CONVOLUTION

$T \in D'(U)$ $\phi \in C_c^\infty(\mathbb{R}^n)$ $V_\phi = \{ x$ s. that $\forall y \in \operatorname{supp}\phi$ $x-y \in U \}$

$x - \operatorname{supp}\phi \subseteq U$

$\forall x \in V_\varphi$ $T * \phi(x) := T(\varphi^x)$ $\varphi^x(y) = \varphi(x-y) \in C_c^\infty(U)$.

$T * \varphi \in C^\infty(V_\phi)$

$$D^\alpha(T * \varphi)(x) = (D^\alpha T) * \varphi(x) = (T * D^\alpha \varphi)(x)$$

$$\frac{\partial}{\partial x_1}[T * \varphi(x)] = \lim_{h \to 0} T\left(\frac{\varphi^{x+he_1} - \varphi^x}{h}\right) \longrightarrow T\left(\frac{\partial}{\partial x_1}\phi(x-y)\right) = T\left(\frac{\partial}{\partial x_1}\phi^x\right)$$

$x \in V_\phi$ $x + he_1 \in V_\phi$

h small

$$\varphi_h(y) := \frac{\phi(x+he_1) - \phi(x-y)}{h} \xrightarrow{h \to 0} \frac{\partial}{\partial x_1}\phi(x-y)$$

in $C_c^\infty(U)$.

$$\left(\frac{\partial}{\partial x_i}T\right) * \phi(x) = \left(\frac{\partial}{\partial x_i}T\right)(\varphi^x) = -T\left(\frac{\partial}{\partial y_i}\phi^x\right) = -T\left(-\frac{\partial}{\partial x_i}\phi^x\right) =$$

$$= T\left(\frac{\partial}{\partial x_i}\phi^x\right) = \frac{\partial}{\partial x_i}(T * \varphi(x))$$

Another property of convolution

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ $\quad T * \phi \in C^\infty(V_\phi)$ $\quad$ let $\psi \in C_c^\infty(V_\phi)$

then for $\tilde{\phi}(x) := \varphi(-x)$ $\qquad \tilde{\phi} * \psi(y) = \int_{V_\phi} \varphi(x-y)\,\psi(x)\,dx$

is well defined for $y \in U$ $\quad (x \in V_\phi \quad x-y \in \operatorname{supp} \phi \Rightarrow y \in U)$

$\tilde{\phi} * \psi \in C_c^\infty(U)$

It is true that $T(\tilde{\varphi} * \psi) = \int_{V_\phi} (T * \phi)(x)\,\psi(x)\,dx$

indeed $\int_{V_\phi} (T*\varphi)(x)\,\psi(x)\,dx = \int_{V_\phi} T(\varphi^x)\,\psi(x)\,dx = \left[y \to \int_{V_\phi} \varphi^x(y)\,\psi(x)\,dx \right]$

$\overset{\text{"linearity"}}{=} T\left[\int_{V_\phi} \phi(x-\cdot)\,\psi(x)\,dx \right]$.

Lemma $\rho(x) = \begin{cases} C e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$ $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$

$\rho_k(x) = k^n \rho(kx)$ $\operatorname{supp} \rho_k = \overline{B(0,\frac{1}{k})}$ $\rho_k \in C^\infty_c(\mathbb{R}^n)$

Let $\phi \in C^\infty_c(U)$. then $\phi * \rho_k \to \phi$ in $C^\infty_c(U)$.

proof Let k_0 such that $\operatorname{supp}(\phi * \rho_{k_0}) = \operatorname{supp}\phi + B(0,\frac{1}{k_0}) = C_0 \subset\subset U$

$\forall k \geq k_0$ $\operatorname{supp}(\phi * \rho_k) \subseteq \operatorname{supp}(\phi * \rho_{k_0}) = C_0$

moreover $\phi * \rho_k \to \phi$ as $k \to 0$ in $C^\infty_c(U)$ $\square$

Theorem $T \in \mathcal{D}'(U)$, (1) $T * \rho_k \to T$ in the sense of distribut.

(2) Moreover $\exists\, \phi_i \in C^\infty_c(U)$ such that $\phi_i \to T$ in the sense of distributions. (That is $C^\infty_c(U)$ is dense in $\mathcal{D}'(U)$).

proof theor (1) $V_{\rho_k} = \{x \mid x + \overline{B(0,\frac{1}{k})} \subseteq U\} = \{x \mid \overline{B(x,\frac{1}{k})} \subseteq U\} =$

$= \{x \in U \;\; d(x, \mathbb{R}^n \setminus U) > \frac{1}{k}\}$

$T * \rho_k \in C^\infty(V_{\rho_k})$ $\quad V_{\rho_k} \subseteq V_{\rho_{k+1}} \subseteq U$

$\bigcup_k V_{\rho_k} = U$

$\forall A \subset\subset U$ open $\exists k_0$ $A \subseteq V_{\rho_k}$ $\forall k \geq k_0 \Rightarrow T * \rho_k \in C^\infty(A)$ $\forall k \geq k_0$

let $\psi \in C_c^\infty(U) \to \exists k_0 \; \mathrm{supp}\,\psi \subseteq V_{\rho_k}$ $\forall k \geq k_0$.

$$T_{T*\rho_k}(\psi) = \int_{V_{\rho_k}} \psi(y) \cdot T * \rho_k(y)\, dy \underset{\text{prop of convolution}}{=} T(\rho_k * \psi)$$

$T_{T*\rho_k}(\psi) = T(\rho_k * \psi)$ $\quad \forall k \geq k_0$ $\quad \mathrm{supp}\,\psi \subseteq V_{\rho_{k_0}}$

$\rho_k * \psi \to \psi \Rightarrow T(\rho_k * \psi) = T_{T*\rho_k}(\psi) \longrightarrow T(\psi)$ $\quad \forall \psi$..

