January 13 (tuesday) 10.30 - 12.

MEASURE THEORY

X set $\quad \mathcal{P}(X) = \{ A \subseteq X \quad A \text{ subsets of } X \}$
(parts of X)

$\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra if

1) $\emptyset \in \Sigma$
2) $A \in \Sigma \Rightarrow X \setminus A \in \Sigma$
3) $A_i \in \Sigma \quad i \in \mathbb{N} \longrightarrow \bigcup_{i=1}^{+\infty} A_i \in \Sigma$

smallest possible σ-algebra $\{\emptyset, X\}$
biggest possible σ-algebra $\mathcal{P}(X)$.

$C \subseteq \mathcal{P}(X)$ $\quad$ C = family of subsets of X

$\Sigma(C)$ = smallest σ-algebra which contains all the elements of C.

→ σ-algebra "generated" by C.

$X = \mathbb{R}$ $\quad$ $C = \{ (a,b] \,,\ a < b \quad a, b \in \mathbb{R} \}$

$(a,b] = \{ x \in \mathbb{R} \quad a < x \leq b \}$

$\Sigma(C)$ = Borel σ-algebra (it contains all possible intervals of $\mathbb{R}$)

$$\bigcup_{n=1}^{+\infty} (a, b+n] \in \Sigma(C)$$

$$(a, +\infty) \in \Sigma(C)$$

$$\bigcup_{n=1}^{\infty} [a-n, b] \in \Sigma(C)$$

$$(-\infty, b] \in \Sigma(C)$$

$$\vdots \quad \cdot \quad \cdot$$

$$\mathbb{R} \setminus (a, b] \cancel{\in} = (-\infty, a] \cup (b, +\infty) \in \Sigma(C)$$

X set Σ σ-algebra of X

we say μ is a MEASURE on (X, Σ) if

$$\mu : \Sigma \longrightarrow [0, +\infty]$$
$$\underbrace{A \in \Sigma} \longrightarrow \mu(A)$$

and moreover $\mu(\emptyset) = 0$ and

μ is σ-additive : if $A_i \in \Sigma$ $i \in \mathbb{N}$ $A_i \cap A_j = \emptyset$ $i \neq j$

$$\mu\Big(\underbrace{\bigcup_{i=1}^{+\infty} A_i}_{\in \Sigma}\Big) = \sum_{i=1}^{+\infty} \mu(A_i) = \lim_{N \to +\infty} \sum_{i=1}^{N} \mu(A_i).$$

Ex $\quad X = \mathbb{R} \quad \Sigma = \mathcal{P}(X)$

$$\mu = \delta_0 : \mathcal{P}(X) \to [0, +\infty)$$

$\boxed{\text{DIRAC-DELTA MEASURE CENTERED AT } x=0}$

$$\delta_{\underline{0}}(A) = \begin{cases} 0 & 0 \notin A \\ 1 & 0 \in A \end{cases}$$

Dirac delta centered at $x = 2$

$$\delta_2(A) = \begin{cases} 0 & 2 \notin A \\ 1 & 2 \in A \end{cases}$$

Ex Ω = set of events

$\mathcal{F}$ = filtration (= σ-algebra) =

= subset of $\mathcal{P}(\Omega)$ which is the events related to some experiments

$(\mu =)\ \mathbb{P}$ = probability measure

$$\mathbb{P} : \mathcal{F} \longmapsto [0,1] \subseteq [0, +\infty]$$

$$A \in \mathcal{F} \longrightarrow \mathbb{P}(A)$$

Def $\mu : \Sigma \to [0, +\infty]$ defined on (X, Σ)

μ is finite if $\mu(X) < +\infty$

- (every probability measure is finite $\mathbb{P}(\Omega) = 1$)
- $\delta_0(\mathbb{R}) = 1$ finite

μ is σ-finite if $X = \bigcup_{i=1}^{+\infty} X_i$ $\quad X_i \in \Sigma$ $\quad \underline{\mu(X_i) < +\infty}$.

