4-z^2, \iff z^2 \leqslant \frac{3}{2}, \overset{z\geqslant 0}{\iff} 0 \leqslant z \leqslant \sqrt{\frac{3}{2}}, \\ 4-z^2, & \text{if } z \geqslant \sqrt{\frac{3}{2}}. \end{cases}$$

Notice also that, since $\rho^2 \geqslant 0$, $\rho^2 = 4 - z^2 \geqslant 0$ iff $0 \leqslant z \leqslant 2$. Therefore,

$$\rho^2 \leqslant \min(1+z^2, 4-z^2) = \begin{cases} 1+z^2, & 0 \leqslant z \leqslant \sqrt{\frac{3}{2}}, \\ 4-z^2, & \sqrt{\frac{3}{2}} \leqslant z \leqslant 2. \end{cases}$$

Therefore

$$\begin{aligned} \lambda_3(D) &= 2\pi\left(\int_0^{\sqrt{3/2}}\int_0^{\sqrt{1+z^2}} \rho\ d\rho\ dz + \int_{\sqrt{3/2}}^2 \int_0^{\sqrt{4-z^2}} \rho\ d\rho\ dz\right) \\ &= 2\pi\left(\int_0^{\sqrt{3/2}} \left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sqrt{1+z^2}} dz + \int_{\sqrt{3/2}}^2 \left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sqrt{4-z^2}} dz\right) \\ &= \pi\left(\int_0^{\sqrt{3/2}} (1+z^2)\ dz + \int_{\sqrt{3/2}}^2 4 - z^2\ dz\right) \\ &= \pi\left(\sqrt{\frac{3}{2}} + 4\left(2 - \sqrt{\frac{3}{2}}\right) + \left[\frac{z^3}{3}\right]_{z=0}^{z=\sqrt{3/2}} - \left[\frac{z^3}{3}\right]_{z=\sqrt{3/2}}^{z=2}\right) \\ &= \pi\left(8 - 3\sqrt{\frac{3}{2}} + \frac{1}{3}\frac{3}{2}\sqrt{\frac{3}{2}} - \frac{1}{3}\left(8 - \frac{3}{2}\sqrt{\frac{3}{2}}\right)\right) = \pi\left(\frac{16}{3} - 3\sqrt{\frac{3}{2}}\right). \quad \square \end{aligned}$$

Exercise 50. i) u and v must be $\mathbb{R}-$differentiable and verify the Cauchy-Riemann equations at each point $(x, y) \in \mathbb{R}^2$.

ii) If u is constant, then by the CR equations we get

$$\begin{cases} \partial_x v = -\partial_y u = 0, \\ \\ \partial_y v = \partial_x u = 0, \end{cases}$$

From the first equation $v(x, y) = \int 0 \, dx + c(y) = c(y)$, and plugging this into the second equation we get $c'(y) = 0$, that is c is constant. Therefore $v(x, y) \equiv c$ so $f = 0 + ic = ic$ is constant.

iii) Following the hint

$$0 = \partial_x (u^2 + v^2) = 2u\partial_x u + 2v\partial_x v = 2(u, v) \cdot (\partial_x u, \partial_x v).$$

Similarly,

$$0 = \partial_y (u^2 + v^2) = 2u\partial_y u + 2v\partial_y v = 2(u, v) \cdot (\partial_y u, \partial_y v).$$

These relations say that $(\partial_x u, \partial_x v)$ and $(\partial_y u, \partial_y v)$ are both perpendicular to (u, v). Assume that $(u, v) \neq (0, 0)$. Since the plane vectors $(\partial_x u, \partial_x v)$ and $(\partial_y u, \partial_y v)$ are both perpendicular to $(u, v) \neq \vec{0}$, they must be parallel, so

$$\exists \alpha \ : \ (\partial_x u, \partial_x v) = \alpha(\partial_y u, \partial_y v), \quad \Longleftrightarrow \quad \begin{cases} \partial_x u = \alpha \partial_y u, \\ \\ \partial_x v = \alpha \partial_y v, \end{cases}$$

Now, by the CR equations we have

$$\partial_x u = \alpha \partial_y u \overset{CR}{=} \alpha(-\partial_x v) = -\alpha(\alpha \partial_y v) \overset{CR}{=} -\alpha^2 \partial_x u$$

that is

$$(1 + \alpha^2)\partial_x u = 0, \implies \partial_x u = 0.$$

In the same way we deduce $\partial_y u = 0$, and from this u is constant. Similarly for v, we conclude that $f = u + iv$ is a constant function. □

