FUNDAMENTAL SOLUTION OF THE LAPLACIAN.

- In $\mathbb{R}^2$ $T = T_f$ $f = -\frac{1}{2\pi} \log|x|$ is the fundamental solution of the Laplacian

- In $\mathbb{R}^n$ $T = T_f$ $f = -\frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}$ is the fundamental solution of the Laplacian.
$n > 2$

proof $n > 2$ $x \neq 0$ $\nabla f = + \frac{1}{n\omega_n} \frac{x}{|x|^n}$

$$\Delta f = \operatorname{div}(\nabla f) = + \frac{1}{n\omega_n} \frac{n}{|x|^n} - \frac{1}{n\omega_n} n \frac{x \otimes x}{|x|^{n-2}} = 0$$

$\forall \phi \in C_c^\infty(\mathbb{R}^n)$

$$\Delta T_f(\phi) = T_f(\Delta\phi) = \int_{\mathbb{R}^n} f \Delta\phi \, dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f\Delta\phi - \underbrace{\Delta f}_{=0} \cdot \phi \, dx =$$

$$= \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} f \cdot \nabla\phi \cdot \left(\frac{-x}{|x|}\right) - \phi \nabla f \cdot \left(\frac{-x}{|x|}\right) d\mathcal{H}^{n-1}(x) =$$

DIVERGENCE THM

$$\left| \int_{\partial B(0,\varepsilon)} f \cdot \nabla\phi\left(\frac{-x}{|x|}\right) d\mathcal{H}^{n-1} \right| \leq \underbrace{\frac{1}{n\omega_n} \frac{1}{\varepsilon^{n-2}}}_{f} \cdot \|\nabla\phi\|_\infty \cdot \mathcal{H}^{n-1}(\partial B(0,\varepsilon))$$

$$= \varepsilon \|\nabla\phi\|_\infty \to 0$$

$$= \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} (-\phi(x)) \cdot \left(+\frac{1}{n\omega_n} \frac{x}{|x|^n}\right) \cdot \left(-\frac{x}{|x|}\right) d\mathcal{H}^{n-1} =$$

$$= \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} \phi(x) \frac{1}{n\omega_n \varepsilon^{n-1}} d\mathcal{H}^{n-1} = \phi(0). \ \square$$

Corollary Let $\phi \in C_c^\infty(\mathbb{R}^n) \Rightarrow$

$$T_f * \phi = f * \phi(x) = \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-2}} \frac{1}{n\omega_n(n-2)} dy = u(x)$$

satisfies $\Delta u(x) = \phi(x)$, $\forall x$

This operator $F : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$

$$\phi \longmapsto F(\varphi) = c_n \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-2}} dy$$

Is a sorto of inverso of the laplacian. $(\Delta F(\varphi) = \varphi)$

Note that it can be extended to $\boxed{F : L^2(U) \to L^2(U)}$

U bdd set. $F(g) = c_n \int_U \frac{g(y)}{|x-y|^{n-2}} dy$

$$g \in L^2(U) \quad \frac{1}{|y|^{n-2}} \in L^1(U) \Rightarrow g * \frac{1}{|y|^{n-2}} \in L^2(U)$$

$$\left\| g * \frac{1}{|y|^{n-2}} \right\|_{L^2} \le \|g\|_{L^2} \left\| \frac{1}{|y|^{n-2}} \right\|_{L^1}$$

(YOUNG INEQUALITY).

$$\| F(g) \|_{L^2} \le \|g\|_{L^2} c_n \left\| \frac{1}{|y|^{n-2}} \right\|_{L^1} = C \|g\|_{L^2}$$

F is linear continuous, also self-adjoint.

it is possible to prove (using Sobolev) that F is COMPACT.

SOBOLEV SPACES

(more suitable in some contexts than distributions → they are Banach spaces)

Ref EVANS chapt 5.