(maybe $\mu(X) = +\infty$ but $\mu(X_i) < +\infty \ \forall i$)

Property of measures (deduced by the definition).

1) $A \subseteq B$ $\quad A, B \in \Sigma$ $\quad \mu(A) \le \underline{\mu(B)}$ (MONOTONICITY)

$B = A \cup \underbrace{(B \setminus A)}$ $\qquad A, B \in \Sigma \Rightarrow B \setminus A \in \Sigma$

$A \cap (B \setminus A) = \emptyset$ $\qquad \mu(B) = \mu(A) + \mu(B \setminus A) \ge 0$.

2) SUB-ADDITIVITY $A_i \in \Sigma$ (not asking $A_i \cap A_j = \emptyset$)

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) \leq \sum_{i=1}^{+\infty} \mu(A_i)$$

3) CONTINUITY

→ $A_i \in \Sigma$ $A_i \subseteq A_{i+1}$ $\forall i \in \mathbb{N}$

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subset \dots$$

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \lim_{n \to +\infty} \mu(A_n)$$

$$\bigcup_{i=1}^{N} A_i = A_N$$

→ $B_i \in \Sigma$ $B_i \supseteq B_{i+1}$ $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$

$$\bigcap_{i=1}^{N} B_i = B_N \qquad \mu\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n \to +\infty} \mu(B_n)$$

IF $\exists\, i_0 \in \mathbb{N}$ SUCH THAT $\mu(B_{i_0}) < +\infty$.

BOREL MEASURES on $\mathbb{R}$

measures defined on the Borel σ-algebra $\mathcal{B}$

$\mathcal{B} = \Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathbb{R})$ $\quad \mathcal{C} = \{ (a,b] \quad a < b \in \mathbb{R} \}$

Every measure $\mu : \mathcal{B} \to [0,+\infty]$ is a Borel measure

Every Borel measure is ASSOCIATED to a function which is called the CUMULATIVE DISTRIBUTION FUNCTION.

Def $F : \mathbb{R} \to \mathbb{R}$

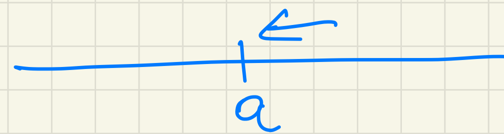
→ is monotone NON DECREASING if $a < b \Rightarrow F(a) \leq F(b)$

→ is right continuous if

$\forall a \in \mathbb{R}$ $$\lim_{n \to +\infty} F\left(a + \frac{1}{n}\right) = F(a)$$

$a + \frac{1}{n} > a$

$$\lim_{n \to +\infty} F\left(a + \frac{1}{n}\right) = \lim_{x \to a^+} F(x)$$



Given a monotone non decreasing right cont. function F → I want to construct a Borel measure μ_F

$(F \longrightarrow \mu_F)$

$\forall\ (a,b] \in \underline{\underline{C}}$ $\qquad$ $(\mathcal{B} = \sigma(C))$

$$\boxed{\mu_F(a,b] := F(b) - F(a)}$$

$\mu_F(a,b] \geq 0$

$a < b \Rightarrow F(b) \geq F(a)$

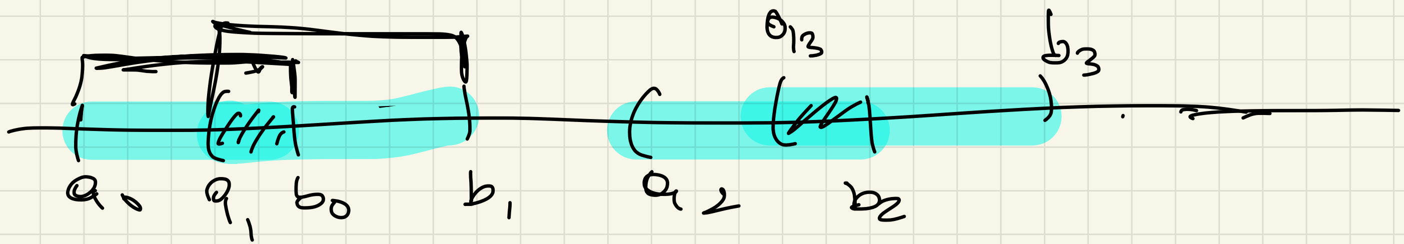
(by MONOTONICITY of F)

