Applicazioni lineare:

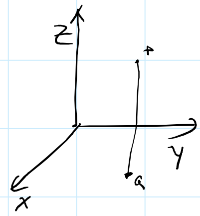
$f: V \to W$ funzione tra $\mathbb{R}$-spazi vettoriali, si dice lineare

⓪ $f(0_V) = 0_W$

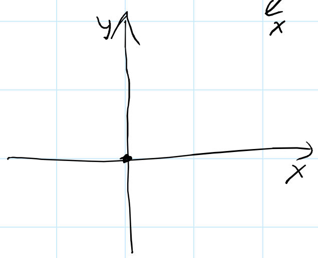
① Rispetta la somma $f(v_1+v_2) = f(v_1) + f(v_2) \quad \forall v_1, v_2 \in V$

② Rispetta il prodotto per scalari $f(av) = a f(v) \quad \forall v \in V \quad \forall a \in \mathbb{R}$.

Esempio: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$



$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$



$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ matrice della proiezione ortogonale sul sottospazio $\langle e_1, e_2 \rangle$ ortogonalmente

Definizione. data $f: V \to W$ lineare si chiama nucleo di f (Kernel)

$$\boxed{\operatorname{Ker} f := \{ v \in V \mid f(v) = 0_W \}} = f^{-1}\{0_W\}$$

Calcoliamo $\operatorname{Ker} f$ con $f: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$$\operatorname{Ker} f = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$\begin{cases} x = 0 \\ y = 0 \\ 0 = 0 \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\boxed{\dim \operatorname{Ker} f = \operatorname{null}(f) = 1}$$

Calcoliamo $\operatorname{Im} f = \{ f(v) \mid v \in V \}$

$$\operatorname{Im} f = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\dim \operatorname{Im} f = \operatorname{rg}(f) = \operatorname{rk}(f) = 2$$

$$\boxed{\begin{array}{l} \dim V = 3 = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f) \\ \quad\;\; 3 = \quad 1 \quad + 2 \end{array}}$$

Formula delle dimensioni

Definizione: data $f: V \to W$ lineare, $B \subseteq W$ si chiama controimmagine o antiimmagine di B tramite f

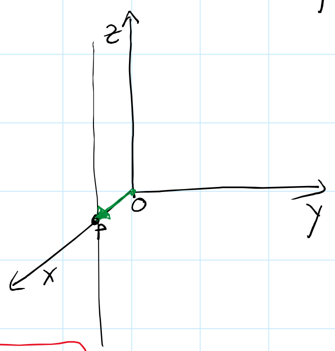
$$f^{-1}B = \{ v \in V \mid f(v) \in B \}$$

Ad esempio se $B = \{w\}$ con $w \in W$, $B \subseteq W$

$$\boxed{f^{-1}\{w\} = \{ v \in V \mid f(v) = w \}}$$

Esempio: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\quad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$



$\begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$

Calcoliamo $\boxed{f^{-1}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}} = \left\{ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} =$ $\quad \begin{cases} x = 1 \\ y = 0 \\ \cancel{0 = 0} \end{cases}$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{z=0} + z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{coeff. del parametro } z} \;\middle|\; z \in \mathbb{R} \right\} =$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$= \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \operatorname{Ker}(f)} \qquad \operatorname{Ker} f = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Calcoliamo $\boxed{f^{-1}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \emptyset}$ $\quad \begin{cases} x = 0 \\ y = 0 \\ 0 = 1 \end{cases}$ impossibile

Domanda 3:

Sia $f: V \to W$ un'applicazione lineare allora $\operatorname{Ker} f \leq V$.

Dim: $\operatorname{Ker} f := \{ v \in V \mid f(v) = 0_W \}$

⓪ $0_V \in \operatorname{Ker} f$ perché $f(0_V) = 0_W$ ⓪ di f appl. lineare.

① Chiusura per la somma: $\forall\, v_1, v_2 \in \operatorname{Ker} f$ dimostriamo che $v_1 + v_2 \in \operatorname{Ker} f$.

$v_1 \in \operatorname{Ker} f \iff f(v_1) = 0_W$

$v_2 \in \operatorname{Ker} f \iff f(v_2) = 0_W$

allora $v_1 + v_2 \in \operatorname{Ker} f$ perché $f(v_1 + v_2) \overset{\text{① } f \text{ appl. lineare}}{=} f(v_1) + f(v_2) = 0_W + 0_W = 0_W$

② Chiusura per prodotto per scalari: $\forall\, v \in \operatorname{Ker} f$ $\forall\, a \in \mathbb{R}$ dimostriamo che $av \in \operatorname{Ker} f$

$v \in \operatorname{Ker} f \iff f(v) = 0_W$

allora $av \in \operatorname{Ker} f$ perché $f(av) \overset{\text{② } f \text{ appl. lineare}}{=} a f(v) = a\, 0_W = 0_W$.