Exercise 56. i) From the general theory, $u+iv$ is $\mathbb{C}-$differentiable on $\mathbb{C}$ iff both u, v are $\mathbb{R}-$differentiable and they fulfill the Cauchy-Riemann equations. Since u is a polynomial, it is $\mathbb{R}-$differentiable. So, we look for v $\mathbb{R}-$differentiable in such a way that

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \partial_x v = -\partial_y u = -(-20x^3 y + 20xy^3) = 20(x^3 y - xy^3), \\ \\ \partial_y v = \partial_x u = 5x^4 - 30x^2 y^2 + 5y^4, \end{cases}$$

From the first equation,

$$v(x, y) = \int 20(x^3 y - xy^3) \, dx + c(y) = 5x^4 y - 10x^2 y^3 + c(y).$$

Imposing the second equation we get,

$$5x^4 - 30x^2 y^2 + c'(y) = 5x^4 - 30x^2 y^2 + 5y^4, \quad \Longleftrightarrow \quad c'(y) = 5y^4, \quad \Longleftrightarrow \quad c(y) = y^5 + c,$$

where $c \in \mathbb{R}$. Conclusion: the unique possible v such that $u + iv$ be $\mathbb{C}-$differentiable on $\mathbb{C}$ is

$$v(x, y) = 5x^4 y - 10x^2 y^3 + y^5 + c.$$

ii) We have

$$\begin{aligned} f(x+iy) &= u(x,y)+iv(x,y) = x^5-10x^3y^2+5xy^4+i(5x^4y-10x^2y^3+y^5+c) \\ &= x^5+i5x^4y-10x^3y2+i10x^2y^3+5xy^4+iy^5+ic \\ &= (x+iy)^5+ic, \end{aligned}$$

from which $f(z)=z^5+ic$. □

Exercise 57. We have $f(x,x)=2x^4 \longrightarrow +\infty$ when $(x,x)\to\infty_2$. So, if a limit exists it must be $=+\infty$. Using polar coordinates

$$f(x,y)=\rho^4\left(\cos^4\theta+\sin^4\theta\right)-2\rho^2(\cos\theta-\sin\theta)^2 \geqslant \rho^4\left(\cos^4\theta+\sin^4\theta\right)-8\rho^2.$$

Now, let $g(\theta):=\cos^4\theta+\sin^4\theta$. g is positive valued and continuous on $[0,2\pi]$. By the Weierstrass theorem there exists a global minimum, that is $\exists\theta_{min}$ such that $g(\theta)\geqslant g(\theta_{min})$ $\forall\theta\in[0,2\pi]$. Let $C:=g(\theta_{min})\geqslant 0$. If $C=0$, $\cos^4\theta_{min}+\sin^4\theta_{min}=0$, that is $\cos\theta_{min}=\sin\theta_{min}=0$, which is impossible. So,

$$f(x,y)\geqslant C\rho^4-8\rho^2 \longrightarrow +\infty,\ \text{when } \rho=\|(x,y)\|\longrightarrow+\infty.$$

We conclude that $\exists\lim_{(x,y)\to\infty_2} f(x,y)=+\infty$.

ii) f is a polynomial, hence it is differentiable on $\mathbb{R}^2$ and

$$\nabla f(x,y)=\left(4x^3-4(x-y),4y^3+4(x-y)\right).$$

A point (x,y) is a stationary point iff

$$\nabla f(x,y)=\vec{0}, \iff \begin{cases} 4x^3-4(x-y)=0, \\ 4y^3+4(x-y)=0, \end{cases} \iff \begin{cases} x-y=x^3, \\ y^3=-x^3. \end{cases}$$

From the second equation we get $y=-x$, and plugging into the first one we get

$$x^3=2x, \iff x(x^2-2)=0, \iff x=0,\pm\sqrt{2}.$$

The stationary points are $(0,0)$, $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$. To classify them we compute the Hessian matrix:

$$\nabla^2 f(x,y)=\begin{bmatrix} 12x^2-4 & 4 \\ 4 & 12y^2-4 \end{bmatrix}$$

From this we see that

$$\nabla^2 f(0,0)=\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix},\quad \nabla^2 f(\pm\sqrt{2},\mp\sqrt{2})=\begin{bmatrix} 20 & 4 \\ 4 & 20 \end{bmatrix}$$

The diagonal sub-determinants of $\nabla^2 f(0,0)$ are $-4,0$ so $\nabla^2 f(0,0)\leqslant 0$ but not <0. It is not possible to conclude anything from the hessian matrix. However, we may notice that $f(x,x)=2x^4$, so $(0,0)$ is a local min point along $y=x$, while $f(x,0)=x^4-2x^2\sim_{x\to 0}-2x^2$, from which we see that $(0,0)$ is a local max point along $x=0$. This shows that $(0,0)$ is a saddle point.

