COMPUTABILITY (04/11/2025)

* DIAGONALISATION

idea : $x_i \qquad i \in I$

$$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \dots \quad \underset{\substack{\uparrow \\ \text{position } k \text{ of } x_i}}{x_i} \quad \dots$$

aim : build $x \qquad x \neq x_i \quad \forall i \in I$

(x different from x_i at position i)

Cantor $\quad \forall X$ set

$$|X| < |2^X| \qquad\qquad 2^X = \{ Y \mid Y \subseteq X \}$$

if X is finite $\qquad X = \{0, 1\} \qquad 2^X = \{ \emptyset, \{0\}, \{1\}, X \}$

$$|X| = 2 < |2^X| = 2^{|X|}$$

Example : $|\mathbb{N}| < |2^{\mathbb{N}}|$

proof

assume by contradiction $|\mathbb{N}| \geq |2^{\mathbb{N}}|$ i.e. $2^{\mathbb{N}}$ countable

($\mathbb{N} \to 2^{\mathbb{N}}$ surjection)

$\overbrace{\qquad\qquad\qquad\qquad}^{2^{\mathbb{N}}}$

	X_0	X_1	X_2	...
0	~~YES~~ NO	NO	YES	
1	NO	~~NO~~ YES	YES	
2	NO	YES	~~YES~~ NO	
3	YES	NO	NO	⋱
⋮	NO	⋮	⋮	
	⋮			

$X_0 = \{0, 3\}$

$D = \{ i \mid i \notin X_i \}$

$\subseteq \mathbb{N}$

$\Rightarrow \exists k$ s.t. $D = X_k$

problem $K \in D$?

- YES : $K \in D \Rightarrow K \notin X_K = D$ contradiction
- NO : $K \notin D \Rightarrow K \in X_K = D$ "

$\Rightarrow$ $2^{\mathbb{N}}$ not countable $\Rightarrow$ $|\mathbb{N}| < |2^{\mathbb{N}}|$

$\square$

<u>EXERCISE</u> : $\mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$ $\quad |\mathcal{F}| > |\mathbb{N}|$

2 possible ways of proving ...

(1) $\mathcal{F}_2 = \{ f \in \mathcal{F} \mid f : \mathbb{N} \to \mathbb{N} \text{ total}, \ \mathrm{cod}(f) \subseteq \{0,1\} \} \subseteq \mathcal{F}$

$\mathcal{F}_2 \to \mathcal{F}$
$f \mapsto f$
injective

$$\begin{array}{ccc} |\mathcal{F}_2| & = & |2^{\mathbb{N}}| \\ \wedge\!| & & \vee \\ |\mathcal{F}| & & |\mathbb{N}| \end{array}$$

bijection $\mathcal{F}_2 \to 2^{\mathbb{N}}$
$f \mapsto \{ n \mid f(n) = 1 \}$

(2) $|\mathcal{F}| > |\mathbb{N}|$

consider an enumeration of functions in $\mathcal{F}$ and argue that there is some function missing

$$\begin{array}{c|cccc} & f_0 & f_1 & f_2 & \dots \\ 0 & f_0(0) & f_1(0) & f_2(0) & \dots \\ 1 & f_0(1) & f_1(1) & f_2(1) & \dots \\ 2 & f_0(2) & f_1(2) & f_2(2) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

$f : \mathbb{N} \to \mathbb{N}$

$$f(n) = \begin{cases} f_n(n) + 1 & \text{if } f_n(n)\downarrow \\ 0 & \text{if } f_n(n)\uparrow \end{cases}$$

we have $f \neq f_n \quad \forall n$, by construction since $f(n) \neq f_n(n)$

Hence there is no full enumeration of elements of $\mathcal{F}$

$\Rightarrow$ $|\mathcal{F}| > |\mathbb{N}|$

$\square$

OBSERVATION: There is a total non-computable function

$f : \mathbb{N} \to \mathbb{N}$

$$f(n) = \begin{cases} \varphi_n(n) + 1 & \text{if } n \in W_n \quad (\text{if } \varphi_n(n)\downarrow) \\ 0 & \text{if } n \notin W_n \quad (\text{if } \varphi_n(n)\uparrow) \end{cases}$$