μ_F is not a measure still (because it is defined on C)

$E \subseteq \mathbb{R}$ $\qquad$ E subset of $\mathbb{R}$

$$\underline{\underline{\mu_F^*}}(E) = \inf \left\{ \sum_{i=1}^{+\infty} F(b_i) - F(a_i) \ , \quad E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

$\bigcup_i (a_i, b_i]$



$$\mu_F\Big(\bigcup_i (a_i,b_i]\Big) \leq \sum_{i=0}^{\infty} \mu_F(a_i,b_i] = \sum_{i=0}^{+\infty} F(b_i)-F(a_i)$$

$$\mu_F(a_i,b_i] = F(b_i)-F(a_i)$$

$$E \subseteq \bigcup_i (a_i,b_i]$$

$$\mu_F(E) \leq \mu_F\Big(\bigcup_i (a_i,b_i]\Big) = \sum_{i=0}^{+\infty} \mu_F(a_i,b_i]$$

$$\mu_F(E) := \inf \Big\{ \sum_{i=0}^{\infty} \mu_F(a_i,b_i] \quad E \subseteq \bigcup_i (a_i,b_i]$$

$\forall E \subseteq \mathbb{R}$ $\qquad \mu_F^*(E) := \inf \left\{ \sum_{i=1}^{+\infty} \mu_F(a_i;b_i] \quad E \subseteq \bigcup_{i=1}^{\infty} (a_i,b_i] \right\}$

OUTER MEASURE
(NOT A MEASURE)

μ_F^* is defined on $\mathcal{P}(\mathbb{R})$.

but it is NOT A MEASURE $\qquad E_i \subseteq \mathbb{R}$, $\quad E_i \cap E_j = \emptyset$

$$\mu_F^*\left(\bigcup_i E_i\right) \underset{\cancel{=}}{\leq} \sum_{i=1}^{\infty} \mu_F^*(E_i)$$

(CARATHEODORY CRITERIUM – beginning of 1900)

$\exists\, \mathcal{M}$ a σ-algebra $\quad \mathcal{B} \subseteq \mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$

such that μ_F^* is a measure when restricted to $\mathcal{M}$.

$F(x) = x \qquad \mu_F(a,b] = b-a =$ length of the interval

$$\mu_F^*(E) = \inf\left\{\sum_{i=1}^{\infty} b_i - a_i \quad E \subseteq \bigcup_i (a_i;b_i]\right\}$$

$\downarrow$

$\exists$ σ-algebra $= \underline{\underline{\mathcal{M}}}$ (σ-algebra of measurable sets)

such that μ_F^* restricted to $\mathcal{M}$ is a MEASURE

$\mu_F^*(A) =$ "length of A".

$A \in \mathcal{M}$

It is not possible to define a NOTION of LENGTH in a <u>COHERENT WAY</u> on all subsets of $\mathbb{R}$.

Theorem (CHARACTERIZATION OF σ-finite BOREL MEASURES on $\mathbb{R}$)

1) Given $F: \mathbb{R} \to \mathbb{R}$ monotone non decreasing and right continuous there exists a UNIQUE Borel measure

$$\mu_F : \mathcal{B} \longrightarrow [0, +\infty]$$

such that $\mu_F(a, b] = F(b) - F(a)$

(ACTUALLY μ_F is DEFINED on a bigger σ-algebra $\mathcal{M} \supseteq \mathcal{B}$)

2) Given $\mu: \mathcal{B} \to \mathbb{R}$ Borel measure which is σ-finite (or finite), there exists $F: \mathbb{R} \to \mathbb{R}$

