BASIC DEFINITION

let $(X, \|\cdot\|_X)$ a Banach space
$(Y, \|\cdot\|_Y)$ a Banach space

$X \subseteq Y$ as a subspace (vectorial ~~sub~~space)

$I : X \longrightarrow Y$, $x \longmapsto x$ is a CONTINUOUS EMBEDDING

if $\exists C > 0 \quad \forall f \in X \subseteq Y \qquad \|f\|_Y \leq C \|f\|_X$.

I is COMPACT if it is continuous and if for any sequence $f_n \in X$ which is bounded ($\|f_n\|_X \leq C \ \forall n$) $\exists f_{n_j}$ subsequence and $g \in Y$

such that $f_{n_i} \to g$ in Y $\|f_{n_i} - g\|_Y \to 0$.

Es 1 SPACE OF HÖLDER CONTINUOUS FUNCTIONS

U bdd open

Recall that if $f : U \to \mathbb{R}$ is uniformly continuous (that is $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $|x-y| \le \delta$ for $x, y \in U \Rightarrow |f(x) - f(y)| \le \varepsilon$)

then f can be extended in a unique way to a continuous function in $\overline{U}$.

(let $x \in \partial U$ $x_n \in U$ $x_n \to x$ $\Rightarrow$ x_n is Cauchy $\Rightarrow$ $f(x_n)$ is Cauchy $\Rightarrow$ $f(x_n) \to y$ $y := f(x)$

y does not depends on the sequence x_n!

$$C^{0,\alpha}(U) = \left\{ f \in C(U) \text{ such that } \exists C>0 \ \frac{|f(x)-f(y)|}{|x-y|^\alpha} \le C \quad \forall x \neq y, \ x,y \in U \right\}$$

$f \in C^{0,\alpha}(U) \rightarrow f$ is uniformly continuous $\Rightarrow$

$f \in C(\bar{U})$, f is bounded.

$$\|f\|_{0\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

$(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha})$ is Banach.

Let f_n be a Cauchy sequence in $C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}$

$\forall \varepsilon \ \exists n_\varepsilon \quad n, m \ge n_\varepsilon$ ① $\|f_n - f_m\|_\infty \le \varepsilon$ ② $\frac{|(f_n-f_m)(x)-(f_n-f_m)(y)|}{|x-y|^\alpha} \le \varepsilon$

by ① f_n is Cauchy in $(C(\bar{U}), \|\cdot\|_\infty) \Rightarrow f_n \to f$

in $C(\bar{U})$ $\quad (\|f_n - f\|_\infty \to 0 \quad f \in C(\bar{U}))$

since $\|f_n - f\|_\infty = \sup_{x \in U} |f_n(x) - f(x)| \to 0 \Rightarrow f_n(x) \to f(x) \ \forall x \in U$

$\Rightarrow f \in C^{0,\alpha}(U)$ $\quad \left(\begin{array}{l} |f_n(x) - f_n(y)| \leq C|x-y|^\alpha \\ \qquad \downarrow \\ |f(x) - f(y)| \leq C|x-y|^\alpha \end{array} \right)$

by ② $\dfrac{|(f_n - f_m)(x) - (f_n - f_m)(y)|}{|x-y|^\alpha} \leq \varepsilon \quad \forall n, m \geq \varepsilon$

$\Rightarrow$ sending $m \to +\infty$ $\quad \dfrac{|(f_n - f)(x) - (f_m - f)(y)|}{|x-y|^\alpha} \leq \varepsilon \quad \forall n \geq n_\varepsilon$

$\Rightarrow \|f_n - f\|_{C^{0,\alpha}} \to 0$. $\quad (C^{0,\alpha}(U), \|\cdot\|_{0,\alpha})$ is BANACH

$(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}) \hookrightarrow (C(\bar U), \|\cdot\|_\infty)$ is CONTINUOUS

since $f \in C^{0,\alpha}(U) \to f \in C(\bar U)$

$\|f\|_\infty \leq \|f\|_{C^{0,\alpha}}$

MOREOVER the embedding is COMPACT.