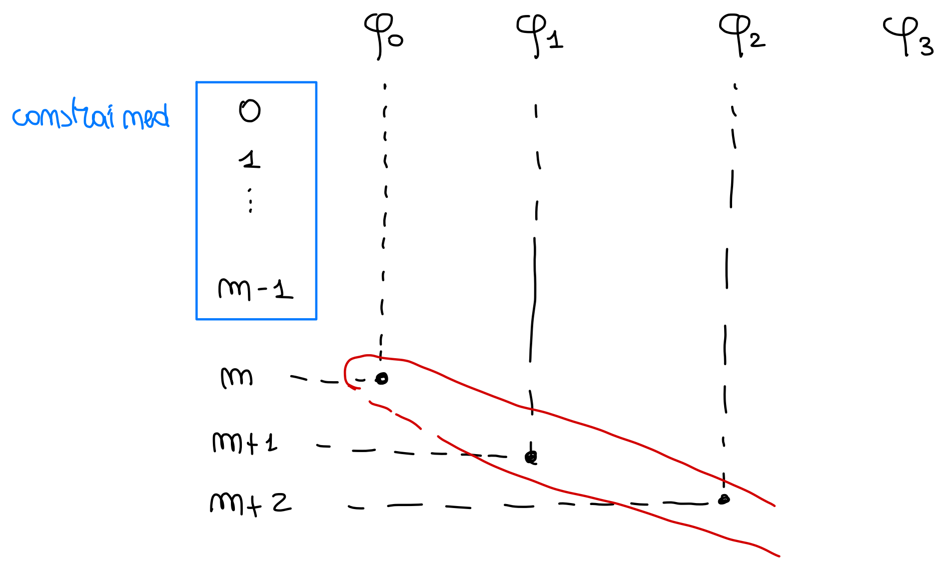
	φ_0	φ_1	φ_2	φ_3	...
0	$\varphi_0(0)$	$\varphi_1(0)$	$\varphi_2(0)$	...	
1	$\varphi_0(1)$	$\varphi_1(1)$	$\varphi_2(1)$	...	
2	$\varphi_0(2)$	$\varphi_1(2)$	$\varphi_2(2)$	...	
3	⋮	⋮	⋮	⋮	

- f is total
- f not computable $\quad \forall n \quad f \neq \varphi_n \quad$ since $\quad f(n) \neq \varphi_n(n)$
  - $\varphi_n(n)\downarrow \qquad f(n) = \varphi_n(n) + 1 \neq \varphi_n(n)$
  - $\varphi_n(n)\uparrow \qquad f(n) = 0 \neq \varphi_n(n)$

EXERCISE: Let $f : \mathbb{N} \to \mathbb{N}$ be a function, $n \in \mathbb{N}$

Show that there is a non computable function $g : \mathbb{N} \to \mathbb{N}$ s.t.

