Foglio 4

ES1

$f: \mathbb{R}^3 \to \mathbb{R}^3$ lineare t.c. $\begin{cases} f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \\ f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\1\end{pmatrix} \\ f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} \end{cases}$

$f\begin{pmatrix}1\\1\\1\end{pmatrix} = ?$ $\quad f\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow f\begin{pmatrix}1\\1\\1\end{pmatrix} = f\begin{pmatrix}1\\0\\0\end{pmatrix} + f\begin{pmatrix}0\\1\\0\end{pmatrix} + f\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}0\\2\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\4\\2\end{pmatrix}$$

$$\begin{pmatrix}-1\\1\\-1\end{pmatrix} = (-1)\begin{pmatrix}1\\0\\0\end{pmatrix} + (1)\begin{pmatrix}0\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow f\begin{pmatrix}-1\\1\\-1\end{pmatrix} = (-1)\cdot f\begin{pmatrix}1\\0\\0\end{pmatrix} + (1)\cdot f\begin{pmatrix}0\\1\\0\end{pmatrix} + (-1) f\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= (-1)\cdot\begin{pmatrix}1\\2\\1\end{pmatrix} + 1\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} + (-1)\begin{pmatrix}-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

ES 3

$$\begin{matrix}1&0&0&-1\\-1&0&1&0\\1&1&0&0\\1&1&1&0\end{matrix} \xrightarrow[\substack{III-I\\IV-I}]{II+I} \begin{matrix}1&0&0&-1\\0&0&1&-1\\0&1&0&1\\0&0&0&+1\end{matrix} \to \begin{matrix}1&0&0&-1\\0&1&0&1\\0&0&1&-1\\0&0&0&+1\end{matrix}$$

$\to$ forma a scala

$rk = 4$

$\Rightarrow$ base di $\mathbb{R}^4$

$f\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

$f\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\1\\1\end{pmatrix}$

$f\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\\2\end{pmatrix}$

$f\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}$

$$\alpha\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + \beta\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \delta\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$\Rightarrow$ risolvo il sistema: $\left(\begin{array}{cccc|c}1&-1&1&1&0\\0&0&1&1&0\\0&1&0&1&1\\-1&0&0&0&0\end{array}\right)$

$$\left(\begin{array}{cccc|c}1&-1&1&1&0\\0&1&0&1&1\\0&0&1&1&0\\0&-1&1&1&0\end{array}\right) \to \left(\begin{array}{cccc|c}1&-1&1&1&0\\0&1&0&1&1\\0&0&1&1&0\\0&0&1&2&1\end{array}\right) \to \left(\begin{array}{cccc|c}1&-1&1&1&0\\0&1&0&1&1\\0&0&1&1&0\\0&0&0&1&1\end{array}\right)$$

$$\Rightarrow \begin{cases}\delta = 1\\ \gamma = -\delta = -1\\ \beta = 1-\delta = 1-1 = 0\\ \alpha = \beta-\gamma-\delta = 0-(-1)-1 = 0\end{cases}$$

$$\Rightarrow f\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 0 f\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + 0\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + (-1) f\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + (+1) f\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$$= 0\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + 0\begin{pmatrix}0\\2\\1\\1\end{pmatrix} + (-1)\begin{pmatrix}-1\\0\\0\\2\end{pmatrix} + (+1)\begin{pmatrix}1\\2\\1\\0\end{pmatrix} = \begin{pmatrix}+2\\+2\\+1\\-2\end{pmatrix}$$

$\mathrm{Im} f = ?$

$$\begin{matrix}1&0&0&1\\0&2&1&1\\-1&0&0&2\\1&2&1&0\end{matrix} \longrightarrow \begin{matrix}1&0&0&1\\0&2&1&1\\0&0&0&3\\0&2&1&1\end{matrix} \longrightarrow \begin{matrix}1&0&0&1\\0&2&1&1\\0&0&0&3\\0&0&0&0\end{matrix}$$

$$\mathrm{Im} f = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}0\\2\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\\3\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}0\\2\\1\\1\end{pmatrix}\begin{pmatrix}-1\\0\\0\\2\end{pmatrix} \right\rangle$$

codominio

$\dim \mathrm{Im} f = 3 < \dim \mathbb{R}^4$

$\Rightarrow f$ non è suriettiva!