$\|f_n\|_{C^{0,\alpha}} \leq C \Rightarrow \|f_n\|_\infty \leq C$

$|f_n(x) - f_n(y)| \leq C|x-y|^\alpha \quad \forall n$

$\Rightarrow$ Ascoli Arzelà thm $\Rightarrow f_{n_i} \to f$ uniformly

that is $\|f_{n_i} - f\|_\infty = \sup_{x \in U} |f_n(x) - f_{n_i}(x)| \to 0$.

Note also that

$f_n(x) \to f(x) \ \forall x \in U \Rightarrow |f(x) - f(y)| \leq C|x-y|^\alpha$.

$\Rightarrow f \in C^{0,\alpha}(U)$.

Finally $\left(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}\right) \longrightarrow \left(C^{0,\beta}(U), \|\cdot\|_{0,\beta}\right)$ is compact $\forall\, \beta < \alpha$.

$f \in C^{0,\alpha}(U) \Longrightarrow f \in C^{0,\beta}(U) \ \forall\, \beta < \alpha$

Indeed $\dfrac{|f(x) - f(y)|}{|x-y|^{\beta}} = \dfrac{|f(x) - f(y)|}{|x-y|^{\alpha}} \cdot |x-y|^{\alpha-\beta} \leq$

$\leq (\operatorname{diam} U)^{\alpha-\beta} \cdot \dfrac{|f(x) - f(y)|}{|x-y|^{\alpha}}$.

therefore $\|f\|_{0,\beta} \leq \max\left(1, (\operatorname{diam} U)^{\alpha-\beta}\right) \|f\|_{0,\alpha}$.

(the immersion is continuous).

Let us prove it is compact.

$f_n \in C^{0,\alpha}(U) \quad \|f_n\|_{C^{0,\alpha}} \leq C \quad \forall n.$

$\Rightarrow \quad f_n \longrightarrow f$ up to subsequence in $(C(\bar{U}), \|\cdot\|_\infty)$

$f_n \to f$ pointwise $\Rightarrow f$ is in $C^{0,\alpha}(U) \subseteq C^{0,\beta}(U)$

$$\frac{|(f_n - f_m)(x) - (f_n - f_m)(y)|}{|x-y|^\beta} =$$

$$= \underbrace{\left(\frac{|(f_n - f_m)(x) - (f_n - f_m)(y)|}{|x-y|^\alpha}\right)^{\beta/\alpha}}_{\leq \|f_n\|_{C^{0,\alpha}} + \|f_m\|_{C^{0,\alpha}} \leq 2C} \cdot \underbrace{|f_n(x) - f_m(y)|^{1-\frac{\beta}{\alpha}}}_{\leq \|f_n - f_m\|_\infty^{1-\beta/\alpha}} \leq 2C \|f_n - f_m\|_\infty^{1-\beta/\alpha}$$

f_n is Cauchy in $C(\bar{U}), \|\cdot\|_\infty \Rightarrow f_n$ is Cauchy in $C^{0,\beta}(U)$

Es 2 Distributional derivative of a characteristic function of a C^1 set.

$E \subseteq \mathbb{R}^n$ E open bold of class C^1

$\chi_E \in L^1(\mathbb{R}^n)$ $\quad |E| = \int_E 1\,dx = \int_{\mathbb{R}^n} \chi_E\,dx$ $\quad \chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$

$$\nabla T_{\chi_E}(\phi) = -\int_E \nabla\phi\,dx = -\left(\int_E \frac{\partial\phi}{\partial x_i}\,dx\right)_{i=1..n}$$

$\downarrow$ component by component

$$\underset{\text{(divergence)}}{=} -\underbrace{\left(\int_{\partial E} \phi\,\nu_E^i\,d\mathcal{H}^{n-1}(x)\right)_i}_{} = -\int_{\partial E} \phi\,\nu_E\,d\mathcal{H}^{n-1}(x)$$

∇T_{χ_E} = (in distributional sense) $= -\nu_E \chi_{\partial E} \cdot d\mathcal{H}^{n-1}$

$D\chi_E$ is a vectorial signed measure supported on ∂E and such that it is absolutely continuous with respect to $\mathcal{H}^{n-1}$ with density $-\nu_E(x)$.
$(-\nu_E(x)$ = interior normal at x to E).