$\forall x < n \qquad g(x) = f(x)$



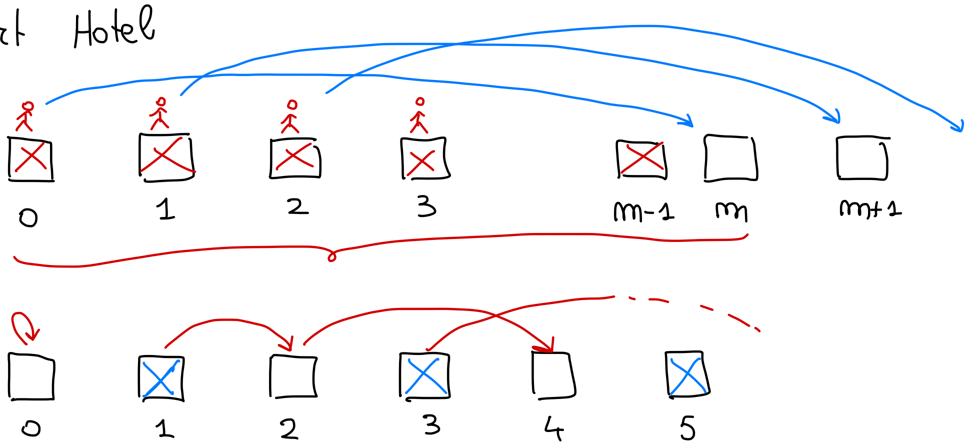
define

$$g(x) = \begin{cases} f(x) & \text{if } x < m \\ \varphi_{x-m}(x) + 1 & \text{if } x \geq m \text{ and } \varphi_{x-m}(x)\downarrow \\ 0 & \text{if } x \geq m \text{ and } \varphi_{x-m}(x)\uparrow \end{cases}$$

not computable

$\forall x \quad \varphi_x \neq g \qquad \varphi_x(x+m) \neq g(x+m)$

Hilbert Hotel



Alternative

$$g(x) = \begin{cases} f(x) & \text{if } x < m \\ \varphi_x(x) + 1 & \text{if } x \geq m \text{ and } \varphi_x(x)\downarrow \\ 0 & \text{if } x \geq m \quad \text{''} \quad \varphi_x(x)\uparrow \end{cases}$$

g is not computable

$$\varphi_0 \quad \varphi_1 \quad \cdots \quad \varphi_{m-1} \quad \overbrace{\varphi_m \quad \varphi_{m+1} \quad \varphi_{m+2} \quad - - - - -}^{g \neq}$$

$g \neq \varphi_x \quad \forall x \geq m$

infinitely many repetitions of all computable functions

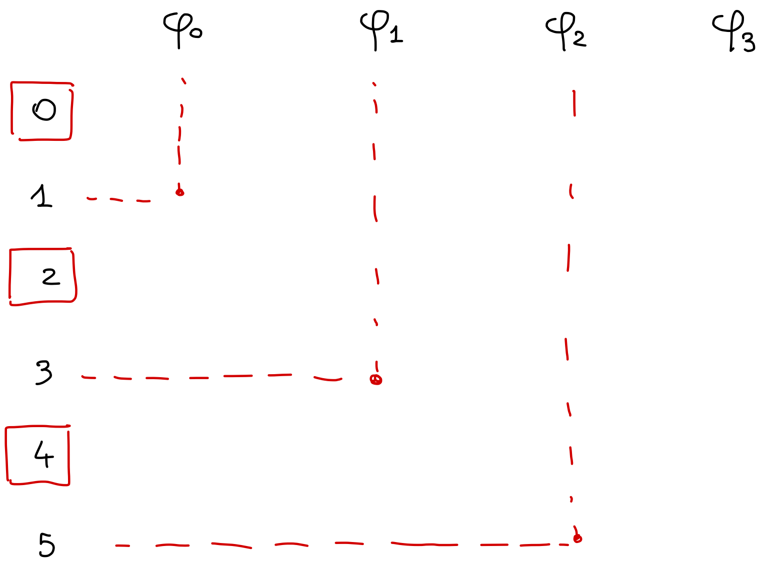
for all computable functions h since there are infinitely many x s.t. $h = \varphi_x$ there is $x \geq m$ s.t. $h = \varphi_x$