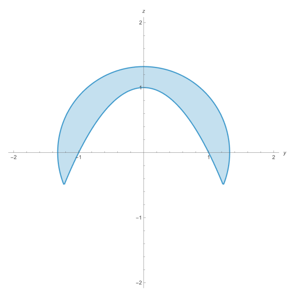
Abot $\nabla^2 f(\pm\sqrt{2}, \mp\sqrt{2})$ the diagonal sub-determinants are 20, 384, both positive, hence $\nabla^2 f(\pm\sqrt{2}, \mp\sqrt{2}) > 0$, both points are local min. points.

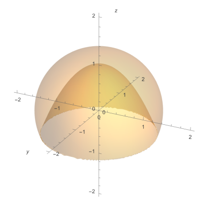
iii) By i), $\nexists \max_{\mathbb{R}^2} f$ while $\exists \min_{\mathbb{R}^2} f$. Let (x, y) be any min point. Since $\mathbb{R}^2$ is open, $(x, y) \in \mathrm{Int}(\mathbb{R}^2)$. According to the Fermat theorem, $\nabla f(x, y) = \vec{0}$, so (x, y) is one of $(0, 0)$, $(\pm\sqrt{2}, \mp\sqrt{2})$. By ii) the unique possibilities are $(\pm\sqrt{2}, \mp\sqrt{2})$, and since $f(\pm\sqrt{2}, \mp\sqrt{2}) = 8$, we conclude that both points are global min points for f. □

Exercise 58. i) We have

$$D \cap \{x = 0\} = \{(0, y, z) \in \mathbb{R}^3 \ : \ z \geqslant 1 - y^2,\ y^2 + z^2 \leqslant \tfrac{7}{4}\}.$$

In the plane yz, this is the part of plane inside the disk $y^2 + z^2 \leqslant \frac{7}{4}$ and above the parabola $z = 1 - y^2$.





D is then the rotation solid obtained by rotation of $D \cap \{x = 0\}$ around the $z-$axis.

ii) We have

$$\lambda_3(D) = \int_D 1 \ dxdydz.$$

Using standard cylindrical coordinates we get

$$(x, y, z) \in D, \quad \Longleftrightarrow \quad z \geqslant 1 - \rho^2,\ \rho^2 + z^2 \leqslant \tfrac{7}{4},\ \theta \in [0, 2\pi].$$

Therefore

$$\lambda_3(D) = \int_{z \geqslant 1-\rho^2,\ \rho^2+z^2 \leqslant \frac{7}{4},\ 0 \leqslant \theta \leqslant 2\pi} \rho \ d\rho d\theta dz \overset{RF}{=} 2\pi \int_{z \geqslant 1-\rho^2,\ \rho^2+z^2 \leqslant \frac{7}{4}} \rho \ d\rho dz.$$

Here we have two possible ways to compute the integral, by integrating first in ρ and then in z or vice versa. Here, we choose to do the first integration in ρ. We notice that

$$z \geqslant 1 - \rho^2,\ \rho^2 + z^2 \leqslant \tfrac{7}{4}, \quad \Longleftrightarrow \quad 1 - z \leqslant \rho^2 \leqslant \tfrac{7}{4} - z^2.$$