$\Rightarrow T * \rho_k \to T$ in the sense of distributions in $D'(A)$ $\forall A \subset\subset U$

(1) is proved

② $T * e_k \notin C_c^\infty(U)$ so I want to reduce to T of compact support. Let $U_h = \{x \in V_{e_h},\ |x| < h\} \subset\subset U$

$U_h \subseteq V_{e_h}$ $U_h \subset\subset U$. $U_h \subseteq U_{h+1}$

Let $\xi_h \in C_c^\infty(U)$ $\xi_h \equiv 1$ on U_h $\xi_h \equiv 0$ on $U \setminus U_{h+1}$, $\xi_h \geq 0$.

$T_h(\varphi) := T(\xi_h \varphi)$

→ $\operatorname{supp} T_h \subseteq U_{h+1}$. [$\forall \phi \in C_c^\infty(U)$ $\operatorname{supp} \phi \subseteq U \setminus U_{h+1}$ $\phi \cdot \xi_h \equiv 0$ $T_h(\phi) = T(\varphi \xi_h) = 0$.]

→ Note that $T_h \longrightarrow T$ in the sense of distribution

(since $\forall \phi \in C_c^\infty(U)$ $\exists h_0$ such that $\operatorname{supp} \phi \subseteq U_{h_0}$ $\Rightarrow$ $\xi_h \phi = \phi$ $\forall h \geq h_0$

$\downarrow$ $T_h \phi = T\varphi$ $\forall h \geq h_0$.

→ CLAIM

$$\operatorname{supp}(T_h * e_k) \subseteq \underbrace{\left\{x \in U,\ d(x, \mathbb{R}^n \setminus U) > \frac{1}{h+1} - \frac{1}{k}\ \ |x| < h+1+\frac{1}{k}\right\}}_{A_{h,k} \subset\subset U} := A_{h,k}$$

NEED TO ASSUME $k > h+1$!

Let $\psi \in C_c^\infty(U)$ $\operatorname{supp} \psi \subseteq U \setminus A_{h,k}$.

$$\int_U T_h * \rho_k (x) \psi(x)\, dx = T_h(\tilde{\rho}_k * \psi) = T_h(\rho_k * \psi) = T(\xi_h(\rho_k * \psi))$$

$$\operatorname{supp}(\rho_k * \psi) \subseteq \operatorname{supp}\psi + \overline{B(0,\tfrac{1}{k})} \subseteq \{x \in U \text{ either } |x| \geq h+1+\tfrac{1}{k} \text{ or } d(x, \mathbb{R}^n \setminus U) \leq \tfrac{1}{h+1} - \tfrac{1}{k}\} + \overline{B(0,\tfrac{1}{k})} \subseteq U \setminus U_{h+1}$$

$$\Rightarrow (\rho_k * \psi)\,\xi_h \equiv 0 \Rightarrow T_h(\rho_k * \psi) = 0 = \int_U (T_h * \rho_k)(x)\, \psi(x)\, dx$$

$$\Rightarrow T_h * \rho_k \in C_c^\infty(U) \Rightarrow T_{T_h * \rho_k} \in \mathcal{D}'(U)$$

(2) $\rho_k * \psi \to \psi$ in $C_c^\infty(U)$ by the ~~lemma~~

$$T_h(\rho_k * \psi) \longrightarrow T_h(\psi) \qquad k \to +\infty \quad \forall \psi$$

$$T_h(\rho_k * \psi) = T_{T_h * \rho_k}(\psi) \longrightarrow T_h(\psi) \quad \text{for } k \to +\infty \quad \forall \psi \in C_c^\infty(U)$$

$T_{T_h * \rho_k} \longrightarrow T_h$ in the sense of distribution

$$T_{T_h * \rho_k} \xrightarrow{k \to +\infty} T_h \xrightarrow{h \to +\infty} T : \; T_{T_h * \rho_k} \longrightarrow T \text{ in the sense of distributions}$$

Another application of convolution.

Def $\mathcal{G}$ is a differential operator of order k with constant coefficients if

$$\forall \phi \in \mathcal{C}^\infty \quad \mathcal{G}(\phi) = \sum_{|\alpha| \leq k} C_\alpha D^\alpha \phi \qquad C_\alpha \in \mathbb{R}$$

Ex $\mathcal{G}(\phi) = \Delta \phi = \frac{\partial^2}{\partial x_1^2}\phi + \dots + \frac{\partial^2}{\partial x_n^2}\phi$

$\forall \phi \in \mathcal{C}^\infty$

$C_\alpha = 1 \iff \alpha = (2,0\dots0)$.
$\alpha = (0,2,\dots)$
otherwise $C_\alpha = 0$

Then $T \in \mathcal{D}'(\mathbb{R}^n)$

$$\mathcal{G}(T) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} C_\alpha D^\alpha (T)$$

ex $\Delta(T) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} T$

Def T is a fundamental solution of the different operator $\mathcal{Y}$ if $\mathcal{Y}(T) = \delta_0$

that is $\mathcal{Y}(T)(\varphi) = \varphi(0) \quad \forall \varphi \in C_c^\infty(U)$.

Proposition If T is a fundamental solution of the operator $\mathcal{Y}$ then

$$\begin{cases} \forall \phi \in C_c^\infty(\mathbb{R}^n) \quad T * \phi \in C^\infty(\mathbb{R}^n) \quad \text{and} \\ \mathcal{Y}(T * \phi) = \phi(x) \end{cases}$$

proof $\mathcal{Y}(T * \phi)(x) = (\mathcal{Y}T) * \phi = \delta_0 * \phi = \phi(x)$ $\square$.