Lezione17b

Matrice associata

① $f: V \longrightarrow W$ lineare

② $B_V = \{v_1, \dots, v_n\}$ base di V $\Longrightarrow A_{B_V, B_W, f} \in M_{m,n}(\mathbb{R})$

③ $B_W = \{w_1, \dots, w_m\}$ base di W

$v_1 \quad f(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$

$\vdots$

$v_n \quad f(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$

$$\begin{array}{ccc} f(v_1) & \dots & f(v_n) \end{array}$$
$$\begin{pmatrix} a_{11} & & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} = A$$

Osservazione:

Se teniamo fissa f e B_W e cambiamo la base del dominio con "operazioni elementari" la matrice ottenuta sarà quella ottenuta agendo con la stessa operazione elementare sulle colonne.

Se teniamo fissa f e B_V e cambiamo la base del codominio con operazioni elementari la matrice ottenuta sarà quella ottenuta agendo con l'operazione elementare inversa sulle righe.

Esempio:

① $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

② $B_V = \{e_1, e_2\}$ $\Longrightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\frac{1}{3} \cdot 1°$ Riga

③ $B_W = \{e_1, e_2\}$

$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$

① $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

② $B_V = \{e_1, e_2\}$ $\Longrightarrow$ $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_{11}\begin{pmatrix} 3 \\ 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$

③ $B'_W = \{3e_1, e_2\} = \{\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3a_{11} \\ a_{21} \end{pmatrix} \quad a_{11} = 1/3, \; a_{21} = 0$

② $B_V = \{e_1, e_2\}$

③ $B' = \{3e_1, e_2\} = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3a_{11} \\ a_{21} \end{pmatrix}$ $\quad a_{11} = 1/3$, $a_{21} = 0$

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_{12} w_1 + a_{22} w_2 = a_{12}\begin{pmatrix} 3 \\ 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3a_{12} + \cancel{0 a_{22}} \\ a_{22} \end{pmatrix}$ $\quad \begin{cases} a_{12} = 0 \\ a_{22} = 1 \end{cases}$

Esempio: $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 3y \\ x - y \end{pmatrix}$ $\quad \begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix} = A_{\mathcal{E},\mathcal{E},f}$

Determinare la matrice $A_{B_V, B_W}(f) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$B_V = \{e_1, e_2\}$

$B_W = \{3e_1, e_2\}$

$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Definizione: data $A \in M_{m,n}(\mathbb{R})$ si chiama

rango righe di A = dimensione del sottospazio generato dalle righe di A $= rg_r(A)$

rango colonne di A = dimensione del sottospazio generato dalle colonne di A $= rg_c(A)$

FATTO: $rg_r(A) = rg_c(A) = n°$ pivot di una matrice a scala ottenuta riduci con Gauss a partire da A.

Se A è in forma a scala $\quad rg_r(A) = rg_c(A) = n°$ pivot

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A$$

$rg_r(A) = \dim \langle (1\ 2\ 0\ 4), (0\ 0\ 1\ 3), (0\ 0\ 0\ 1), \cancel{(0\ 0\ 0\ 0)} \rangle = 3$

$rg_c(A) = \dim \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \cancel{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 3$

Sia $A \in M_{m,n}(\mathbb{R})$

① $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

② $B_{\mathbb{R}^n} = \{e_1, \dots, e_n\}$

③ $B_{\mathbb{R}^m} = \{e_1, \dots, e_m\}$

$$\boxed{f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$$

$A = (A_1\ A_2 \dots A_n)$

$\dim \operatorname{Im} f = rg_c A$

$\operatorname{Im} f = \langle f(e_1), \dots, f(e_n) \rangle$
$= \langle A_1, \dots, A_n \rangle$

Se operiamo con operazioni elementari sulle righe di A corrisponde a cambiare base in $\mathbb{R}^m$

$rg_c A = rg_c S$ con S matrice ridotta a scala a partire dalla matrice A.

$ \parallel$

$rg_r S = rg_r A$ perché op. elem. sulle righe non cambiano il sottosp. generato dalle righe.

Domanda 7:

Teorema di Rouché-Capelli: Teorema R-C

Dato un sistema lineare di m equazioni in n incognite con coefficienti reali:

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$A|b = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$ matrice completa

$A \in M_{m,n}(\mathbb{R})$

$$\boxed{A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}} \qquad Ax = b$$

$m \times \cancel{n} \quad \cancel{n} \times 1 \qquad m \times 1$

① Ha soluzioni se e solo se $rg A = rg(A|b)$

② Se $rg A = rg(A|b)$ allora

$$Sol_{(A|b)} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \right\} = v + \operatorname{Ker} A \quad \text{con} \quad v \in \mathbb{R}^n$$

soluzione $Av = b$ e $\operatorname{Ker} A = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{0} \right\}$ soluzione del sistema omogeneo associato del sistema omogeneo associato $Ax=\vec{0}$. $\boxed{\dim \operatorname{Ker} A = n - \operatorname{rg} A}$