Notice that $1 - z \leqslant \frac{7}{4} - z^2$ iff $z^2 - z - \frac{3}{4} \leqslant 0$ or $4z^2 - 4z - 3 \leqslant 0$. This happens iff $-\frac{1}{2} \leqslant z \leqslant \frac{3}{2}$. Actually, by the sphere constraint, $z^2 \leqslant \frac{7}{4}$, that is $|z| \leqslant \frac{\sqrt{7}}{2} < \frac{3}{2}$, so $-\frac{1}{2} \leqslant z \leqslant \frac{\sqrt{7}}{2}$. For such z, $1 - z < 0$ for $z > 1$. Therefore,

$$1 - z \leqslant \rho^2 \leqslant \frac{7}{4} - z^2, \quad \Longleftrightarrow \quad \begin{cases} -\frac{1}{2} \leqslant z \leqslant 1, & \sqrt{1 - z} \leqslant \rho \leqslant \sqrt{\frac{7}{4} - z^2}, \\ \\ 1 \leqslant z \leqslant \frac{\sqrt{7}}{2}, & 0 \leqslant \rho \leqslant \sqrt{\frac{7}{4} - z^2}. \end{cases}$$

Consequently,

$$\begin{aligned}
\int_{z\geqslant 1-\rho^2,\ \rho^2+z^2\leqslant \frac{7}{4}} \rho\, d\rho dz &= \int_{-1/2}^{1}\int_{\sqrt{1-z}}^{\sqrt{\frac{7}{4}-z^2}} \rho\, d\rho\, dz + \int_{1}^{\sqrt{7}/2}\int_{0}^{\sqrt{\frac{7}{4}-z^2}} \rho\, d\rho\, dz \\
&= \int_{-1/2}^{1} \left[\frac{\rho^2}{2}\right]_{\rho=\sqrt{1-z}}^{\rho=\sqrt{\frac{7}{4}-z^2}} dz + \int_{1}^{\sqrt{7}/2} \left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sqrt{\frac{7}{4}-z^2}} dz \\
&= \frac{1}{2}\left(\int_{-1/2}^{1}\left(\tfrac{7}{4}-z^2-(1-z)\right) dz + \int_{1}^{\sqrt{7}/2} \tfrac{7}{4}-z^2\, dz\right) \\
&= \frac{1}{2}\left(\int_{-1/2}^{\sqrt{7}/2} \tfrac{7}{4}-z^2\, dz - \int_{-1/2}^{1}(1-z)\, dz\right) \\
&= \tfrac{1}{2}\left(\tfrac{7}{4}\tfrac{\sqrt{7}+1}{2} - \tfrac{1}{3}\left[z^3\right]_{z=-1/2}^{z=\sqrt{7}/2} - \left(\tfrac{3}{2}-\tfrac{1}{2}\left[z^2\right]_{z=-1/2}^{z=1}\right)\right) \\
&= \tfrac{7}{48}\left(2\sqrt{7}-1\right)
\end{aligned}$$

from which $\lambda_3(D) = \frac{7}{24}\left(2\sqrt{7}-1\right)\pi$. □

Exercise 59. i) The equation has form $y' = f(t, y)$ where

$$f(t, y) := \log(1 - ty).$$

The domain of definition for f is

$$D := \{(t, y) \in \mathbb{R}^2 \ :\ 1 - ty > 0\}.$$

Now,

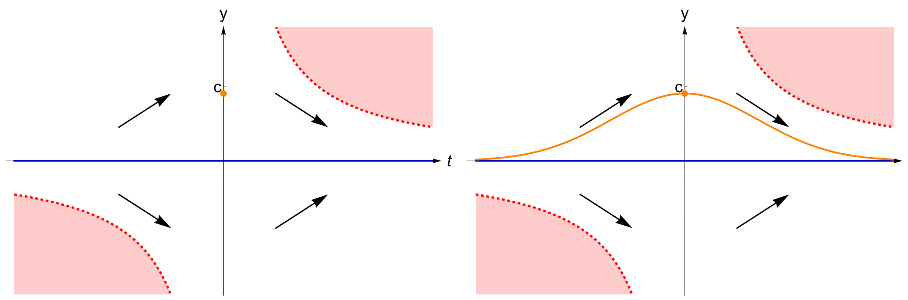
$$1 - ty > 0, \iff ty < 1, \iff \begin{cases} t > 0,\ y < \frac{1}{t}, \\ t = 0,\ \forall y, \\ t < 0,\ y > \frac{1}{t}. \end{cases}$$

On its natural domain f is continuous and also $\partial_y f = \frac{-t}{1-ty}$ is continuos on D. We conclude that local existence and uniqueness holds on D.