$\underset{4}{\dim \mathbb{R}^4} = \dim \ker f + \underset{3}{\dim \mathrm{Im} f} \Rightarrow \dim \ker f = 1 \Rightarrow$ non è iniettiva!

infatti cerco $v \in \mathbb{R}$ t.c. $f(v) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$\left(\begin{array}{cccc|c}1&0&-1&1&0\\0&2&0&2&0\\0&1&0&1&0\\1&1&2&0&0\end{array}\right) \longrightarrow \left(\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&1&0\\0&0&0&0&0\\0&1&+3&-1&0\end{array}\right) \longrightarrow \left(\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&1&0\\0&0&3&-2&0\\0&0&0&0&0\end{array}\right)$$

$$\begin{cases}\alpha - \gamma + \delta = 0\\ \beta + \delta = 0\\ 3\gamma - 2\delta = 0\end{cases} \qquad \begin{cases}\alpha = \gamma - \delta = \frac{2}{3}\delta - \delta = -\frac{\delta}{3}\\ \beta = -\delta\\ \gamma = \frac{2}{3}\delta\end{cases} \xrightarrow{\delta = 3} \begin{cases}\alpha = -1\\ \beta = -3\\ \gamma = 2\end{cases}$$

$$(-1)\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + (-3)\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + 3\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}7\\5\\0\\1\end{pmatrix} \qquad \ker f = \left\langle \begin{pmatrix}7\\5\\0\\1\end{pmatrix} \right\rangle$$

$$f\underbrace{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}}_{v_1}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\quad,\quad f\underbrace{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}}_{v_2}=\begin{pmatrix}0\\2\\1\\1\end{pmatrix}\quad,\quad f\underbrace{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{v_3}=\begin{pmatrix}-1\\0\\0\\2\end{pmatrix}\quad,\quad f\underbrace{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}_{v_4}=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$$

FOGLIO 4 ES 3

$$\left(\begin{array}{c|c} v_1 & f(v_1)\\ v_2 & f(v_2)\\ v_3 & f(v_3)\\ v_4 & f(v_4)\end{array}\right) \xrightarrow[\text{fino ad avere}]{\text{riduco con Gauss}} \left(\begin{array}{c|c} I_n^{4\times4} & \begin{matrix} f(e_1)\\ f(e_2)\\ f(e_3)\\ f(e_4)\end{matrix}\end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1&0&0&-1&1&0&0&1\\ -1&0&1&0&0&2&1&1\\ 1&1&0&0&-1&0&0&2\\ 1&1&1&0&1&2&1&0\end{array}\right) \xrightarrow{\begin{matrix}II+I\\ III-I\\ IV-I\end{matrix}} \left(\begin{array}{cccc|cccc} 1&0&0&-1&1&0&0&1\\ 0&0&1&-1&1&2&1&2\\ 0&1&0&1&-2&0&0&1\\ 0&1&1&1&0&2&1&-1\end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1&0&0&-1&1&0&0&1\\ 0&1&0&1&-2&0&0&1\\ 0&0&1&-1&1&2&1&2\\ 0&0&1&0&2&2&1&-2\end{array}\right) \xrightarrow{IV-III} \left(\begin{array}{cccc|cccc} 1&0&0&-1&1&0&0&1\\ 0&1&0&1&-2&0&0&1\\ 0&0&1&-1&1&2&1&2\\ 0&0&0&1&1&0&0&-4\end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1&0&0&0&2&0&0&-3\\ 0&1&0&0&-3&0&0&-5\\ 0&0&1&0&2&2&1&-2\\ 0&0&0&1&1&0&0&-4\end{array}\right)\begin{matrix}=f(e_1)\\ =f(e_2)\\ =f(e_3)\\ =f(e_4)\end{matrix} \qquad A_{f\,\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^4}}=\big(f(e_1)\ f(e_2)\ f(e_3)\ f(e_4)\big)$$

$$f\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}2&-3&2&1\\0&0&2&0\\0&0&1&0\\-3&5&-2&-4\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}2x-3y+2z+t\\2z\\z\\-3x+5y-2z-4t\end{pmatrix}$$

$$f\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\2\\1\\-2\end{pmatrix} \qquad \operatorname{Ker} f:\ \left(\begin{array}{cccc|c}2&-3&2&1&0\\0&0&1&0&0\\0&0&2&0&0\\-3&5&-2&-4&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}2&-3&2&1&0\\0&1/2&1&-5/2&0\\0&0&1&0&0\\0&0&0&0&0\end{array}\right)$$