③ $Sol_{(A|b)} \leq \mathbb{R}^n$ se e solo $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

Dimostrazione:

$\boxed{\text{Sia } f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$ cioè $A_{\mathcal{E}_n,\mathcal{E}_m,f} = A$

allora $\boxed{f^{-1}\left\{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\right\}} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \right\} = \boxed{Sol_{(A|b)}}$

$$\underline{f^{-1}\left\{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\right\}} = \begin{cases} \emptyset & \text{se } \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \notin \operatorname{Im} f = \langle A_1, \dots, A_n \rangle \\ v_0 + \operatorname{Ker} f = v_0 + \operatorname{Ker} A \text{ con } f(v_0) = Av_0 = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} & \text{se } \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \operatorname{Im} f \end{cases}$$

$A = (A_1, \dots, A_n)$

① $Sol_{(A|b)} \neq \emptyset \iff \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \langle A_1, \dots, A_n \rangle \iff$

$\operatorname{rg}(A) = \dim \langle A_1, \dots, A_n \rangle = \dim \langle A_1, \dots, A_n ; b \rangle = \operatorname{rg}(A|b)$

perché $\langle A_1, \dots, A_n \rangle \subseteq \langle A_1, \dots, A_n ; b \rangle$

② Se $\operatorname{rg} A = \operatorname{rg}(A|b) \Longrightarrow Sol_{A|b} = f^{-1}\{b\} = v_0 + \operatorname{Ker} f$ con $f(v_0) = b$

$\operatorname{Ker} f = \{ v \in \mathbb{R}^n \mid f(v) = \vec{0} \} = \{ v \in \mathbb{R}^n \mid Av = \vec{0} \} = \operatorname{Ker} A$ $\qquad Av_0 = b$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$Sol_{A|b} = v_0 + \operatorname{Ker} A$ $\qquad \dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

$\boxed{\dim \operatorname{Ker} A = n - \operatorname{rg}(A)}$ $\qquad n = \dim \operatorname{Ker} A + \operatorname{rg} A$

③ Se $Sol_{(A|b)} = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; Ax = b \right\} \leq \mathbb{R}^n$ allora

$\vec{0} \in Sol_{(A|b)}$ $\qquad$ $\vec{0} = A\vec{0} = b \implies b = \vec{0}$

Se $b = \vec{0}$ $\quad Sol_{(A|b)} = \{ x \in \mathbb{R}^n \mid Ax = \vec{0} \} = \operatorname{Ker} f \leq \mathbb{R}^n$. $\square$

Esempio

Determinare un sistema lineare (dipendente dal parametro $h \in \mathbb{R}$) avente un numero minimo di equazioni con soluzioni

$$S_h: \begin{pmatrix}1\\h\\0\\0\end{pmatrix} + \underbrace{\left\langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle}_{\operatorname{Ker} A}$$

$$Sol_{A|b} = \left\{ \underbrace{\begin{pmatrix}1\\h\\0\\0\end{pmatrix} + a\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\\1\end{pmatrix}} \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1+a\\h+a+b\\b+c\\c\end{pmatrix} \qquad \begin{cases} x_1 = 1+a \\ x_2 = h+a+b \\ x_3 = b+c \\ \cancel{x_4 = c} \end{cases} \qquad \text{eliminare } \cancel{a}, b, \cancel{c}$$

$$c = x_4$$

$$\begin{cases} \cancel{x_1 = 1+a} \\ x_2 = h+a+b \\ x_3 = b+x_4 \end{cases} \qquad a = x_1 - 1 \qquad \begin{cases} x_2 = h + x_1 - 1 + b \\ \cancel{x_3 = b + x_4} \end{cases} \qquad b = x_3 - x_4$$

$$x_2 = h + x_1 - 1 + x_3 - x_4 \qquad \boxed{x_1 - x_2 + x_3 - x_4 = 1-h}$$

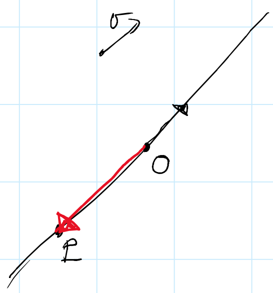
$$(1 \quad -1 \quad 1 \quad -1 \mid 1-h) \qquad \operatorname{rg} A = \operatorname{rg}(A|b) = 1$$

2) Per quali valori di $h \in \mathbb{R}$ $\quad S_h \leq \mathbb{R}^4$? $\quad$ Se e solo se $h = 1$

$$\boxed{\overset{P}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} + \left\langle \overset{v_1}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}, \overset{v_1}{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}} \right\rangle}$$

$P + \operatorname{Ker} A$

$O + \operatorname{Ker} A$



$O + \langle v \rangle = P + \langle v \rangle$

$P - O \in \langle v \rangle$

$x_1 - x_2 + x_3 - x_4 = 0 \qquad \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \begin{pmatrix}1\\1\\0\end{pmatrix}$

$x_1 - x_2 + x_3 - x_4 = 0$ $\quad \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\a\end{pmatrix}$

$$\boxed{\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle}$$

Definizione: un'applicazione lineare $f: V \to V$ si dice **endomorfismo** di V.