ii) $y \equiv C$ is a solution iff $0 = \log(1 - tC)$, that is $1 - tC \equiv 1$, or $tC \equiv 0$. Since C is a constant and t is variable, the previous holds iff $C = 0$, so $y \equiv 0$ is the unique constant solution. Moreover

$$0 \leqslant y' = \log(1 - ty), \iff 1 - ty \geqslant 1, \iff ty \leqslant 0, \iff t \leqslant 0,\ y \geqslant 0, \text{ or } t < 0,\ y \geqslant 0.$$

We get the following picture.



iii) We show that $y(t) > 0$ for every t. Indeed, if there exists a t^* such that $y(t^*) \leqslant 0$ then

- if $y(t^*) = 0$, y would cross the constant solution. By uniqueness, $y(t) \equiv 0$ so also $y(0) = 0$, but $y(0) = c > 0$, so we get a contradiction.
- if $y(t^*) < 0$ then, by continuity, there would be a t^{**} such that $y(t^{**}) = 0$, so we fall back to the previous contradiction.

Since $y(t) > 0$ for every t, by ii) we deduce that $y \nearrow$ for $t < 0$ and $y \searrow$ for $t > 0$.

iv) Since $y \searrow$ for $t > 0$, there exists $\ell := \lim_{t \to \beta} y(t)$. Since $y > 0$ it follows that $\ell \geqslant 0$. We check that $\beta = +\infty$. If $\beta < +\infty$, we claim that $\ell < \frac{1}{\beta}$. Indeed: $1 - ty(t) > 0$ so, letting $t \longrightarrow \beta$, $1 - \beta\ell \geqslant 0$. If $1 - \beta\ell = 0$, then

$$y'(t) \longrightarrow \log(1 - \beta\ell) = \log 0+ = -\infty,$$

but then the solution would have a vertical asymptote at β. This is impossible being $y(t) > 0$ for every t. Thus $1 - \beta\ell > 0$, that is the final point $(\beta, \ell) \in D$, and this is not possible by the argument of exit from compact sets. Contradiction.

Since all this follows by the assumption $\beta < +\infty$ we conclude that $\beta = +\infty$. We claim that $\ell = 0$. If $\ell > 0$, then $1 - ty(t) \to 1 - (+\infty)\ell = -\infty$, so for t large $1 - ty(t) < 0$, impossible for a solution.

Similarly, we can prove that $\alpha = -\infty$ and $\lim_{t \to \alpha} y(t) = 0$.

v) Figure: see above. □

Exercise 60. We can set the problem as a constrained optimization problem. Let $f(\vec{x}) := (\vec{a} \cdot \vec{x})^2 = \left(\sum_{j=1}^{d} a_j x_j\right)^2$. We see that f is polynomial, hence it is continuous. The optimization domain

$$D := \{\vec{x} \in \mathbb{R}^d \ : \ \|\vec{x}\| = 1\} = \left\{\vec{x} \in \mathbb{R}^d \ : \ \sqrt{\sum_{j=1}^{d} x_j^2} = 1\right\}$$

is closed (defined by an equation on a continuous function) and bounded (obviously because $\|\vec{x}\| = 1$ on D). Thus, D is compact, so f attains both min and max on D because of the Weierstrass theorem. To determine these points we apply the Lagrange multipliers theorem. Before we start, to simplify calculations it is convenient to notice that

$$D = \{\vec{x} \in \mathbb{R}^d \ : \ x_1^2 + \cdots + x_d^2 = 1\}.$$

We notice that $D = \{g = 0\}$ where $g(x_1, \ldots, x_d) = x_1^2 + \cdots + x_d^2 - 1$, and g is a submersion on D. This because g is not a submersion iff $\nabla g = 0$. Since $\nabla g(\vec{x}) = 2\vec{x}$, we have

$$\nabla g(\vec{x}) = \vec{0}, \quad \Longleftrightarrow \quad 2\vec{x} = \vec{0}, \quad \Longleftrightarrow \quad \vec{x} = \vec{0} \notin D.$$

Now, at any min/max point for f on D we have

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x}).$$

We have

$$\partial_k f(\vec{x}) = \partial_k (\vec{a}\cdot\vec{x})^2 = 2(\vec{a}\cdot\vec{x})\partial_k(\vec{a}\cdot\vec{x}),$$

and since

$$\partial_k \vec{a}\cdot\vec{x} = \partial_k \sum_{j=1}^{d} a_j x_j = a_k,$$

we conclude that

$$\nabla f(\vec{x}) = 2(\vec{a}\cdot\vec{x})\vec{a}.$$

So,

$$\nabla f(\vec{x}) = \lambda\nabla g(\vec{x}), \iff 2(\vec{a}\cdot\vec{x})\vec{a} = 2\lambda\vec{x}, \iff \lambda\vec{x} = (\vec{a}\cdot\vec{x})\vec{a}.$$