$U \subseteq \mathbb{R}^n$ $\quad W^{1,p}(U) = \{ f \in L^p(U) \quad f$ admits weak

$p \in [1, +\infty]$ $\quad$ derivatives $\frac{\partial f}{\partial x_i} \ \forall i, \quad \frac{\partial f}{\partial x_i} \in L^p(U) \}$

$k \in \mathbb{N}$ $\quad W^{k,p}(U) = \{ f \in L^p(U), \quad f$ admits weak

derivatives $\frac{\partial f}{\partial x_i} \in W^{1,p}(U) \} =$

$= \{ f \in L^p(U) \quad \forall \alpha \in \mathbb{N}^n \quad |\alpha| \le k \quad \exists\, v_\alpha \in L^p(U)$ such

that $T_{v_\alpha} = D^\alpha T_f \}$

$\forall \phi \in C^\infty_c(U) \quad D^\alpha T_f(\phi) = \int_U v_\alpha \phi \, dx = (-1)^{|\alpha|} \int_U f D^\alpha \phi \, dx$

Observe that we may define Sobolev spaces without using distributions.

We say that $v_i \in L^1_{loc}(U)$ is a weak derivative of $f \in L^1_{loc}(U)$ in the direction e_i if

$$\forall \phi \in C^\infty_c(U) \qquad \int_U \phi \, v_i \, dx = - \int_U \frac{\partial \phi}{\partial x_i} f \, dx$$

By De-Bois Reymond lemma v_i is unique (if it exists!)

$$W^{1,p}(U) = \{ f \in L^p(U) \;\; \forall i \; f \text{ has weak derivative } v_i \text{ and } v_i \in L^p(U) \}$$

$$W^{k,p}(U) = \{ f \in L^p(U) \;\; \forall \alpha \; |\alpha| \le k \;\; \exists v_\alpha \in L^p(U) \text{ such that } \forall \phi \in C^\infty_c(U) \; \int_U \phi \, v_\alpha \, dx = (-1)^{|\alpha|} \int_U D^\alpha \phi \, f \, dx \}$$

$k \in \mathbb{N}$.

Prop $(W^{k,p}(U), \|\cdot\|_{W^{k,p}})$ is BANACH SPACE

→ $p \in [1,+\infty)$ $W^{k,p}(U)$ is separable

→ $p \in (1,+\infty)$ $W^{k,p}(U)$ is reflexive

→ $p = 2$ $W^{k,2}(U)$ is Hilbert $\quad W^{k,2}(U) = H^k(U)$

$$(f,g)_{W^{1,2}(U)} = \int_U f \cdot g \, dx + \sum_{i=1}^{n} \int_U \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \, dx$$

proof $k=1$ $\quad W^{1,p}(U) \subseteq L^p(U) \times [\underbrace{L^p(U) \times \dots \times L^p(U)}_{n \text{ time}}]$

$$f \longmapsto \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

Continuous embedding

$W^{1,p}(U)$ is closed in $(L^p(U))^{n+1}$ (the image of $W^{1,p}(U)$ is closed in $(L^p(U))^{n+1}$)

$f_k \in W^{1,p}$ and Cauchy

$\Downarrow$

f_k is Cauchy in $L^p(U)$ $\qquad$ $f_k \to f$ in $L^p(U)$

$\frac{\partial}{\partial x_i} f_k$ is Cauchy in $L^p(U)$ $\qquad$ $\frac{\partial}{\partial x_i} f_k \to g_i$ in $L^p(U)$

Moreover $\forall \phi \in C_c^\infty(U)$ $\qquad$ $\int_U \phi \frac{\partial}{\partial x_i} f_k \, dx = - \int_U \frac{\partial}{\partial x_i}\phi \, f_k \, dx$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_U \phi \, g_i \, dx = - \int_U \frac{\partial}{\partial x_i} \phi \, f \, dx$$

$$g_i = \frac{\partial}{\partial x_i} f$$

$\Rightarrow f \in W^{1,p}(U)$.

So $W^{1,p}$ is closed $\Rightarrow W^{1,p}(U)$ is Banach space and

$1 < p < +\infty$ $\left(L^p(U)\right)^{n+1}$ is reflexive $\Rightarrow$ $W^{1,p}(U)$ is closed subspace of a reflexive space $\Rightarrow$ it is reflexive

$1 \leq p < +\infty$ $\left(L^p(U)\right)^{n+1}$ is separable $\Rightarrow$ $W^{1,p}(U)$ is a subspace of a separable space then it is separable.