SOBOLEV SPACE IN DIMENSION 1

Definit $f \in AC(a,b)$, $(a,b) \subseteq \mathbb{R}$ — f ABSOLUTELY CONTINUOUS

$\iff$ $f \in C(a,b)$ and f is differentiable almost everywhere with $f' \in L^1(a,b)$ and

$$f(x) = f(y) + \int_y^x f'(t)\,dt \quad \forall x,y \in [a,b].$$

In particular derivative a.e = weak derivative.

(indeed let $f \in AC(a,b)$, $\forall \phi \in C_c^\infty(a,b)$

$$\int_a^b \phi'(x) f(x)\,dx = \int_a^b \phi'(x)\left[f(a) + \int_a^x f'(t)\,dt\right] dx = \cancel{\int_a^b \phi'(x) f(a)\,dx}_{=0} +$$

$$+ \int_a^b \phi'(x) \int_a^x f'(t)\,dt\,dx = \int_a^b f'(t) \int_t^b \phi'(x)\,dx\,dt =$$

$a < t < x < b$

$$= \int_a^b f'(t)\left[\phi(b) - \phi(t)\right] dt = - \int_a^b f'(t)\,\phi(t)\,dt.$$

f' is the weak derivative of f.

Theorem Let $(a,b) \subseteq \mathbb{R}$ bdd.

1) Then $W^{1,1}(a,b) = AC(a,b)$ $\quad f \in W^{1,1}(a b) \Leftrightarrow \exists \tilde{f}$ $\tilde{f} = f$ a.e $\tilde{f} \in AC(a,b)$

2) If f $f \in W^{1,p}(a b)$ $p > 1$ $(p \in (1, +\infty])$

Then $\exists \tilde{f} = f$ a.e $\tilde{f} \in AC(a,b)$, $\tilde{f} \in C^{0,1-\frac{1}{p}}(a,b)$, $p = +\infty$ $\tilde{f} \in C^{0,1}(a,b)$ (Lipschitz)

$W^{1,p}(a,b) \subseteq AC(a,b) \cap C^{0,1-\frac{1}{p}}(a,b)$

$\exists C > 0$ such that

3) $\|f\|_\infty \leq C \|f\|_{W^{1,p}}$ and $\|f\|_{C^{0,1-\frac{1}{p}}} \leq C \|f\|_{W^{1,p}}$
$\forall p \in [1, +\infty]$

4) Finally $f \in W^{1,p}(\mathbb{R}) \Rightarrow f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, f absolutely continuous. (moreover $f \in C^{0,1-\frac{1}{p}}(a,b)$ $\forall (a,b) \subset\subset \mathbb{R}$)

PROOF $f \in AC(a,b) \rightarrow f \in W^{1,1}(a b)$ by definition

$f \in W^{1,p}(a b) \Rightarrow v$ weak derivative $\in L^p(a,b)$
$p \in [1, +\infty]$ $\quad \int_a^b f \phi' dx = -\int_a^b v \phi \, dx \quad \forall \phi \in C_c^\infty(a,b)$

Define $g(x) := \int_a^x v(t)\,dt \qquad g \in \mathcal{C}(a,b) \quad g(a)=0$

$$\int_a^b [f(x)-g(x)]\,\phi'(x)\,dx = \int_a^b f(x)\phi'(x)\,dx - \int_a^b \int_a^x v(t)\,dt\,\phi'(x)\,dx$$

$a<t<x<b$

$$= -\int_a^b v(x)\phi(x)\,dx - \int_a^b v(t)\int_t^b \phi'(x)\,dx\,dt =$$

$$= -\int_a^b v(x)\,\phi(x)\,dx - \int_a^b v(t)\left[\cancel{\phi(b)} - \phi(t)\right]dt = -\int_a^b v(x)\phi(x)\,dx +$$

$$+\int_a^b v(t)\phi(t)\,dt = 0$$

$\forall\, \phi \in \mathcal{C}_c^\infty(a,b) \quad \int_a^b [f(x)-g(x)]\,\phi'(x)\,dx = 0 \implies \exists\, c \in \mathbb{R}$ such that $f(x) = g(x)+c$ a.e. $\quad \tilde{f}(x) = c + g(x) \in \mathcal{C}[a,b]$