$\square$

Domanda 4:

Sia $f: V \to W$ lineare allora

$$\operatorname{Im} f = \{ f(v) \mid v \in V \} \leq W.$$

Dimostrazione:

⓪ $0_W \in \operatorname{Im} f$ $\qquad 0_W = f(0_V) \in \operatorname{Im}(f)$

⓪ f è appl. lineare.

① Chiusura per la somma: $\forall\, f(v_1), f(v_2) \in \operatorname{Im} f$ la loro somma

$$f(v_1) + f(v_2) = f(v_1 + v_2) \in \operatorname{Im} f \qquad v = v_1 + v_2$$

① f appl. lineare

② Chiusura per prodotto per scalari: $\forall\, f(v) \in \operatorname{Im} f$ $\forall\, a \in \mathbb{R}$

$$a f(v) = f(av) \in \operatorname{Im} f$$

② f appl. lineare

$\square$

Osservazione: se $f: V \longrightarrow W$ e $V = \langle v_1, \dots, v_n \rangle$ allora
$\operatorname{Im} f = f(V) = \langle f(v_1), \dots, f(v_n) \rangle$ perché

$$\operatorname{Im} f = \{ f(v) \mid v \in V \} = \left\{ f\left(\sum_{i=1}^{n} a_i v_i \right) \;\middle|\; a_i \in \mathbb{R} \ \forall i = 1, \dots, n \right\} =$$

$$= \left\{ \sum_{i=1}^{n} a_i f(v_i) \;\middle|\; a_i \in \mathbb{R} \ \forall i = 1, \dots, n \right\} = \langle f(v_1), \dots, f(v_n) \rangle$$

$$f(a_1 v_1 + a_2 v_2 + \dots + a_n v_n) = f(a_1 v_1) + \dots + f(a_n v_n) =$$
$$= a_1 f(v_1) + \dots + a_n f(v_n)$$

Esempio:

Esercizio 3. Si consideri l'applicazione lineare $f : \mathbb{R}^4 \to \mathbb{R}^3$ tale

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 2x_4 \\ -x_1 + 3x_2 - x_3 + x_4 \\ -x_1 - x_2 + 3x_3 + x_4 \end{pmatrix}$$

i) Determinare la matrice M associata ad f rispetto alle basi canoniche di $\mathbb{R}^4$ ed $\mathbb{R}^3$. Determinare una base $\mathcal{B}_{\operatorname{Ker}(f)}$ di $\operatorname{Ker}(f)$ e una base $\mathcal{B}_{\operatorname{Im}(f)}$ di $\operatorname{Im}(f)$. L'applicazione f è iniettiva; è suriettiva; è biiettiva? (3,5 pti)

ii) Determinare $f^{-1}\left\{\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}\right\}$. (1 pti)

$$f: \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \qquad f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 2x_4 \\ -x_1 + 3x_2 - x_3 + x_4 \\ -x_1 - x_2 + 3x_3 + x_4 \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 0 & 0 & -2 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 3 & 1 \end{pmatrix}$$

Calcoliamo $\operatorname{Ker} f = \{ v \in V \mid f(v) = 0_W \} =$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 2x_1 - 2x_4 \\ -x_1 + 3x_2 - x_3 + x_4 \\ -x_1 - x_2 + 3x_3 + x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \begin{array}{c} M \qquad\qquad\quad 0 \\ \left(\begin{array}{cccc|c} 2 & 0 & 0 & -2 & 0 \\ -1 & 3 & -1 & 1 & 0 \\ -1 & -1 & 3 & 1 & 0 \end{array} \right) \end{array}$$

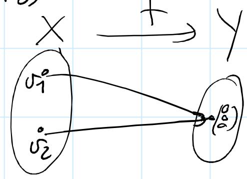
$$\begin{cases} x_1 - x_4 = 0 \\ -\cancel{x_4} + 3x_2 - x_3 + \cancel{x_4} = 0 \\ -\cancel{x_4} - x_2 + 3x_3 + \cancel{x_4} = 0 \end{cases} \qquad \begin{cases} x_1 = x_4 \\ x_3 = 3x_2 \\ -x_2 + 9x_2 = 0 \end{cases} \qquad \begin{cases} x_1 = x_4 \\ x_3 = 0 \\ x_2 = 0 \end{cases}$$

$$\operatorname{Ker} f = \left\{ \begin{pmatrix} x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix} \middle| \; x_4 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\boxed{\begin{array}{l} B_{\operatorname{Ker} f} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \\ \dim \operatorname{Ker} f = 1 \end{array}}$$

$\Rightarrow$ Non è iniettiva $\qquad \underbrace{f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_2}$ $\quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_1 \neq v_2$ ma $f(v_1) = f(v_2)$

$\Rightarrow$ Non è biiettiva $\qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$



Calcoliamo Im f.