$$f\begin{pmatrix}-1\\-1\\2\\-1\end{pmatrix}=\begin{pmatrix}4\\4\\2\\-2\end{pmatrix}$$

$$\Rightarrow\begin{cases}z=0\\ \frac{y}{2}+z-\frac{5}{2}t=0\\ 2x-3y+2z+t=0\end{cases}\qquad\begin{cases}z=0\\ y=5t\\ x=7t\end{cases}$$

$$\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}7t\\5t\\0\\t\end{pmatrix}=t\begin{pmatrix}7\\5\\0\\1\end{pmatrix}\qquad \operatorname{Ker} f=\left\langle\begin{pmatrix}7\\5\\0\\1\end{pmatrix}\right\rangle$$

ES 4

$$\begin{matrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} rk = 3 \\ \dim \mathcal{B} = 3 \end{matrix} \Rightarrow B_{\mathbb{R}^3} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$f\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \left.\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{matrix}\right| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \rightarrow \left.\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{matrix}\right| \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \rightarrow \left.\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}\right| \begin{matrix} 1 \\ 1 \\ -1 \end{matrix}$$

$$f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{matrix} \alpha + \beta + \gamma = 1 \\ \beta = 1 \\ 2\gamma = -1 \end{matrix} \qquad \begin{cases} \alpha = 1 - 1 + \frac{1}{2} = \frac{1}{2} \\ \beta = 1 \\ \gamma = -1/2 \end{cases}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$\dim Im = 2 = \dim \mathbb{R}^2 \Rightarrow f$ é suriettiva

$\dim \mathbb{R}^3 = \dim \ker f + \dim Im \Rightarrow \dim \ker f = 1 \Rightarrow$ non é iniettiva

cerco v t.c. $f(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{matrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{matrix}\right| \begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \left.\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & -8 \end{matrix}\right| \begin{matrix} 0 \\ 0 \end{matrix} \qquad \begin{cases} \alpha = -3\gamma \\ \beta = 8\gamma \end{cases} \xrightarrow{\gamma = 1} \begin{matrix} \alpha = -3 \\ \beta = 8 \end{matrix}$$

$$-3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 8 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 12 \end{pmatrix} \qquad \ker f = \left\{ \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} \right\}$$

ESERCIZIO 5

$$f: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+t \\ z-y+2t \\ x+y-z-t \end{pmatrix}$$

$$A_{\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^3}} f = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$\ker f: \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 1 & 1 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & -1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = -t \\ y = z+2t \end{cases} \qquad \begin{pmatrix} -t \\ z+2t \\ z \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathcal{B}_{\ker f} = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$ dim ker = 2 $(\neq 0) \Rightarrow f$ no iniettiva

per completare a una $\mathcal{B}\,\mathbb{R}^4$ aggiungo $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

formula dimensioni: $\dim \mathbb{R}^4 = \dim \operatorname{Im} f + \dim \ker f \Rightarrow \dim \operatorname{Im} f = 2$

$\Rightarrow$ prendo 2 vettori colonna l.i. da $A_{\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^3}}$

$$\mathcal{B}_{\operatorname{Im} f} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$ $\dim \operatorname{Im} f = 2 \;(\neq \dim \mathbb{R}^3) \Rightarrow f$ no suriettiva

per completare a una base di $\mathbb{R}^3$ aggiungo $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ESERCIZIO 9

d) $f : \mathbb{R}^4 \to \mathbb{R}^3 \qquad f\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}x+t\\y-2x+z\\x-y\end{pmatrix} \qquad f^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$A\underline{x} = \underline{b}$ ammette soluzioni $\iff$ $rkA = rk(A|b)$

e $S_{A|\underline{b}} = v + S_{A|0}$ con $v \in S_{A|b}$ soluzione particolare

$S_{A|0} = \ker f = \ker A$ soluzione del sistema omogeneo associato

$$A_{E_{\mathbb{R}^4}E_{\mathbb{R}^3}} = \begin{pmatrix}1 & 0 & 0 & 1\\ -2 & 1 & 1 & 0\\ 1 & -1 & 0 & 0\end{pmatrix}$$