$\tilde{f}(a) = c$

$$\tilde{f}(x) = \tilde{f}(a) + \int_a^x v(t)\,dt \implies \tilde{f}(x) = \tilde{f}(y) + \int_y^x v(t)\,dt \quad \forall x,y$$

Moreover $v = \tilde{f}'$ a.e. $\quad \tilde{f} \in AC(a,b)$.

$p>1 \quad |\tilde{f}(y) - \tilde{f}(x)| \le \int_y^x |\tilde{f}'| \, dt \underset{\text{HÖLDER}}{\le} \|\tilde{f}'\|_{L^p} \cdot |x-y|^{1-\frac{1}{p}}$

$(\tilde{f}', \chi_{(x,y)})$ $\quad \forall x, y \in [a,b]$.

$p=1 \quad |\tilde{f}(y) - \tilde{f}(x)| \le \int_y^x |\tilde{f}'| \, dt \le \|\tilde{f}'\|_{L^1} \cdot 1$.

$p > 1 \quad \tilde{f} \in C^{0, 1-1/p}(a,b)$

Let $\bar{x}$ such that $|\tilde{f}(\bar{x})| = \min_{x \in [a,b]} |\tilde{f}(x)|$

$$|\tilde{f}(\bar{x})| = \frac{1}{b-a} \int_a^b |\tilde{f}(\bar{x})| \, dt \le \frac{1}{(b-a)} \int_a^b |\tilde{f}(t)| \, dt \le \frac{\|f\|_p \cdot (b-a)^{1-\frac{1}{p}}}{(b-a)}$$

$$|\tilde{f}(x)| = |\tilde{f}(\bar{x}) + \int_{\bar{x}}^x \tilde{f}'(t) \, dt| \le |\tilde{f}(\bar{x})| + \int_a^b |\tilde{f}'(t)| \, dt \le (\text{Hölder})$$

$$\le \|f\|_{L^p} \cdot (b-a)^{-1/p} + \|\tilde{f}'\|_{L^p} (b-a)^{1-\frac{1}{p}} \le C \|f\|_{W^{1,p}(a,b)}$$

therefore $f \in W^{1,p}(a,b) \Longrightarrow f \in AC(a,b)$ (up to choosing a representative)

Moreover $\exists\, C>0$ s. that $\|f\|_\infty \leq C\|f\|_{W^{1,p}}$ $\forall f \in W^{1,p}(a,b)$

and if $p>1$ $f \in C^{0,1-\frac{1}{p}}(a,b)$ $\left(1-\frac{1}{p}=1 \text{ if } p=+\infty\right)$

and $\|f\|_{C^{0,1-\frac{1}{p}}(a,b)} = \|f\|_\infty + \sup\limits_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{1-\frac{1}{p}}} \leq C\|f\|_{W^{1,p}} + C\|f'\|_{L^p} \leq \bar{C}\|f\|_{W^{1,p}}$

PART (3)

Let $f \in W^{1,p}(\mathbb{R})$ $\rightarrow$ $\begin{cases} f \in C^{0,1-\frac{1}{p}}(a,b) \ \forall (a,b) \subset\subset \mathbb{R} & p>1 \\ f \in AC(a,b) \ \forall (a,b) \subset\subset \mathbb{R} & p=1 \end{cases}$ $\Rightarrow f \in C(\mathbb{R})$