This equation for $\vec{x}$ yields the following alternatives:

- $\lambda = 0$: in this case $(\vec{a}\cdot\vec{x})\vec{a} = \vec{0}$, and since $\vec{a} \neq \vec{0}$, this is possible iff $\vec{a}\cdot\vec{x} = 0$, that is $\vec{x} \perp \vec{a}$. For these $\vec{x}$ we have $f(\vec{x}) = (\vec{a}\cdot\vec{x})^2 = 0$.
- $\lambda \neq 0$: in this case $\vec{x} = \mu\vec{a}$ (where we set $\mu := \frac{\vec{a}\cdot\vec{x}}{\lambda} \in \mathbb{R}$). Imposing $\vec{x} \in D$ we have $\|\mu\vec{a}\| = 1$ that is $|\mu|\|\vec{a}\| = 1$, from which

$$\mu = \pm\frac{1}{\|\vec{a}\|}, \implies \vec{x} = \pm\frac{\vec{a}}{\|\vec{a}\|}.$$

  In this case

$$f(\vec{x}) = \left(\vec{a}\cdot \pm\frac{\vec{a}}{\|\vec{a}\|}\right)^2 = \left(\pm\frac{\|\vec{a}\|^2}{\|\vec{a}\|}\right)^2 = \|\vec{a}\|^2 > 0.$$

Conclusion: the maximum is attained at $\vec{x} = \pm\frac{\vec{a}}{\|\vec{a}\|}$ with value $\|\vec{a}\|^2$.

Alternative solution. This is based on the Cauchy-Schwarz inequality and does not require any use of differential calculus. We recall that

$$|\vec{x}\cdot\vec{y}| \leqslant \|\vec{x}\|\|\vec{y}\|, \ \forall \vec{x},\vec{y} \in \mathbb{R}^d.$$

So, if $\|\vec{x}\| = 1$,

$$(\star)\ |\vec{a}\cdot\vec{x}| \leqslant \|\vec{a}\|\|\vec{x}\| \leqslant \|\vec{a}\|, \implies (\vec{a}\cdot\vec{x})^2 = |\vec{a}\cdot\vec{x}|^2 \leqslant \|\vec{a}\|^2, \ \forall\vec{x} \ : \ \|\vec{x}\| = 1.$$

If we take $\vec{x}_\pm := \pm\frac{\vec{a}}{\|\vec{a}\|}$. These $\vec{x}_\pm$ are well defined because $\vec{a} \neq \vec{0}$ and

$$\|\vec{x}_\pm\| = \left\|\pm\frac{\vec{a}}{\|\vec{a}\|}\right\| = \frac{1}{\|\vec{a}\|}\|\vec{a}\| = 1.$$

Moreover,

$$(\star\star)\ (\vec{a}\cdot\vec{x}_\pm)^2 = \left(\vec{a}\cdot\frac{\vec{a}}{\|\vec{a}\|}\right)^2 = \frac{1}{\|\vec{a}\|^2}(\vec{a}\cdot\vec{a})^2 = \frac{1}{\|\vec{a}\|^2}\|\vec{a}\|^4 = \|\vec{a}\|^2.$$

Combining $(\star)$ and $(\star\star)$ we get that

$$\max_{\|\vec{x}\|=1\}(\vec{a}\cdot\vec{x})^2=\|\vec{a}\|^2,}$$

and the maximum is achieved at $\vec{x}_\pm = \pm\frac{\vec{a}}{\|\vec{a}\|}$. Are there other max points? If $\vec{x}^*$ is any max point s.t. $\|\vec{x}^*\| = 1$, then

$$(\vec{a} \cdot \vec{x}^*)^2 = \|\vec{a}\|^2.$$

If this happens, it means that the CS is actually an equality, and this happens iff $\vec{x}^* = \lambda\vec{a}$. Imposing $\|\vec{x}^*\| = 1$ we get $|\lambda|\|\vec{a}\| = 1$, that is $\lambda = \pm\frac{1}{\|\vec{a}\|}$, that is $\vec{x}^*$ is either $\vec{x}_+$ or $\vec{x}_-$. □

□