$\mathbb{R} = \bigcup B_i \quad \mu(B_i) < +\infty$

$\mu(\mathbb{R}) < +\infty$.

monotone non decreasing and right continuous

such that $\mu = \mu_F$ $\quad (\mu(a,b] = F(b) - F(a))$

F is UNIQUE up to ADDITION of CONSTANT

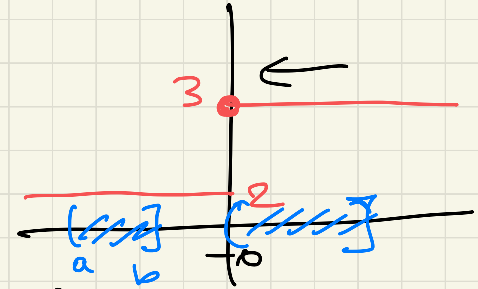
$$\mu = \mu_F = \mu_G \iff \exists k \in \mathbb{R} \ \ \forall x \in \mathbb{R} \quad F(x) = G(x) + k$$

$F = G + k$

(F is called CUMULATIVE DISTRIBUTION FUNCTION ASSOCIATED TO μ.)

$$\left\{ \begin{array}{l} \mu \text{ Borel measure} \\ \text{either finite} \\ \text{or } \sigma\text{-finite} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} F : \mathbb{R} \to \mathbb{R} \\ \text{monotone non decr.} \\ \text{right continuous} \end{array} \right\}$$

Ex $F(x) = \begin{cases} 2 & x < 0 \\ 3 & x \geq 0 \end{cases}$



F monotone non decreasing

right continuous $\lim_{x \to 0^+} F(x) = F(0) = 3$

$\mu_F(a,b] = \underbrace{F(b) - F(a)}$

$= 0$ if $\boxed{b < 0} \to 2-2$

if $a \geq 0 \to 3-3$

$= 1$ if $0 \in (a,b]$

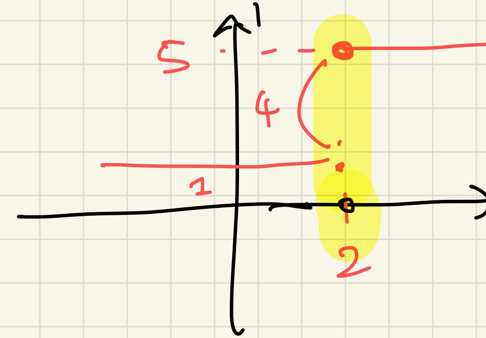
$a < 0 \leq b$

$E \subseteq \mathbb{R}$

$E \subseteq \cup_i (a_i, b_i]$

$\mu_F^*(E) = \begin{cases} 0 := \inf \{ \sum (F(b_i) - F(a_i)) \} & 0 \notin E \\ 1 & 0 \in E \end{cases}$

ex $F(x) = \begin{cases} 5 & x \geq 2 \\ 1 & x < 2 \end{cases}$



$$\mu_F(A) = \begin{cases} 4 & 2 \in A \\ 0 & 2 \notin A \end{cases}$$

$$\mu_F\{2\} = 4 > 0$$
$$= F(2) - \lim_{x \to 2^-} F(x)$$

$$\mu_F = 4 \cdot \delta_{\{2\}}$$

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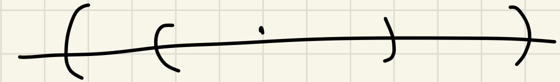
ex $F(x) = x$ $\quad \mu_F : \mathcal{B} \longrightarrow [0, +\infty]$

μ_F is called LEBESGUE MEASURE

(associate to every $A \in \mathcal{B}$ the "length" of A)

Observe that the Lebesgue is σ-finite

$$\mathbb{R} = \bigcup_{n=0}^{+\infty} (-n, +n]$$



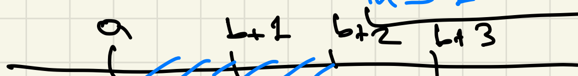
$$\mu_F(-n, n] = F(n) - F(-n) = 2n < +\infty$$

COMMENTS on PART 1 of the theorem.