Fix $x \in \mathbb{R}$ ~~let~~ $x_1 \in [x-1,x+1]$ such that $|f(x_1)| = \min\limits_{[x-1,x+1]} |f(y)|$

$$|f(x)| = \left|f(x_1) + \int_{x_1}^{x} f'(y)\,dy\right| \leq |f(x_1)| + \int_{x-1}^{x+1} |f'(y)|\,dy \leq$$

$$\leq \frac{1}{2}\int_{x-1}^{x+1} |f(y)|\,dy + \int_{x-1}^{x+1} |f'(y)|\,dy \underset{\text{HÖLDER}}{\leq} \frac{1}{2}\|f\|_{L^p}\, 2^{1-\frac{1}{p}} + \|f'\|_{L^p}\, 2^{1-1/p} \leq C\|f\|_{W^{1,p}}$$

$f \in L^\infty(\mathbb{R})$

Observe that

$C^{0,1-\frac{1}{p}}(a,b) \not\subseteq W^{1,p}(a,b)$ in general

ex $f(x)=\sqrt{x} \in C^{0,\frac{1}{2}}(0,1)$

but $f'(x) = \dfrac{1}{2\sqrt{x}} \notin L^2 \Rightarrow f \notin W^{1,2}(0,1)$

$C^{0,1-\frac{1}{p}}(a,b) \not\subseteq AC(a,b)$ in general

(ex Cantor function is Hölder continuous but not absolutely continuous).

IMPORTANT CONSEQUENCE OF THE THEOREM

$$W^{1,p}(a,b), \|\cdot\|_{1,p} \hookrightarrow \mathcal{C}([a,b], \|\cdot\|_\infty)$$

$$f \longmapsto f$$

$$\|f\|_\infty \leq C\|f\|_{W^{1,p}}$$ CONTINUOUS EMBEDDING

Moreover the immersion is compact for $p>1$

$$W^{1,p}(a,b), \|\cdot\|_{1,p} \xhookrightarrow{\text{CONTINUOUS}} \left(\mathcal{C}^{0,1-\frac{1}{p}}(a,b), \|\cdot\|_{0,1-\frac{1}{p}}\right) \xhookrightarrow{\text{COMPACT}} \mathcal{C}([a,b], \|\cdot\|_\infty)$$

$$f \in W^{1,p}(a,b) \to f \in \mathcal{C}^{0,1-\frac{1}{p}}(a,b) \qquad \left(\begin{array}{l} |f(x)-f(y)| \leq \|f'\|_{L^p} |x-y|^{1-\frac{1}{p}} \\ \|f\|_\infty \leq C\|f\|_{W^{1,p}} \end{array} \right)$$

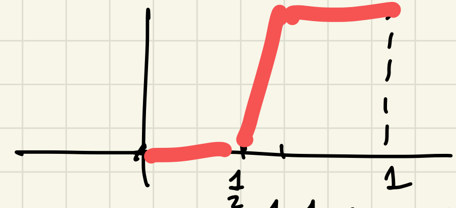
$$\|f\|_{0,1-\frac{1}{p}} \leq C'\|f\|_{W^{1,p}}$$

$$\left(W^{1,p}(a,b), \|\cdot\|_{1,p}\right) \xhookrightarrow{\text{COMPACTLY}} \mathcal{C}([a,b], \|\cdot\|_\infty) \qquad p>1.$$

$p = 1$? $\quad (W^{1,1}(a,b), \|\cdot\|_{1,1}) \hookrightarrow (\mathcal{C}[a,b], \|\cdot\|_\infty)$

CONTINUOUSLY

but NOT COMPACT



Ex $\quad f_k(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ k(x-\frac{1}{2}) & \frac{1}{2} \leq x < \frac{1}{2}+\frac{1}{k} \\ 1 & x \geq \frac{1}{2}+\frac{1}{k} \end{cases}$

$f_k \in W^{1,1}(0,1)$

Indeed $f_k \in L^1(0,1)$ $\quad f_k' = \begin{cases} 0 & x < \frac{1}{2} \\ k & \frac{1}{2} < x < \frac{1}{2}+\frac{1}{k} \\ 0 & x > \frac{1}{2}+\frac{1}{k} \end{cases}$