$$S_{A|0}: \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & 0\\ -2 & 1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\end{array}\right) \to \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 2 & 0\\ 0 & 1 & 0 & 1 & 0\end{array}\right) \to \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 2 & 0\\ 0 & 0 & 1 & 1 & 0\end{array}\right)\end{array}$$

$$\begin{cases} z = -t \\ y = -2t - z = -t \\ x = -t \end{cases} \Rightarrow \begin{pmatrix}-t\\-t\\-t\\t\end{pmatrix} = -t\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \qquad \ker f = \left\langle \begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \right\rangle$$

$$S_{A|b}: \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & x_1\\ -2 & 1 & 1 & 0 & x_2\\ 1 & -1 & 0 & 0 & x_3\end{array}\right) \to \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & x_1\\ 0 & 1 & 1 & 2 & x_2 + 2x_1\\ 0 & +1 & 0 & 1 & -x_3 + x_1\end{array}\right)$$

$$\to \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & x_1\\ 0 & 1 & 1 & 2 & x_2 + 2x_1\\ 0 & 0 & -1 & -1 & -x_3 + x_1 - x_2 - 2x_1\end{array}\right) \to \left(\begin{array}{cccc|c}1 & 0 & 0 & 1 & x_1\\ 0 & 1 & 1 & 2 & x_2 + 2x_1\\ 0 & 0 & 1 & 1 & x_3 + x_1 + x_2\end{array}\right)$$

$$\begin{cases} x + t = x_1 \\ y + z + 2t = x_2 + 2x_1 \\ z + t = x_3 + x_1 + x_2 \end{cases} \xrightarrow{t=0} \begin{cases} x = x_1 \\ y = \cancel{x_2} + 2x_1 - (x_3 + x_1 + \cancel{x_2}) \\ z = x_3 + x_1 + x_2 \\ t = 0 \end{cases}$$

$$f^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_1 - x_3\\x_1 + x_2 + x_3\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \right\rangle$$

$$f^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \right\rangle$$

ES. 9

d) $f: \mathbb{R}^4 \longmapsto \mathbb{R}^3$

$$f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+t \\ y-2x+z \\ x-y \end{pmatrix} \qquad f^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \ f^{-1}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} ; \ f^{-1}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} ; \ f^{-1}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ cerco $v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4$ t.c. $f(v) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+t = 1 \\ y-2x+z = 0 \\ x-y = 0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{III} - \text{I}]{\text{II} + 2\text{I}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & -1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{\text{III} + \text{II}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

rkA = rkA|b il sistema ammette soluzioni

$$\begin{cases} z+t = 1 \\ y+z+2t = 2 \\ x+t = 1 \end{cases} \qquad \begin{cases} z = 1-t \\ y = 2-2t-(1-t) = 1-t \\ x = 1-t \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1-t \\ 1-t \\ 1-t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$f^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

soluzione particolare

soluzione del sistema omogeneo associato $\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{array}\right)$

$A\underline{x} = \underline{b}$ ammette soluzioni

$\Leftrightarrow$ rkA = rkA|$\underline{b}$ e

$S_{A|b} = v + S_{A|\underline{0}}$

con $v \in S_{A|b}$ (soluzione particolare)

$S_{A|0} = \ker f = \ker A$ (soluzione del sistema omogeneo associato)

$$f^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v + \ker f \overset{!}{=} v' + \ker f \rightarrow v - v' \in \ker f$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow v' = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \ker f \overset{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \ker f \quad \text{con } \ker f = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

ESERCIZIO 10

f) $f: \mathbb{R}^4 \to \mathbb{R}^3$

$$f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} y+4t+2x \\ 3t-2y+4z \\ x \end{pmatrix}$$

→ Considero la base canonica di $\mathbb{R}^4$ per il dominio $\mathcal{E}_{\mathbb{R}^4} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

e la base canonica di $\mathbb{R}^3$ per il codominio $\mathcal{E}_{\mathbb{R}^3} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

→ Cerco la matrice $A_{\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^3}f}$:

1^a colonna: $f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \underline{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{0}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underline{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}_{1^a \text{ colonna}}$

↓

è data dai coefficienti della combinazione lineare dei vettori di $\mathcal{E}_{\mathbb{R}^3}$