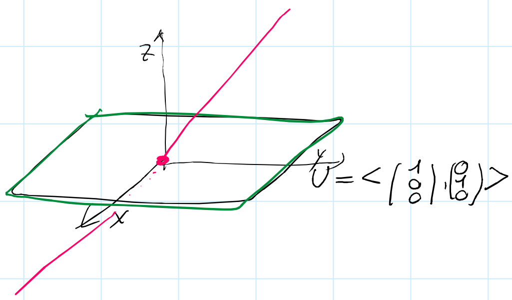
Definizione: dato V uno spazio vettoriale e $V = U \oplus W$ si chiama proiezione su U con direzione W l'applicazione lineare definita come:

$P_U^W : V \longrightarrow V \qquad v \in V \qquad v = u + w$ con $u \in U, w \in W$ (è unico)

$P_U^W(v) = u$

$V = \mathbb{R}^3 = U \oplus W = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

$\qquad\qquad 2 \qquad\qquad 1$



$$rg \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 3$$

$U = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \qquad W = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

$U: \ z = 0 \qquad W = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

$v = \begin{pmatrix}x\\y\\z\end{pmatrix} = u + w$ con $u \in U$, $w \in W$

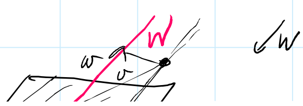
$\begin{pmatrix}x\\y\\z\end{pmatrix} = u + \begin{pmatrix}a\\a\\a\end{pmatrix} \qquad w = \begin{pmatrix}a\\a\\a\end{pmatrix} \in \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

$u = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}a\\a\\a\end{pmatrix} = \begin{pmatrix}x-a\\y-a\\z-a\end{pmatrix} \qquad z - a = 0$ perché $u \in U \qquad \boxed{a = z} \Rightarrow w = \begin{pmatrix}z\\z\\z\end{pmatrix}$

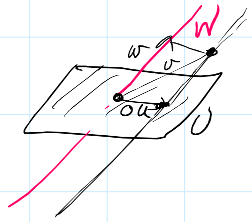
$\boxed{u = \begin{pmatrix}x-z\\y-z\\0\end{pmatrix}} \qquad P_U^W(v) = \underline{\underline{u}} \qquad P_U^W\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x-z\\y-z\\0\end{pmatrix} \Rightarrow P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

Proprietà delle proiezioni: $p = P_U^W : V \longrightarrow V \qquad V = U \oplus W$

① p è lineare



① p è lineare



② $\mathrm{Ker}\, p = W$ $\quad p(v)=\vec{0} \quad v=\vec{0}+w \in W$

③ $\mathrm{Im}\, p = U$

④ $\boxed{p\circ p = p}$ $\quad p^2 = p$

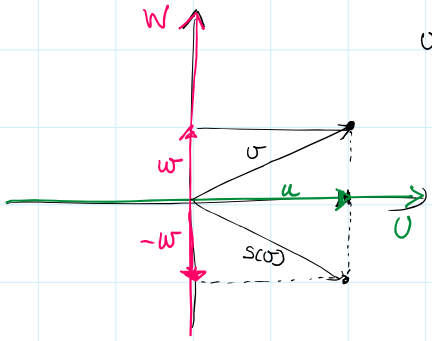
⑤ $\forall\, w\in W \quad p(w) = 0\cdot w = \vec{0}$

$\forall\, u\in U \quad p(u) = u = 1\cdot u$

$$\mathbb{R}^2 = \left\langle \binom{1}{0}\right\rangle \oplus \left\langle\binom{0}{1}\right\rangle$$
$$U \oplus W$$

$v = u + w$

$p_U^W(v) = u$



$S = S_U^W$ simmetria di asse U e direzione W

$S(v) = u - w =$ $\qquad S^2(v) = v$

$=$ $\qquad S^2 = S\circ S = id_V \qquad S^2 = \mathbb{I}_n$

$S(v) = u - w \qquad \underline{p}(v) = u \qquad \underline{v} = u+w$

$\overset{!}{=} 2u - u - w = 2u - (u+w) = 2p(v) - v$

$$\boxed{S = 2P - \mathbb{I}_n}$$