$f_k' \in L^1(0,1)$

$$\|f_k\|_{W^{1,1}(0,1)} = \|f_k\|_{L^1} + \|f_k'\|_{L^1} = \left(\frac{1}{2} + \frac{1}{2k}\right) + 1 \leq 2 \quad \forall k$$

$f_k \longrightarrow \chi_{(\frac{1}{2},1)}$ in $L^1(0,1)$

$\chi_{(\frac{1}{2},1)} \notin W^{1,1}(0,1)$ indeed the derivative in the sense of distributions of $\chi_{(\frac{1}{2},1)}$ is $\delta_{\frac{1}{2}}$.

Obs that $W^{1,p}(\mathbb{R}) \hookrightarrow \tilde{L}^{\infty}(\mathbb{R})$ continuous embedding

but NEVER COMPACT

$\phi \in C_c^{\infty}(\mathbb{R}) \subseteq W^{1,p}(\mathbb{R})$

$\phi_n(x) := \phi(x-n) \qquad \|\phi_n\|_{W^{1,p}} = \|\phi\|_{W^{1,p}} \quad \forall n$

but ϕ_n is not converging to 0 in L^{∞} $\left(\|\phi_n\|_{\infty} = \|\phi\|_{\infty} \neq 0\right)$

Finally for $n>1$ we cannot expect that

$f \in W^{1,p}(U)$ $\quad U \subseteq \mathbb{R}^n$ is CONTINUOUS.

Ex 1 Take $f(x) = \frac{1}{|x|^r}$ $\quad f \in W^{1,p}(U)$ $\forall\, U$ bdd in $\mathbb{R}^n$

$\iff$ $rp < n$ $\quad$ $p<n$
$(r+1)p < n \iff$ $\quad$ $r < \frac{n-p}{p}$

Ex 2 take $f(x) = \lg\left(\lg\left(1+\frac{1}{|x|}\right)\right) \in W^{1,n}(U)$ $\quad \forall\, U \subseteq \mathbb{R}^n$ bdd.

$$|\nabla f(x)| \simeq \left| \frac{1}{\lg \frac{|x|+1}{|x|}} \cdot \frac{1}{\frac{|x|+1}{|x|}} \cdot \left(-\frac{1}{|x|^2}\right) \cdot \frac{|x|}{|x|} \right| = \frac{1}{|\lg(|x|+1) - \lg|x||} \, \frac{\cancel{|x|}}{(|x|+1)} \, \frac{1}{|x|^2}$$

for $|x| \to 0$ $\quad |\nabla f(x)| \simeq \frac{1}{\lg|x|} \, \frac{1}{|x|}$

$$\frac{1}{\lg|x|}\,\frac{1}{|x|} \in L^n\left(B\left(0,\tfrac{1}{2}\right)\right)$$

$$\int_{B(0,\frac{1}{2})} \frac{1}{(\lg|x|)^n}\,\frac{1}{|x|^n}\,dx = \int_0^{1/2} \frac{1}{|\lg r|^n}\,\frac{1}{r^n}\, n\,\omega_n r^{n-1}\,dr$$

$\downarrow$ integration on spheres

$$= \int_0^{1/2} \frac{1}{|\lg r|^n}\, n\omega_n \frac{1}{r}\,dr = \int_{-\infty}^{\lg(\frac{1}{2})} \frac{1}{|s|^n}\, n\omega_n\, ds < +\infty$$

$\downarrow s = \lg r$

since $n > 1$

$W^{1,p}(U)$ $p \leq n$ contains NON CONTINUOUS FUNCTIONS (with singularity.)

$W^{1,p}(U)$ $p > n$ are $\left(1-\frac{n}{p}\right)$ Hölder continuous functions.