2^a colonna $f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-2)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}_{2^a \text{ colonna}}$

analogamente per $f\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}_{3^a \text{ colonna}}$ $\quad f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}_{4^a \text{ colonna}}$

$$\Rightarrow A_{\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^3}f} = \begin{pmatrix} 2 & 1 & 0 & 4 \\ 0 & -2 & 4 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

ESERCIZIO 11

f) $A_f = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = A_{\mathcal{E}_{\mathbb{R}^4}\mathcal{E}_{\mathbb{R}^3}f}$

$$f : \mathbb{R}^{4 = n^o \text{ colonne}} \longrightarrow \mathbb{R}^{3 = n^o \text{ righe}}$$

$$f : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x-y+2z \\ 2x+4z \\ z-t \end{pmatrix}$$

ESERCIZIO $\to$ $\phi_h : \mathbb{R}^3 \to \mathbb{R}^3$

$$\phi_h \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+2z \\ y+hz \\ x+hy+2z \end{pmatrix} \quad \text{con } h \in \mathbb{R}$$

a) Determinare la matrice A_h associata a ϕ_h rispetto alla base canonica $\mathcal{E}_{\mathbb{R}^3}$ di $\mathbb{R}^3$

b) Per ogni $h \in \mathbb{R}$ determinare una base di $\ker(\phi_h)$ e una base di $\mathrm{Im}(\phi_h)$

c) Per ogni $h \in \mathbb{R}$, determinare la controimmagine $\phi_h^{-1}\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right]$ tramite ϕ_h del vettore $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

a) $$A_{\mathcal{E}_3 \mathcal{E}_3} \phi_h = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & h \\ 1 & h & 2 \end{pmatrix}$$

b) Risolviamo il sistema lineare

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h & 0 \\ 1 & h & 2 & 0 \end{array}\right) \xrightarrow{III - I} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h & 0 \\ 0 & h-1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{III + (1-h) II} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h & 0 \\ 0 & 0 & h(1-h) & 0 \end{array}\right)$$

caso $h = 1$: $$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right)$$

$$\ker \phi_1 = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \qquad \mathrm{Im}\, \phi_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$\dim \ker \phi_1 = 1$

$\dim \mathrm{Im}\, \phi_1 = 2$

caso $h = 0$: $$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\ker \phi_0 = \left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rangle \qquad \mathrm{Im}\, \phi_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$\dim \ker \phi_0 = 1$

$\dim \mathrm{Im}\, \phi_0 = 2$

per $h \neq 0, 1$

$$\ker \phi_h = \{ 0_{\mathbb{R}^3} \} \qquad \mathrm{Im}\, \phi_h = \mathbb{R}^3$$

$\dim \ker \phi_h = 0$

$\dim \mathrm{Im}\, \phi_h = 3$

c) Si ha $\begin{pmatrix}1\\1\\0\end{pmatrix} \notin \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle = \operatorname{Im} \phi_1$

$\Rightarrow \quad \phi^{-1}\left(\begin{pmatrix}1\\1\\0\end{pmatrix} \right) = \emptyset$

Si ha $\begin{pmatrix}1\\1\\0\end{pmatrix} \in \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle = \operatorname{Im} \phi_0$

inoltre $\phi_0 \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow \phi_0^{-1}\left(\begin{pmatrix}1\\1\\0\end{pmatrix} \right) = \begin{pmatrix}0\\1\\0\end{pmatrix} + \ker \phi_0$

$= \begin{pmatrix}0\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}2\\0\\-1\end{pmatrix} \right\rangle$

per $k \neq 0,1$ risolvo il sistema

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & k & 1 \\ 1 & k & 2 & 0 \end{array}\right) \xrightarrow{III-I} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & k & 1 \\ 0 & k-1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{III-(1-k)II} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & k(1-k) & -k \end{array}\right) \xrightarrow{-\frac{1}{k} III} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & k-1 & 1 \end{array}\right)$$

$$\begin{cases} x + y + 2z = 1 \\ y + kz = 1 \\ (k-1)z = 1 \end{cases} \qquad \begin{cases} x = 1 - 2(1/k-1) + (1 - k(1/k-1)) \\ y = 1 - k(1/k-1) \\ z = 1/k-1 \end{cases}$$

dunque $\phi_k^{-1}\left(\begin{pmatrix}1\\1\\0\end{pmatrix} \right) = \left(\frac{k-2}{k-1} ; \frac{-1}{k-1} , \frac{1}{k-1} \right)$