Se $V = \langle v_1, \dots, v_n \rangle$ allora

$f(V) = \langle f(v_1), \dots, f(v_n) \rangle$

$f: \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \qquad V = \mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ allora

$\operatorname{Im} f = \langle f(e_1), f(e_2), f(e_3), f(e_4) \rangle =$

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 2x_4 \\ -x_1 + 3x_2 - x_3 + x_4 \\ -x_1 - x_2 + 3x_3 + x_4 \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{array}{c} M = \begin{pmatrix} 2 & 0 & 0 & -2 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 3 & 1 \end{pmatrix} \\ \quad\;\; \uparrow \quad\;\; \uparrow \quad\;\; \uparrow \quad\;\; \uparrow \\ \quad f(e_1) \; f(e_2) \; f(e_3) \; f(e_4) \end{array}$$

$f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \quad f(e_3) = f\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$$\operatorname{Im} f = \left\langle \underbrace{\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}}_{w_2}, \underbrace{\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}}_{w_3}, \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{w_4} \right\rangle = \left\langle \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\rangle$$

$w_1 + w_4 = \vec{0}$ relazione di dipendenza eliminiamo w_4

$B_{\operatorname{Im} f} = \left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$ oppure $\dim \operatorname{Im} f = 3 \quad \operatorname{Im} f \subseteq \mathbb{R}^3$

$\Rightarrow \boxed{\operatorname{Im} f = \mathbb{R}^3} \Rightarrow$ <u>f è suriettiva</u>

Oppure potevamo prendere come base di Im f $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

2) Calcoliamo $f^{-1}\left\{\underbrace{\begin{pmatrix}0\\2\\2\end{pmatrix}}_{\underline{w}}\right\} = \left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \mathbb{R}^4 \;\middle|\; f\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\2\\2\end{pmatrix}\right\}$

$$f\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}2x_1-2x_4\\-x_1+3x_2-x_3+x_4\\-x_1-x_2+3x_3+x_4\end{pmatrix} = \begin{pmatrix}0\\2\\2\end{pmatrix} \qquad \begin{cases}2x_1-2x_4=0\\-x_1+3x_2-x_3+x_4=2\\-x_1-x_2+3x_3+x_4=2\end{cases}$$

$x_1=0$
$x_2=1$
$x_3=1$
$x_4=0$

$$\left(\begin{array}{cccc|c}2&0&0&-2&0\\-1&3&-1&1&2\\-1&-1&3&1&2\end{array}\right) \xrightarrow{\frac{1}{2}\cdot 1^\circ R} \left(\begin{array}{cccc|c}1&0&0&-1&0\\-1&3&-1&1&2\\-1&-1&3&1&2\end{array}\right) \xrightarrow[\substack{2^\circ R+1^\circ R\\3^\circ R+1^\circ R}]{}$$

$$\left(\begin{array}{cccc|c}1&0&0&-1&0\\0&3&-1&0&2\\0&-1&3&0&2\end{array}\right) \xrightarrow[S_{2,3}]{} \left(\begin{array}{cccc|c}1&0&0&-1&0\\0&-1&3&0&2\\0&3&-1&0&2\end{array}\right) \xrightarrow[3^\circ+3\cdot 2^\circ R]{}$$

$$\left(\begin{array}{cccc|c}1&0&0&-1&0\\0&-1&3&0&2\\0&0&8&0&8\end{array}\right) \qquad \begin{cases}x_1=x_4\\-x_2+3x_3=2\\8x_3=8\end{cases} \qquad \begin{cases}x_1=x_4\\x_2=3x_3-2=3-2=1\\x_3=1\end{cases}$$

$$\boxed{f^{-1}\left\{\begin{pmatrix}0\\2\\2\end{pmatrix}\right\} = \left\{\begin{pmatrix}x_4\\1\\1\\x_4\end{pmatrix} \;\middle|\; x_4\in\mathbb{R}\right\} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \underbrace{\left\langle\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\rangle}_{\mathrm{Ker} f} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \mathrm{Ker} f}$$