$F : \mathbb{R} \to \mathbb{R}$ right cont & monotone non decreasing

$\to$ $\boxed{\exists!\, \mu_F : \mathcal{B} \to \mathbb{R}}$ $\quad (\mu_F(a,b] = F(b) - F(a)).$

$$\mu_F(a, +\infty) = \mu_F\Big(\bigcup_{n=1}^{+\infty} (a, b+n]\Big) = \lim_{n \to +\infty} \mu(a, b+n]$$



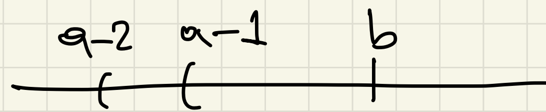
$$(a, b+n] \subseteq (a, b+n+1]$$

$$\mu_F(a,+\infty) = \lim_{n\to+\infty} \mu_F(a, b+n] =$$

$$= \lim_{n\to+\infty} F(b+n) - F(a) = \lim_{x\to+\infty} F(x) - F(a)$$

$$\mu_F(-\infty, b] = \lim_{n\to+\infty} \mu_F[a-n, b] = \lim_{n\to+\infty} F(b) - F(a-n)$$

$$(-\infty, b] = \bigcup_{n=1}^{\infty} (a-n, b]$$



$$(a-n, b] \subseteq (a-(n+1), b]$$

$$= F(b) - \lim_{x\to-\infty} F(x)$$

$$\lim_{x\to+\infty} F(x) - \lim_{y\to-\infty} F(y)$$

$$\mu_F(\mathbb{R}) = \mu_F(-\infty,+\infty) = \mu_F\big((-\infty,a] \cup (a,+\infty)\big) =$$

μ_F is finite $\Longleftrightarrow$ $\lim\limits_{x \to +\infty} F(x) < +\infty$

$\lim\limits_{x \to -\infty} F(x) > -\infty$

In general $\underline{\mu_F \text{ is } \sigma\text{-finite}}$

$\mathbb{R} = \bigcup\limits_{n=1}^{\infty} (-n, n]$ $\qquad \mu_F(-n, n] = F(n) - F(-n) < +\infty$

$\sim$.

$\underline{\mu_F(a, b] = F(b) - F(a)}$ $\qquad (a, b) = \bigcup\limits_{n=1}^{+\infty} \left(a, b - \frac{1}{n}\right]$

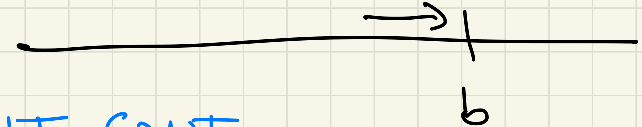
$b \notin \bigcup\limits_{n} \left(a, b - \frac{1}{n}\right]$

$\mu_F \underbrace{(a, b)}_{(a,]} = \mu_F\left(\bigcup\limits_{n=1}^{\infty} (a, b - \frac{1}{n}]\right) =$

$= \lim\limits_{n} \mu_F\left(a, b - \frac{1}{n}\right] = \lim\limits_{n \to +\infty} F\left(b - \frac{1}{n}\right) - F(a)$

$$\mu_F(a,b) = \lim_{x\to b^-} F(x) - F(a)$$

$$= \lim_{n\to+\infty} F\left(b-\frac{1}{n}\right) - F(a)$$



by ass. I know F is RIGHT CONT

$$\lim_{x\to b^+} F(x) = F(b)$$

$$\lim_{x\to b^-} F(x) \le F(b)$$

If F is CONTINOUOUS in b $\left(\lim_{x\to b^+} F(x) = F(b) = \lim_{x\to b^-} F(x) \right)$