Ex 3 Distributions with null derivatives on intervals are constants

$T \in \mathcal{D}'(a,b) \qquad T'(\varphi)=0 \quad \forall \phi \in C_c^\infty(a,b) \quad (a,b) \in \mathbb{R}$

$T'(\varphi) = -T(\varphi') = 0$.

Observ. if $\phi \in C_c^\infty(a,b)$ with $\int_a^b \phi(x)\,dx = 0 \implies$
$\exists\, \psi \in C_c^\infty(a,b)$ such that $\psi'(x) = \phi(x)$
(sufficient to choose $\psi(x) = \int_a^x \phi(t)\,dt$)

Let $\psi \in C_c^\infty(a,b)$ $\quad \int_a^b \psi = 1$

For $\phi \in C_c^\infty(a,b)$ $\quad \phi - \psi \int_a^b \phi \, dx = \xi \in C_c^\infty(a,b)$

$$\int_a^b \xi(t)\,dt = 0$$

$\to \exists\, w \in C_c^\infty(a,b)$ $\quad w' = \xi$

$$T(w') = -T(w) = 0 = T(\xi) = T(\varphi) - T(\psi)\int_a^b \phi \, dx$$

$$\Longrightarrow \quad T(\varphi) = T(\psi)\int_a^b \phi(x)\,dx \quad \forall \varphi \in C_c^\infty(a,b)$$

$\to T$ is constant $\quad T = C$ - $\quad \left(C = T(\psi)\right)$.

Es 4 Hausdorff measures

$f : \mathbb{R}^n \to \mathbb{R}^k$ s. that $\sup_{\substack{x \neq y \\ x,y \in \mathbb{R}^n}} \dfrac{|f(x)-f(y)|}{|x-y|}$:

Then $\forall A \in \mathcal{B}(\mathbb{R}^n)$
$$\mathcal{H}^s(f(A)) \leq C^s \mathcal{H}^s(A).$$

$$\mathcal{H}^s(f(A)) = \sup_{\delta > 0} \mathcal{H}^s_\delta(f(A))$$

$$\mathcal{H}^s_\delta(f(A)) = \inf \left\{ \frac{\omega_s}{2^s} \sum_i (\operatorname{diam} E_i)^s \quad \begin{array}{l} f(A) \subseteq \cup_i E_i \\ \operatorname{diam} E_i \leq \delta \end{array} \right\}$$

$$\leq \inf \left\{ \frac{\omega_s}{2^s} \sum_i (\operatorname{diam} E_i)^s \quad \begin{array}{l} A \subseteq \cup_i V_i \\ E_i = f(V_i) \end{array} , \begin{array}{l} f(A) \subseteq \cup_i f(V_i) \\ \operatorname{diam} E_i \leq \delta \end{array} \right.$$

we restrict the class where computing the infimum.

$$\operatorname{diam} E_i = \operatorname{diam} f(V_i) \leq C \cdot \operatorname{diam} V_i$$

$$\leq \inf \Big\{ \frac{\omega_s}{2^s} \underbrace{\sum_i (\operatorname{diam} E_i)^s}_{\leq \sum_i C^s (\operatorname{diam} V_i)^s} \quad E_i = f(V_i) \quad A \subseteq \bigcup_i V_i \quad \operatorname{diam} V_i \leq \frac{\delta}{C} \Big\}$$

(Note that ~~th~~is implies $\operatorname{diam} E_i \leq \delta$)

$$\leq \inf \Big\{ \frac{\omega_s}{2^s} C^s \sum_i (\operatorname{diam} V_i)^s \quad A \subseteq \bigcup_i V_i \quad \operatorname{diam} V_i \leq \delta/C \Big\} =$$

$$= C^s \mathcal{H}^s_{\delta/C}(A)$$

$$\mathcal{H}^s_\delta(f(A)) \leq C^s \mathcal{H}^s_{\delta/C}(A)$$

$$\downarrow \delta \to 0$$

$$\mathcal{H}^s(f(A)) \leq C^s \mathcal{H}^s(A). \quad \square$$