$\leadsto \quad g \neq \varphi_x = h$

$\Rightarrow$ g not computable $\square$

EXERCISE : show that there exists a non-computable total function such that

$g(n) = 0 \quad \forall\, n$ even



Idea : $g \neq \varphi_n$

on input $2n+1$

$$g(n) = \begin{cases} 0 & \text{if } n \text{ even} \\ \varphi_{\frac{n-1}{2}}(n) + 1 & \text{if } n \text{ odd } \& \ \varphi_{\frac{n-1}{2}}(n)\downarrow \\ 0 & \text{if } n \text{ odd and } \varphi_{\frac{n-1}{2}}(n)\uparrow \end{cases}$$

- g total
- $g(n) = 0$ if n is even
- g is not computable since $\forall n \ \ \varphi_n \neq g$

$$g(2n+1) \neq \varphi_n(2n+1)$$

EXERCISE : Given $f_0, f_1, f_2 \dots$ countable sequence of functions. $\mathbb{N} \to \mathbb{N}$

Define $f : \mathbb{N} \to \mathbb{N}$ s.t. $dom(f) \neq dom(f_i)$ $\ i \in \mathbb{N}$.

PARAMETRISATION (SMN) THEOREM

Let $e \in \mathbb{N}$ and consider the function computed $P_e = \gamma^{-1}(e)$

$$f = \varphi_e^{(2)} : \mathbb{N}^2 \to \mathbb{N}$$

$$f(x, y) = \varphi_e^{(2)}(x, y)$$

Let $x \in \mathbb{N}$ fixed and consider

$$f_x : \mathbb{N} \to \mathbb{N}$$

$$f_x(y) = f(x,y) = \varphi_e^{(2)}(x, y)$$

computable
(constant x composed with f)

example $f(x,y) = y^x$

$$f_0(y) = y^0 = 1$$

$$f_1(y) = y^1 = y$$

$$f_2(y) = y^2 = y^2$$

$\vdots$

since f_x is computable for all $x \in \mathbb{N}$, there is $d \in \mathbb{N}$

$$f_x = \varphi_d \quad \ldots \quad = \varphi_{s(e,x)}$$

depends on $e \in \mathbb{N}$, $x \in \mathbb{N}$

hence there is a function $s : \mathbb{N}^2 \to \mathbb{N}$

$$s(e, x) = d$$

The smn theorem says that s is computable

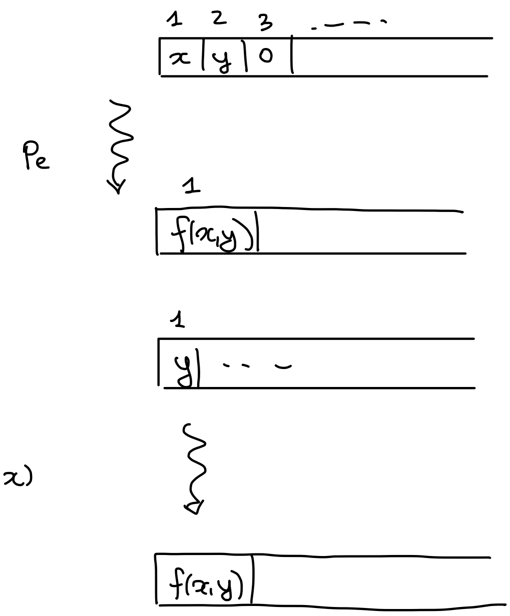
$f(x,y)$ {
def $P_e(x,y)$:
 - - .
 — —
 — _ y
 x
return - -

fix $x = 1$

def $P_e(\cancel{x}, y)$
 - - .
 — —
 — _ y
 $\cancel{x}$ 1
return - -

idea

given $e \in \mathbb{N}$



for each $x \in \mathbb{N}$ fixed

we want a program $P_{s(e,x)}$

how can we obtain $P_{s(e,x)}$?

$$P_e \overset{x}{\rightsquigarrow} P_{s(e,x)} \begin{cases} \text{move } y \text{ in } R_2 \\ \text{write } x \text{ to } R_1 \\ P_e \end{cases}$$

$$e, x \rightsquigarrow P_{s(e,x)}$$

[TO BE CONTINUED]