$$\mu_F(a,b) = \mu_F(a,b] \Rightarrow \mu_F\{b\} = 0$$

$$\mu_F\{b\} = F(b) - \lim_{x\to b^-} F(x)$$

$\sim \cdot \sim$

part 2 of the theorem

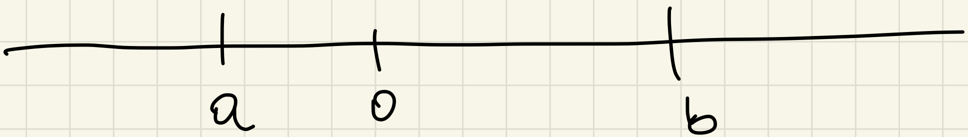
$\rightarrow$ given μ σ-finite / finite Borel measure

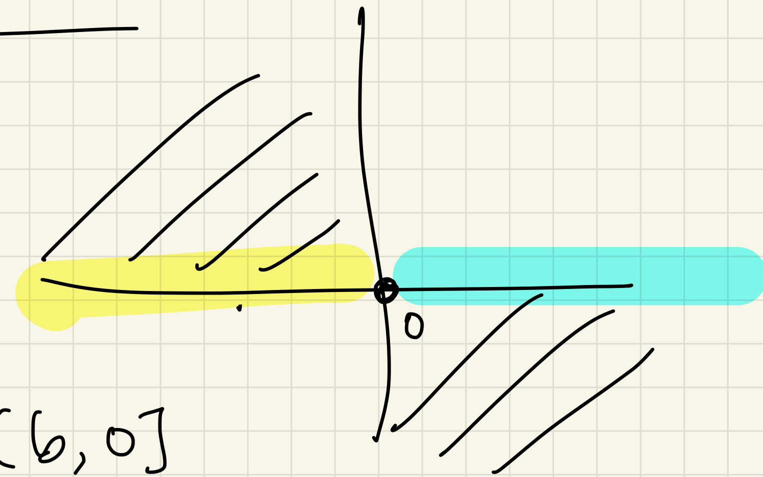
$\exists$ F right continuous + monotone non decreas.

$$\mu : \mathcal{B} \to [0,+\infty] \quad \sigma\text{- finite}$$

$$\underline{\underline{F(x)}} = \begin{cases} 0 & x=0 \\ \mu(0,x] \geq 0 & x>0 \\ -\mu(x,0] \leq 0 & x<0 \end{cases}$$

① $b > a \implies F(b) \geq F(a)$ (I want to prove F is monotone non decreasing



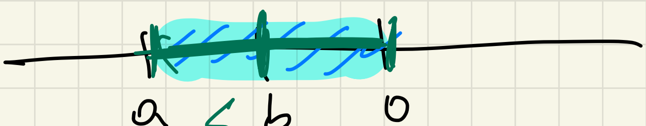


$a \leq 0 \to F(a) \leq 0$

$b \geq 0 \to F(b) \geq 0$

$a < b \leq 0$

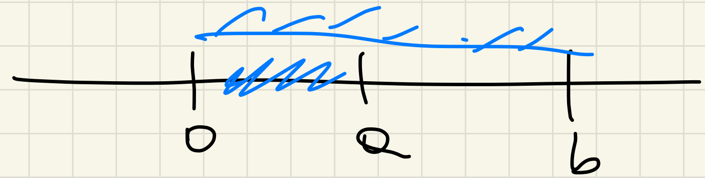
$(a, 0] \supseteq (b, 0]$



$\mu(a, 0] \geq \mu(b, 0]$

$F(a) = -\mu(a, 0] \leq -\mu(b, 0] = F(b)$

$0 < a < b$



$$(0, a] \subseteq (0, b]$$

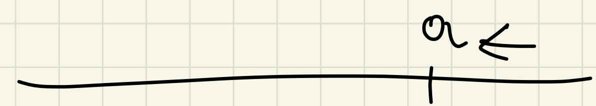
$$\mu(0, a] \le \mu(0, b]$$

F is monotone non decreasing.

I have F right continuous

$\downarrow$

$$\lim_{x \to a^+} F(x) = F(a)$$



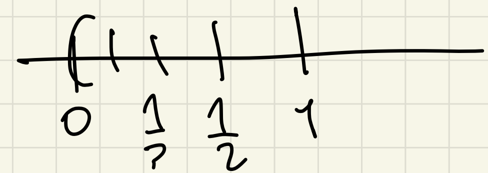
$a = 0$ $\quad F(0) = 0 \quad \lim_{x \to 0^+} F(x) = \lim_{n \to +\infty} \mu(0, \frac{1}{n}]$

$$= \lim_{n\to+\infty} \mu\left(0,\frac{1}{n}\right] = \mu\left[\bigcap_{n=1}^{+\infty}\left(0,\frac{1}{n}\right]\right] = \mu(\phi) = 0$$

$$\boxed{\lim_{x\to 0^+} F(x) = 0 = F(0)}$$

$$(0,1] \supseteq \left(0,\frac{1}{2}\right] \supseteq \left(0,\frac{1}{3}\right] \supseteq \dots$$

$$\bigcap_{n=1}^{+\infty}\left(0,\frac{1}{n}\right] = \phi$$

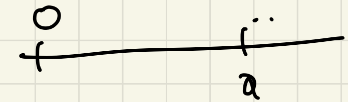


$$x \in \bigcap_{n=N}^{+\infty}\left(0,\frac{1}{n}\right] \qquad 0 < x \le \frac{1}{N}$$

$$x \in \left(0,\frac{1}{n}\right] \qquad 0 < x \le \frac{1}{n}$$

$a>0$ $\quad \lim_{n\to+\infty} F\left(a+\frac{1}{n}\right) = F(a)$

$\underline{a>0}$ $\quad F(a) = \mu\,(0,a]$



$$\lim_{n} F\left(a+\frac{1}{n}\right) = \lim_{n} \mu\left(0, a+\frac{1}{n}\right] =$$

$$= \mu\Big(\underbrace{\bigcap_{n=1}^{\infty} \left(0, a+\frac{1}{n}\right]}_{0<x\le a+\frac{1}{n}}\Big) = \mu\,(0,a] = F(a)$$

$a<0$ ...

If $\mu_F = \mu_G$ for F, G right cont ~~d~~ and increasing . non decreasing

$$\Downarrow$$

$\exists\, k \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $F(x) = G(x) + k$

$$\mu_F(a,b] = F(b) - F(a) \overset{\underline{\underline{a<b}}}{=} \mu_G(a,b] = G(b) - G(a)$$

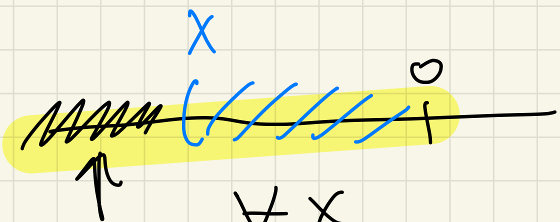
$$\boxed{\forall a, b \qquad F(b) - F(a) = G(b) - G(a)}$$

Fix $a = 0$ $\quad F(x) - F(0) = G(x) - G(0)$

$$F(x) + \underbrace{G(0) - F(0)}_{\parallel\; k} = G(x) \qquad \forall x \in \mathbb{R}$$

μ is finite $\qquad G(x) = \mu(-\infty, x] < +\infty$

$$F(x) = \begin{cases} 0 & x = 0 \\ \mu(0, x] & x > 0 \\ -\mu(x, 0] & x < 0 \end{cases}$$



$\forall x$

$$\boxed{G(x) = F(x) + \underbrace{\mu(-\infty, 0]}_{\text{constant}}}$$

$G(0) = 0 + \mu(-\infty, 0]$

$x > 0 \qquad \mu(-\infty, x] = \mu(0, x] + \mu(-\infty, 0] = \mu(-\infty, x]$

$x < 0 \qquad \mu(-\infty, x] = -\mu(x, 0] + \mu(-\infty, 0] = \mu(-\infty, x]$